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APPLICATION OF GROUP THEORY TO MUSICAL NOTES

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ABSTRACT: This paper critically analysis the behavior and the relationship that exist between musical notes and group theory. The musical notes form additive abelian group modulo 12. Finally, the work came up with some propositions due to the musical notes behavior and their proofs one of which was name Dido's Theorem.

KEYWORDS: Group Theory, Abelian Group Modulo 12, Musical Notes, Dido's Theorem, Transposition, Inversion

INTRODUCTION

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra. Other well-known algebraic structures, such as rings, fields, and vectors space can all be seen as groups endowed with additional operations and axioms. Various physical systems, such as crystals and the hydrogen atom, can be modeled by symmetry groups. Thus, group theory has many important applications in Physics, chemistry, and materials science. Group theory is also central to public key cryptography.

The modern concept of abstract group developed out of several fields of mathematics (Wussing, 2007). The idea of group theory although developed from the concept of abstract algebra, yet can be applied in many other areas of mathematical areas and other field in sciences and as well as in music.

Music

Music theory is a big field within mathematics and lots of different people have taken it in different directions. Music is that one of the fine arts which is concerned with the combination of sounds with a view to beauty of form and the expression of thought or feeling. Music itself is not complete without musical notes. And the musical notes are: $C, C\#, D, D\#\ E, F, F\#\ , G, G\#, A, A\#, B.$ For years, many people find it difficult to comprehend some concept in group theory satisfactorily, but with the behavior of these musical notes, group theory can be studied.

"All is number" is the motto of the Pythagorean School. This school was founded by the Greek mathematician and philosopher Pythagoras (ca. 580-500 B.C.) The members of the school pursued the study of mathematics, philosophy, astronomy and music. [Heath, p. 67].

Group Theory

Group theory is the branch of pure mathematics which is emanated from abstract algebra. Due to its abstract nature, it was seeming to be an arts subject rather than a science subject. In fact, was considered pure abstract and not practical.

Even students of group theory after being introduced to the course seems not to believe as to whether the subject has any practical application in real life, because of its abstract nature (Tsok, 2013).

The problem prompts the researchers to study the different ways in which group can be express concretely both from theoretical and practical point of view, with intention of bringing its real-life application in musical notes.

This paper aim at taking some concepts of group theory to study and understand musical notes in relation to the group's axioms. The main objective is to see these musical notes interpretation algebraically as regard to their behavior. This work focus on the behavior of musical notes which largely depend on groups axioms, theorems such as two left cosets, cyclic groups, Langrange's and Sylow's first theorem.

Definition of Terms

We present here some few definitions that will help us to be familiar with concepts in music and abstract algebra.

Musical Notes

Musical notes are the following notes $C\ \ C\#\ \ D\ \ D\#\ \ E\ \ F\ F\#\ \ G\ \ G\#\ \ A\ \ A\#\ \ B$ when logically combined, give out pleasant sound to the ear.

The first note which is C, is called the root note.

C# is called the 2nd note D is called the 3rd note

D# is called the 4th note E is called the 5th note

F is called the 5th note F# is called the 6th note

G is called the 7th note G# is called the 8th note

A is called the 10th note A# is called the 11th note

B is called the 12th note

Musical Flat b

Musical flats can be defined as the movement of sound from one pitch to the one lower, and it is donated to b. For example, movement from F to any other note to the left.

Musical Sharp #

This can be considered as the movement of sound from a pitch (note) to another pitch higher, and it bis denoted by #. For example, movement from F to any other note to the right on the musical notes.

Tone

This simply meant any movement from a musical note to the next note two steps forward or backward on the musical notes. For example, movement from F to G or to D#.

Semitone

This can be defined as any movement from a musical note to the next note a step forward or backward on the musical notes (Scales). For example, movement from F to F# or F to E.

Chord

A chord is produced when two, three or more notes are sounded together.

Transposition

Transposition involves playing or writing a given melody at a different pitch higher or lower other than the original.

