



A NEW APPROACH ON THE RESPONSE OF NON-UNIFORM PRESTRESSED TIMOSHENKO BEAMS ON ELASTIC FOUNDATION SUBJECTED TO HARMONIC LOADS

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ABSTRACT: The dynamic response of the Timoshenko beam resting on an elastic foundation subjected to harmonic moving load using modal analysis (MA) was investigated. The method of MA was employed to obtain a closed form solution to this class of dynamical systems. In order to use MA, accurate information is needed on the natural frequencies, mode shapes and orthogonality of the mode shapes. A thorough literature survey reveals that the method has not been reported in existing literature to solve non-prestressed Timoshenko beams. Thus, we present complete information on how to use MA to derive the forced vibration responses of a simply thick beam subjected to harmonic moving loads. The effects of axial force and foundation parameters on the dynamic characteristics of the beams are studied and described in detail. In order to validate the accuracy of this method, we compare the frequency parameter with the existing literature which appears to compare favorably.

KEYWORDS: Elastic Foundation; Harmonic Moving Load; Modal Analysis; Non-Prestressed; Timoshenko Beam.



INTRODUCTION

The dynamic analysis of elastic structures on an elastic foundation is a major research field in the realms of structural Engineering, Physics and Applied Mathematics [1-7]. So far and during these years, many researchers have conducted different studies in this field. When moving loads are applied to a structure, dynamic deflections and stresses may become considerably higher than those induced by static loads. When structural members are under the passage of moving loads, the interaction between the passing load and the structure makes the dynamic response analysis very complex [1]. The work of Timoshenko [8] gave impetus to research work in this area of study. The analysis of the dynamic response of a simple beam continuously supported by a viscoelastic foundation to a moving load moving at variable speed was considered. The analysis reveals several resonance conditions depending on the visco-elasticity of the foundation. Also, a theory for the response to an arbitrary number of concentrated moving masses of a rectangular plate continually supported by an elastic Pasternak type foundation was developed [9]. It was found that the critical speeds of the system increased with an increase in the values of the foundation moduli, whether the inertia effect of the system is considered or not. By virtue of the relevance in the analysis and design of railway tracks, bridges, elevated roadways, decking slabs, etc., the dynamic response of structural members under the passage of moving loads have been extensively investigated and a number of experimental and numerical studies have been reported in literature in recent years [2-7]. In this study, the concern is beam-type flexure under harmonic moving loads. Many researchers have developed various solution techniques to the transverse vibration of Timoshenko beam, which include semi-analytical method [10-12], transform matrix method [13, 14], integral transform method [15, 16], Galerkin's methods [17], finite element method [18], finite difference method [19], frequency-domain spectral method [20], and time-domain spectral element method [21]. The analytical closed-form solution for a moving load problem using MA can be obtained when the information regarding natural frequencies, mode shapes and orthogonality properties of the mode shapes are derived. Many researchers have developed general solutions of the transverse vibrations of a Timoshenko beam. This include Han et al. [22], the general solution is obtained for two frequencies ranges, $\omega < \omega_c$ and $\omega > \omega_c$, excluding the cutoff frequency ω_c . Taehyum Kim et al. [23] and van Rensburg and van der Merwe [24] developed a general solution that includes the three frequencies ranges i.e $\omega \leq \omega_c$ and $\omega \geq \omega_c$ including the cutoff frequency ω_c .

However, their method of solution cannot handle the variable prestressed Timoshenko beam. Thus, in this study, the discussion based on the mathematical formulation of the general solutions of prestressed variable simply supported Timoshenko beam, resting on an elastic foundation subjected to harmonic moving loads considering the frequency $\omega \leq \omega_c$ using MA.

Problem Formulation

The problem of prestressed Timoshenko beam of length L on a bi-parametric foundation subjected to moving loads is governed by an initial boundary value system of equations. This system of equation can be written in matrix form as

$$\mathbf{M} \frac{\partial^2 \mathbf{v}(x, t)}{\partial t^2} + \mathbf{K} \mathbf{v}(x, t) = \mathbf{Q}(x, t) \quad (1)$$

where

$$\begin{aligned}
 \mathbf{v}(x,t) &= \begin{Bmatrix} u(x,t) \\ \phi(x,t) \end{Bmatrix}; \quad \mathbf{Q}(x,t) = \begin{Bmatrix} f(x,t) \\ 0 \end{Bmatrix} \\
 \mathbf{M} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{K} = \begin{bmatrix} K_1 - (\eta_1 + K_N(x)) \frac{\partial^2}{\partial x^2} & \eta_1 \frac{\partial}{\partial x} \\ -\eta_2 \frac{\partial}{\partial x} & \eta_2 - r_0 \frac{\partial^2}{\partial x^2} \end{bmatrix} \\
 \eta_1 &= \frac{\varphi G}{\rho}; \quad \eta_2 = \frac{\varphi GA}{\rho I}; \quad K_1 = \frac{K_\omega}{\rho A}; \quad K_N(x) = \frac{N(x)}{\rho A}; \quad f(x,t) = \frac{F(x,t)}{\rho A}; \quad r_0 = \frac{E}{\rho}
 \end{aligned} \quad (2)$$

