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THE CONSTANT OF BELTRAMI FLOW IN THE HELICITY OF PEKERIS, ACCAD AND SHKOLAR FLOW

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Abstract: *In this paper will be showed that the constant C of Beltrami is equal to Λ, the positive root of the second order of Spherical Bessel $j_2(r)$ in Pekeris, Accad and Shkoler (PAS) flow. [1]*

KEYWORDS: Constant C, Positive Root

INTRODUCTION

Since Beltrami is PAS flow, then

$$C\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} \tag{1.1}$$

The value of C in (1.1) will be derived from the following equations

$$\boldsymbol{v} = \boldsymbol{s} + \boldsymbol{t} \tag{1.2}$$

$$\nabla \times \boldsymbol{v} = \nabla \times \boldsymbol{s} + \nabla \times \boldsymbol{t} \tag{1.3}$$

$$C\boldsymbol{v} = \nabla \times \boldsymbol{s} + \nabla \times \boldsymbol{t} \tag{1.4}$$

In PAS flow

$$\boldsymbol{t} = \left(0, \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}, -t_2^2\frac{\partial Y_2^2}{\partial\theta}\right) \tag{1.5}$$

$$\boldsymbol{s} = \left(\frac{6}{r}s_2^2Y_2^2, \frac{1}{r}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}\right) \tag{1.6}$$

MAIN RESULTS

Substitute (1.5) and (1.6) to (1.2) leads to

$$\boldsymbol{v} = \left(\frac{6}{r}s_2^2Y_2^2, \frac{1}{r}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta} + \frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}, \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi} - t_2^2\frac{\partial Y_2^2}{\partial\theta}\right) \tag{2.1}$$

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while $$\nabla\times\boldsymbol{t}=\left(\frac{6}{r}t_2^2Y_2^2,\frac{1}{r}\frac{\partial}{\partial r}(rt_2^2)\frac{\partial Y_2^2}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rt_2^2)\frac{\partial Y_2^2}{\partial\phi}\right) \tag{2.2}$$

and $$\nabla\times\boldsymbol{s}=\left(0,\frac{1}{r\sin\theta}\left(\frac{6}{r}s_2^2-\frac{\partial^2}{\partial r^2}(rs_2^2)\right)\frac{\partial Y_2^2}{\partial\phi},\frac{1}{r}\left(\frac{\partial^2}{\partial r^2}(rs_2^2)-\frac{6}{r}s_2^2\right)\frac{\partial Y_2^2}{\partial\theta}\right) \tag{2.3}$$

Substitute (2.2) and (2.3) to (1.3) and to (1.4) lead to

$$C\boldsymbol{v}=C\left(\frac{6}{r}s_2^2Y_2^2,\frac{1}{r}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}+\frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi},\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}-t_2^2\frac{\partial Y_2^2}{\partial\theta}\right) \tag{2.4}$$

$$\nabla\times\boldsymbol{v}=\left(\frac{6}{r}t_2^2Y_2^2,\frac{1}{r\sin\theta}\left(\frac{6}{r}s_2^2-\frac{\partial^2}{\partial r^2}(rs_2^2)\right)\frac{\partial Y_2^2}{\partial\phi}+\frac{1}{r}\frac{\partial}{\partial r}(rt_2^2)\frac{\partial Y_2^2}{\partial\theta},\frac{1}{r}\left(\frac{\partial^2}{\partial r^2}(rs_2^2)-\frac{6}{r}s_2^2\right)\frac{\partial Y_2^2}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(rt_2^2)\frac{\partial Y_2^2}{\partial\phi}\right) \tag{2.5}$$

Substitute (2.4) and (2.5) to (1.1) leads to the following identities

$$C\left(\frac{6}{r}s_2^2Y_2^2\right)=\frac{6}{r}t_2^2Y_2^2 \tag{2.6}$$

$$C\left(\frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}+\frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}\right)=\frac{1}{r\sin\theta}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right)\frac{\partial Y_2^2}{\partial\phi}+\frac{1}{r}\frac{d}{dr}(rt_2^2)\frac{\partial Y_2^2}{\partial\theta} \tag{2.7}$$

$$C\left(\frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}-t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)=\frac{1}{r}\left(\frac{d^2}{dr^2}(rs_2^2)-\frac{6}{r}s_2^2\right)\frac{\partial Y_2^2}{\partial\theta}+\frac{1}{r\sin\theta}\frac{d}{dr}(rt_2^2)\frac{\partial Y_2^2}{\partial\phi} \tag{2.8}$$