Abstract Group

A group is a non- empty set $(G, *)$ together with an operation $(*)$ on it which satisfies the following axioms:

$G_1 \quad \forall x, y \in G, \quad x * y \in G \qquad Closure$

$G_2 \quad \forall x, y, z \in G, \quad (x * y) * z = x * (y * z) \qquad Associative$

$G_3 \quad \forall x, y \in G, \exists\, e \in G \ni \quad x * e = e * x = x \qquad Identity$

$G_4 \quad \forall x \in G \; \exists\, x^{-1} \in G \ni \quad x * x^{-1} = x^{-1} * x = e \qquad Inverse$

Furthermore, a group is said to be abelian if

$G_5 \quad \forall x, y \in G, \quad x * y = y * x \in G \qquad Abelian$

Integers Modulo m

This is a finite group that is called the additive group of the residue class of integers modulo m. it is denoted by z_m.

P- Group

Let p be an arbitrary but fixed prime number. A finite group is said to be a p-group if its order is power of p.

P- Subgroup

Let G be a group. If $T \leq G$ and $|T| = p^r$ for some $r \geq 0$ then T is called p - subgroup of G.

THEORETICAL UNDERPINNING

Several authors have work on the application of group theory to many fields in sciences, games and many more other fields but only few have ventured the field of music.

Pythagoras (428-347 B.C.). Who is considered as founder of the first school of mathematics as a purely deductive science is also the founder of a theoretical music. He used to say that “All is number “and musical notes are not exceptional, that is C, C#, D, D#, E, F, F#, G, G#, A, A#, B. But why “All is number”. The Pythagoreans associated certain meanings and characters to numbers. They considered odd numbers as males and even numbers as females. To the Pythagoreans, one is the number of reason, two is the number of opinion, three is the number of harmony, four is the number of justice, five is the number of marriage, six is the number of creation, seven is the number of awe, and ten is the number of the universe

A couple of possible reasons were given. The first one is the Eastern influence. Having traveled to Egypt and Babylon, Pythagoras might have been influenced by numerology, which deals with numbers and mystical relations among them, that was common in these two regions. A second possible reason is to give an alternative view to the contemporary belief in Greek concerning the principles of things. At the time, it was believed that earth, air, fire and water are the four basics principles of things. This did not convince Pythagoras in explaining the principles of immaterial things. A third possibility comes from astronomy, a subject that was studied by Pythagoras. In studying stars, one observes that each constellation can be characterized by the number of stars composing it and the geometrical figure that they form. The fourth possible reason comes from music. The members of the school practiced music. Pythagoras observed that musical notes produced from a vibrating string of some length could be characterized by (ratios of) numbers. Dividing a vibrating string by some movable object into two different lengths produced different types of musical notes. These notes are then described by the ratios of the lengths of the parts of the vibrating string. Explaining musical notes and describing stars by numbers may have then led the Pythagoreans to think that numbers can also be used to explain other phenomena. (Heath, 1965)

Thomas M. Flore (1993). He referred to C, C#, D, D#, E, F, F#, G, G#, A, A#, B. As the Z_{12} Model of pitch class. He constructed a musical clock as below:

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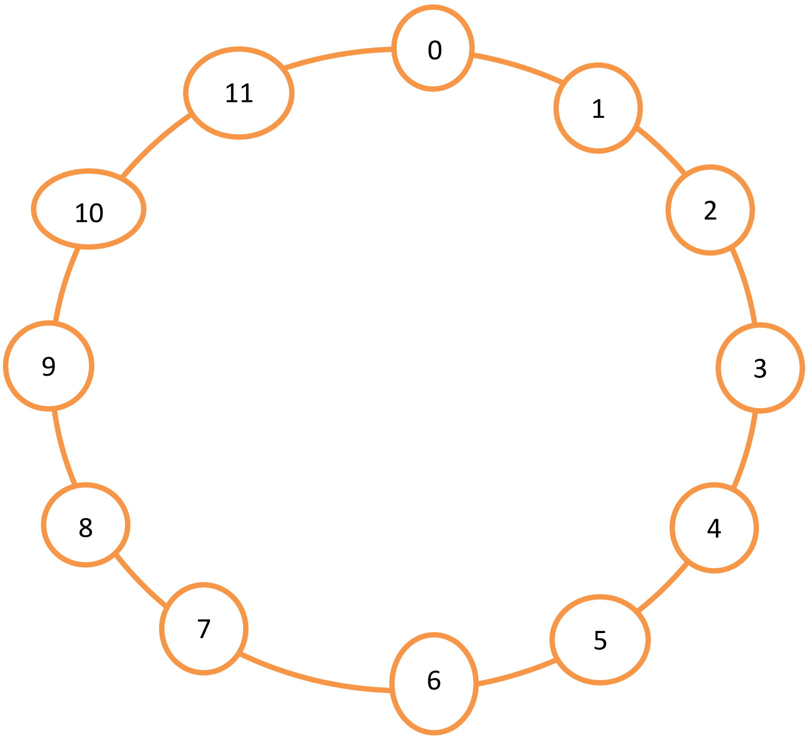