Equation (2.6) can be reduced to $Cs_2^2=t_2^2$ (2.9)

Equations (2.7) and (2.8) are separated based on $\frac{\partial Y_2^2}{\partial\theta},\frac{\partial Y_2^2}{\partial\phi}$ leads to

Equations (2.7) $$C\frac{1}{r}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\theta}=\frac{1}{r}\frac{d}{dr}(rt_2^2)\frac{\partial Y_2^2}{\partial\theta}$$

$$C\frac{d}{dr}(rs_2^2)=\frac{d}{dr}(rt_2^2) \tag{2.10}$$

$$C\frac{t_2^2}{\sin\theta}\frac{\partial Y_2^2}{\partial\phi}=\frac{1}{r\sin\theta}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right)\frac{\partial Y_2^2}{\partial\phi}$$

$$C\frac{t_2^2}{\sin\theta}=\frac{1}{r\sin\theta}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right)$$

$$Ct_2^2=\frac{1}{r}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right) \tag{2.11}$$

Equations (2.8) $$C\frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)\frac{\partial Y_2^2}{\partial\phi}=\frac{1}{r\sin\theta}\frac{d}{dr}(rt_2^2)\frac{\partial Y_2^2}{\partial\phi}$$

$$C\frac{1}{r\sin\theta}\frac{d}{dr}(rs_2^2)=\frac{1}{r\sin\theta}\frac{d}{dr}(rt_2^2)$$

$$C\frac{d}{dr}(rs_2^2)=\frac{d}{dr}(rt_2^2)$$ equal to equation (2.10)

$$C\left(-t_2^2\frac{\partial Y_2^2}{\partial\theta}\right)=\frac{1}{r}\left(\frac{d^2}{dr^2}(rs_2^2)-\frac{6}{r}s_2^2\right)\frac{\partial Y_2^2}{\partial\theta}$$

$$C(-t_2^2)=\frac{1}{r}\left(\frac{d^2}{dr^2}(rs_2^2)-\frac{6}{r}s_2^2\right)$$

$$Ct_2^2=\frac{1}{r}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right)$$ equal to equation (2.11)

Next, the magnitude of constant C from equation (2.9), (2.10), (2.11)

Since in the model of PAS flow, $s_2^2=K\Lambda j_2(\Lambda r)$ and $t_2^2=\Lambda s_2^2$, then from (2.9) we have $CK\Lambda j_2(\Lambda r)=K\Lambda^2 j_2(\Lambda r)$. Hence $C=\Lambda$.

From (2.10) we have

$$C\frac{d}{dr}(rs_2^2)=\frac{d}{dr}(rt_2^2)$$

$$C\frac{d}{dr}\left(r\,K\Lambda j_2(\Lambda r)\right)=\frac{d}{dr}\left(r\Lambda K\Lambda j_2(\Lambda r)\right)$$

$$CK\Lambda\frac{d}{dr}\left(rj_2(\Lambda r)\right)=K\Lambda^2\frac{d}{dr}\left(rj_2(\Lambda r)\right)$$

$$C=\Lambda$$

From (2.11), we have

$$Ct_2^2=\frac{1}{r}\left(\frac{6}{r}s_2^2-\frac{d^2}{dr^2}(rs_2^2)\right)$$

$$CK\Lambda^2 j_2(\Lambda r)=\frac{1}{r}\left(\frac{6}{r}K\Lambda j_2(\Lambda r)-\frac{d^2}{dr^2}\left(rK\Lambda j_2(\Lambda r)\right)\right)$$

$$=\frac{1}{r}\left(\frac{6}{r}K\Lambda j_2(\Lambda r)-\frac{d}{dr}\left(\frac{d}{dr}\left(rK\Lambda j_2(\Lambda r)\right)\right)\right)$$