Fig. 2.1 Musical Clock

He also said, we have a bijection between the set of pitch classes and Z_{12}. He defined transposition as:

$T_n : Z_{12} \rightarrow Z_{12} \ni \quad T_n(X): X + n$ and invertion was also defined as $I_n : Z_{12} \rightarrow Z_{12} \ni$ $I_n\,(X) : -x + n$ where n is in mod12

Ada Zhang, (2009). Considered possibly musical notes with their corresponding integers as:

C	C#	D	D#	E	F	F#	G	G#	A	A#
0	1	2	3	4	5	6	7	8	9	10

He defined transposition, T_n as that which moves a pitch-class or pitch-class set up by n (mod12).

And inversion was also defined here as T_nI as the pitch (A) about C(0) and then transposes it by n. that is , $T_nI(a) = -a + n(mod12)$.

Then further, laid out all the pitches in a circular pattern on a 12-sided polygon. That is, consider the transposition T_{11}. It sends C to B, C# to C,

Alissa (2009). Assert that the musical actions of the dihedral groups. This paper considers two ways in which the dihedral groups act on the set of major and minor triads.

According to Emma, (2011), referred to the musical notes with their corresponding integers as in Ada Zhang, (2009) as M_{12}, that is the Mathieu group. He asserts that this can be generated by just two permutations Expressed below in both two-line notation and cycle notation. We denote these generating permutations as P_1 and P_0

$$P_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & 4 & 4 & 7 & 8 & 2 & 9 & 1 & 10 & 0 & 11 \end{bmatrix} = (0\ 5\ 8\ 1\ 6\ 2\ 4\ 3\ 7\ 9\ 10)(11)$$

$$P_0 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 5 & 7 & 4 & 8 & 3 & 9 & 2 & 10 & 1 & 11 & 0 \end{bmatrix} = (0\ 6\ 9\ 1\ 5\ 3\ 4\ 8\ 10\ 11)(2\ 7)$$

Adam, (2011) defined transposition and inversion as : Transposition is define as $T_n : Z_{12} \to Z_{12} \ni \quad T_n(X) : x + n \mod 12$ and he also define Inversion as $I_n : Z_{12} \to Z_{12} \ni \quad I_n\,(X) : -x + n$ where n is in mod12. This operation was composite function.

METHODOLOGY

Theorem 3.1

Let $H \leq G$ be groups and $g \in G$.

Then: (i) $g \in gH$

(ii) Two left cosets of H in G are either identical or disjoint.

(iii) The number of elements in gH is $|H|$

Proof:

(i) Since, $1 \in H$,

We have that $g^{-1} \in gH$

(ii) Take the left coset aH of H in G. By (i) above, $a \in aH$.

Suppose that $a \in bH$ for some $b \in G$. Then we have to show that $aH = bH$ since

Since$a \in bH$. We have that, $a = bh_1$ for some $h_1 \in H$, so that for any $h \in H$,.

$$ah = (bh_1)h$$

$$= b(h_1 h) \in bH$$

That is, $aH \subseteq bH$: and, Thus $bh = (ah_1^{-1})h \ = a(h_1^{-1}h) \in aH$

That is, $bH \subseteq aH$. Thus $aH = bH$. It follows that if $aH \cap bH \neq \emptyset$, then $aH = bH$ and as such distinct left cosets are disjoint

(iii) The map $H \to gH$ defined by $h \to gh$ is bijective. Thus, $|H| = |gH|$

Theorem 3.2 (Langrange's Theorem)

The order of a subgroup of a finite group is a factor of the order of the group.

Proof: Let $|G| = n < \infty$ $let\ H \leq G$ and let $|H| = m$ Now, G is the union of pairwise disjoint cosets of H. Let there be j distinct cosets of H in G. We know that for any $a \in G\ |aH| = |H| = m$. Therefore, the total number of elements in G is mj So n=mj, that is, m divides n and the result follow $|G| = |G: H||H|$.