The next step we used the following recurrence formula

$$\left.\begin{aligned}h_n'(x)&=h_{n-1}(x)-\frac{n+1}{x}h_n(x)\\ h_{n+1}(x)&=\frac{2n+1}{x}h_n(x)-h_{n-1}(x)\end{aligned}\right\}\qquad[2]$$

$$\frac{1}{r}\left(\frac{6}{r}K\Lambda j_2(\Lambda r)-\frac{d}{dr}\left(\frac{d}{dr}\left(rK\Lambda j_2(\Lambda r)\right)\right)\right)$$

$$=\frac{6}{r^2}K\Lambda j_2(\Lambda r)-\frac{1}{r}K\Lambda\frac{d}{dr}\left(\frac{d}{dr}rj_2(\Lambda r)\right)$$

$$=\frac{6}{r^2}K\Lambda j_2(\Lambda r)-\frac{1}{r}K\Lambda\frac{d}{dr}\left(j_2(\Lambda r)+r\frac{d}{dr}(j_2(\Lambda r))\right)$$

$$=\frac{6}{r^2}K\Lambda j_2(\Lambda r)-\frac{1}{r}K\Lambda\frac{d}{dr}\left(j_2(\Lambda r)+r\left(\Lambda j_1(\Lambda r)-\frac{3}{r}j_2(\Lambda r)\right)\right)$$

$$=\frac{6}{r^2}K\Lambda j_2(\Lambda r)-\frac{1}{r}K\Lambda\frac{d}{dr}\left(j_2(\Lambda r)+\Lambda rj_1(\Lambda r)-3j_2(\Lambda r)\right)$$

$$=\frac{6}{r^2}K\Lambda j_2(\Lambda r)-\frac{1}{r}K\Lambda\frac{d}{dr}\left(\Lambda rj_1(\Lambda r)-2j_2(\Lambda r)\right)$$

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$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{1}{r} K\Lambda \left(\Lambda \frac{d}{dr}\big(r j_1(\Lambda r)\big) - 2\frac{d}{dr}\big(j_2(\Lambda r)\big) \right)$$

$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{1}{r} K\Lambda \left(\Lambda \left(j_1(\Lambda r) + r\frac{d}{dr}\big(j_1(\Lambda r)\big) \right) - 2\left(\Lambda j_1(\Lambda r) - \frac{3}{r} j_2(\Lambda r) \right) \right)$$

$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r)$$
$$- \frac{1}{r} K\Lambda \left(\Lambda \left(j_1(\Lambda r) + r\left(\Lambda j_0(\Lambda r) - \frac{2}{r} j_1(\Lambda r) \right) \right) - 2\Lambda j_1(\Lambda r) + \frac{6}{r} j_2(\Lambda r) \right)$$

$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{1}{r} K\Lambda \left(\Lambda j_1(\Lambda r) + \Lambda^2 r j_0(\Lambda r) - 2\Lambda j_1(\Lambda r) - 2\Lambda j_1(\Lambda r) + \frac{6}{r} j_2(\Lambda r) \right)$$

$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{1}{r} K\Lambda \left(\Lambda^2 r j_0(\Lambda r) - 3\Lambda j_1(\Lambda r) + \frac{6}{r} j_2(\Lambda r) \right)$$

$$= \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{6}{r^2} K\Lambda j_2(\Lambda r) - \frac{1}{r} K\Lambda\big(\Lambda^2 r j_0(\Lambda r) - 3\Lambda j_1(\Lambda r)\big)$$

$$= -\frac{1}{r} K\Lambda\big(\Lambda^2 r j_0(\Lambda r) - 3\Lambda j_1(\Lambda r)\big)$$

$$= -K\Lambda \left(\Lambda^2 j_0(\Lambda r) - \frac{3}{r}\Lambda j_1(\Lambda r) \right)$$

$$= -K\Lambda^3 \left(j_0(\Lambda r) - \frac{3}{\Lambda r} j_1(\Lambda r) \right)$$

$$= K\Lambda^3 j_2(\Lambda r)$$

$$CK\Lambda^2 j_2(\Lambda r) = K\Lambda^3 j_2(\Lambda r)\,, \text{so } C = \Lambda$$

CONCLUSION

The magnitude of constant C in Beltrami flow is equal to Λ, the positive roots of second order of Spherical Bessel function $j_2(r)$.

REFERENCES

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