Theorem 3.3

Every subgroup of a cyclic group is cyclic.

Proof: Let $H \leq G = \langle g \rangle$ If $H = \{1\},\ then\ H = \{g^0\}$ is trivially cyclic. Then $H \neq \{1\}$ and choose $h \in H$. Then $h = g^s$ for some $s \in Z$. And $h^{-1} = g^{-s}$ Thus there are positive integers $t \ni g^t \in H$.

Take the least of such positive integers and call it I. By the well-ordering principle of natural number, any set of positive integers contains a smallest number. By division algorithm we may write $S = ql + r,\ 0 \leq r < l.\ Then\ h = q^s = g^{ql+r} = (g^l)^q g^r$ so that $g^r = (g^l)^{-q} h \in H$ If $r \neq 0, then\ r < l$ which contradicts the choice of l. thus, $r = 0\ and\ so\ h = (g^l)^q$ Hence H$\subseteq \langle g^l \rangle.\ now\ g^l \in H\ and\ so\ \langle g^l \rangle \in H.$ Accordingly, $H = \langle g^l \rangle$ and the theorem follows.

Theorem 3.4 (First Sylow's Theorem)

Let G be a finite group, p a prime and p^r the highest power of P diving the order of G. Then there is a subgroup of G of order of G. Then there is a subgroup of G of order p^r.

Proof; We will prove the theorem by induction on the order n of G. For $|G| = 1$ the theorem is trivial. Assume $n > 1$ and the theorem is true for groups of order $< n$. Suppose $|Z(G)| = c$

We have two possibilities;

(i) $c|p$ or

(ii) $p \dagger c$

i. Suppose $c|p\ \ Z(G)$ is an abelian group. Therefore Z(G) has an element of order p.

Let N be a cyclic subgroup of Z(G) is normal in G consider G/N. Then $|G/N| = n/p$ by theorem 3.1 Hence by our induction assumption, G/N has a subgroup H of order p^{r-1} therefore $\exists$ a subgroup H of $G\exists\ H/N = \overline{H}$ As $p^{r-1} = |\overline{H}| = |H|/|N| = |H|/p,$ We conclude that $|H| = q^r$.

Thus, in this case, G has a subgroup of order p^r .

ii. Suppose $p|c$. The class equation for G is of the form $|G| = |Z(G)| + \sum_{R \in \mathfrak{R}} [G: C(R)]$.

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Since $p||G|$ $and\ p \dagger c$, we have $p \dagger \sum_{R\in\Re}[G: C(R)]$. Therefore for at least one $R \in \Re^*$, $p \dagger |G: C(R)|$ But $|G| = |G: C(R)|/|C(R)|$ by Theorem 3.1 Hence $p^r\big||C(R)|$ $, Since\ p^r\big||G|$. Now $|C(R)| \neq |G|; for\ if\ |C(R)| = |G|$, Then C(R) = G and $R \cap Z(G) = \phi$.

Thus, by the induction assumption, C(R) has a subgroup H of order P^r. Consequently, so does G.

In either case we have found a subgroup H of order P^r. AS required

RESULTS AND DISCUSSION

Introduction

Numbering of the Musical Notes

C	$C\#$	D	$D\#$	E	F	$F\#$	G	$G\#$	A	$A\#$	B
0	1	2	3	4	5	6	7	8	9	10	11

Note that $B\# = C$

It shows that the musical notes form a group of integers of Modulo 12.

That is $Z_{12} = \{C,\ C\#,\ D,\ D\#, E, F, F\#, G, G\#, A, A\#, B\ \}$. Let the operation be $* = \# = +$

Result of the behavior of the musical note on Groups

Musical Notes as it related to groups axiom. Without loss of generality

i. Closureness

$E, F \in Z_{12}$, hence $E * F = A \in Z_{12}$

ii. Associativity

$E, F\ and\ F\# \in Z_{12}$, hence $(E * F) * F\# = E * (F * F\#)$

$$= A * F\# = E * B$$

$$= D\# = D\#$$

iii. Identity

$F \in Z_{12} \exists\, C \in Z_{12}$, hence $F * C = C * F = F$

iv. Inverse

$F \in Z_{12} \exists G \in Z_{12}$, hence $F * G = G * F = C \in Z_{12}$

-Therefore, musical note behavior satisfied all the mathematical group axioms.

v. Furthermore, $\forall F, G \in Z_{12} \quad F * G = G * F = C \in Z_{12}$

This shows that it is not just a group, but also an abelian group.

With the behavior of the musical notes we have just seen, we personally suggest for the root note of musical scales (notes) to be algebraically named as the identity note.

Table 1: List of Musical Notes and their inverse

Note	Inverse
C	C
C#	B
D	A#
D#	A
E	G#
F	G
F#	F#

The behavior of the musical note as related to table 1. Gave us insight to formulate this Proposition which we intend call it Dido's theorem.

Proposition 4.1 (Dido's theorem)

If G is cyclic, then there is at least an element which is unique with its inverse

Proof

Suppose G is cyclic

$\Longrightarrow \forall x \in G,$ each $x \in G$ can be written in the form $x = g^m$ for some $g \in G$ Where $m \in Z \ \exists \, some \ y \in G \ni x * y = e \in G$

$\Longrightarrow x = y$ where e is the identity element $y = x^{-1} \Longrightarrow x = x^{-1}$

This completes the proof.

Remark table 1 give better understanding of the proposition above

The Result of theorem 3.1

$Z_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$. $\quad H = \{C, C\#, B\} \ \Longrightarrow H \leq Z_{12}$

$$DH = \{D * C, D * C\#, D * B\}$$

$$DH = \{D, D\#, C\#\}$$

Clearly, $D \in DH$ And again, $|H| = |DH| = 3$. Furthermore, for some $A, F \in Z_{12}$ $(A * F)H = DH$

$\Longrightarrow$ two left cosets are identical in this case for some $A, G \in Z_{12}$

$A * G \in Z_{12}$ but $A * G = E \neq D$, $EH \neq DH$ Two left cosets are disjoint in this case.

The Result of Theorem 3.2

$|Z_{12}| = 12$ Since $|Z_{12}| = 12$ $|H| = 3$ $\Rightarrow |H| = Z_{12}$ $|Z_{12}|/|H| = 12/3 = 4$ It is true that the order of a subgroup divides the order of a group.

The Result of theorem 3.3

From $Z_{12} = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$ For $C \in Z_{12}$

$C^0 = C$ $\quad C^1 = C\#$

$C^2 = D$ $\quad C^3 = D\#$

$C^4 = E$ $\quad C^5 = F$

$C^6 = F\#$ $\quad C^7 = G$

$C^8 = G\#$ $\quad C^9 = A$

$C^{10} = A\#$ $\quad C^{11} = B$

$C^{12} = C$

Musical notes are cyclic. That is $Z_{12} = \langle C \rangle$. Consider the subgroup $H = \{C,\ C\#, B\}$

$$B \in H \leq Z_{12}$$

$$B^0 = B$$

$$B^1 = C$$

$$B^2 = C\#$$

Clearly, H is cyclic. Using musical notes we are satisfied with the theorem which states that "every cyclic group has a subgroup which is also cyclic.

Proposition: Every musical note is a generator of Z_{12}

Proof

$C^0 = C$ $\quad C^1 = C$

$C^2 = D$ $\quad C^3 = D\#$

$C^4 = E$ $\quad C^5 = F$

$C^6 = F\#$ $\quad C^7 = G$

$C^8 = G\#$ $\quad C^9 = A$

$$C^{10} = A\# \qquad C^{11} = B$$

$$C^{12} = C \qquad -$$

Therefore, C has generated Z_{12}. Similarly, every other note can behave as such.

The Result of Theorem 3.4 Recall that the $|Z_{12}| = 12$, $12 = 2x2x3 = 2^2x3$ $\exists H \leq Z_{12} \ni |H| = 2^2$ Which is sylow-2subgroup of Z_{12} It is true that every finite group has a sylow-Psubgroup which is in line with the first sylow's theorem.

Group theory as a Structure for Atonal Music Theory

The numbering of the pitch classes reveals their isomorphism to Z_{12}. More interestingly, the group of transpositions and inversions, denoted $T_n/T_n I$ is isomorphic to the dihedral group D_{12} .

SUMMARY AND CONCLUSSION

The results of this findings in our paper, shows that musical notes behaviors satisfied all group axioms and are related to group theory. Since music is food for the soul and mind, we suggest that a good understanding of group theory to musician can help in composing best musical composition that will give satisfaction to Audience and as well bring healings to their minds.

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