$u(x,t)$ is the transverse displacement, $\phi(x,t)$ is the rotation of the cross section due to bending,

$f(x,t)$ is the external transverse force, $N(x) = N_0 \left(1 + \sin \frac{\pi x}{L} \right)$ is the variable axial force, N_0 is the axial force at $x = 0$, K_ω is the Winkler foundation, E is the young modulus, G is the shear modulus, ρ is the mass density, A is the cross-sectional area, I is the moment of inertia and φ is the shear correction factor. The natural and geometric boundary conditions relevant to Equation (1) are given by

$$\begin{aligned}
 C(0,t) &= -C_1(t) \quad \text{or} \quad u(0,t) = u_1(t); \quad C(L,t) = -C_2(t) \quad \text{or} \quad u(L,t) = u_2(t); \\
 M(0,t) &= C_1(t) \quad \text{or} \quad \phi(0,t) = \phi_1(t); \quad M(L,t) = M_2(t) \quad \text{or} \quad \phi(L,t) = \phi_2(t);
 \end{aligned} \quad (3)$$

where $C(x,t)$ and $M(x,t)$ are the transverse shear force and bending moment respectively, given as

$$C(x,t) = \varphi GA \left(\frac{\partial u}{\partial x} - \phi \right); \quad M(x,t) = EI \frac{\partial \phi}{\partial x} \quad (4)$$

and the initial conditions are given as

$$\mathbf{v}(x,0) = \mathbf{g}(x); \quad \frac{\partial \mathbf{v}(x,0)}{\partial t} = \mathbf{h}(x); \quad (5)$$

THE FREE AND FORCED VIBRATION

General Solution

In order to obtain the eigenfunctions (natural modes) for the model under discussion, the general solutions for the free vibration with constant axial force must first be obtained. Thus, we considered the homogeneous equation of Equation (1) when the axial force is constant as follows:

$$\mathbf{M} \frac{\partial^2 \mathbf{v}(x,t)}{\partial t^2} + \mathbf{K} \mathbf{v}(x,t) = 0 \quad (6)$$



We assume the solution of Equation (6) is in the following form:

$$v(x, t) = \begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} e^{i\omega t} = \mathbf{V}(x) e^{i\omega t} \quad \text{where } \mathbf{V}(x) = \mathbf{d} e^{rx} \quad (7)$$

where i is an imaginary number, ω is the angular frequency and r denotes the wave number. Therefore, substituting Equation (7) into Equation (6) yields the following algebraic equations:

$$\begin{bmatrix} -\beta_1 r^2 + \beta_3 & r\eta_1 \\ r\eta_2 & -r_0 r^2 + \beta_2 \end{bmatrix} \mathbf{d} e^{rx} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (8)$$

where

$$\beta_1 = (\eta_1 + K_N); \beta_2 = \eta_2 - \omega^2; \beta_3 = K_1 - \omega^2; K_N = \frac{N_0}{\rho A} \quad (9)$$

from which we obtain eigenvalues r . For the existence of non-trivial solutions, the determinant of the 2×2 matrix in Equation (8) must vanish at certain values of r . Thus, a dispersion equation is obtained as follows:

$$r^4 - \frac{1}{r_0 \beta_1} (\beta_1 \beta_2 + r_0 \beta_3 - \eta_1 \eta_2) r^2 + \frac{\beta_2 \beta_3}{r_0 \beta_1} = 0 \quad (10)$$

The eigenvalues are given by

$$r_i = \pm \sqrt{\frac{1}{2r_0 \beta_1} \left[(\beta_1 \beta_2 + r_0 \beta_3 - \eta_1 \eta_2) \pm \sqrt{(\beta_1 \beta_2 + r_0 \beta_3 - \eta_1 \eta_2)^2 - 4r_0 \beta_1 \beta_2 \beta_3} \right]} \quad (11)$$

for $i = 1, 2, 3, 4$

The corresponding eigenvectors \mathbf{V}_i are given by

$$\mathbf{V}_i = \begin{bmatrix} r\eta_1 \\ -\beta_1 r^2 + \beta_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -r_0 r^2 + \beta_2 \\ r\eta_1 \end{bmatrix} \quad (12)$$

Of the four roots, the two given by

$$r_j = \pm \sqrt{\frac{1}{2r_0 \beta_1} \left[(\beta_1 \beta_2 + r_0 \beta_3 - \eta_1 \eta_2) \pm \sqrt{(\beta_1 \beta_2 + r_0 \beta_3 - \eta_1 \eta_2)^2 - 4r_0 \beta_1 \beta_2 \beta_3} \right]} \quad (13)$$

for $j = 1, 2$

are either real or imaginary depending on the frequency ω (for a given material and geometry), and the other two roots given by

$$r_j = \pm \sqrt{\frac{1}{2r_0\beta_1} \left[(\beta_1\beta_2 + r_0\beta_3 - \eta_1\eta_2) \pm \sqrt{(\beta_1\beta_2 + r_0\beta_3 - \eta_1\eta_2)^2 - 4r_0\beta_1\beta_2\beta_3} \right]} \quad (14)$$

for $j = 3, 4$

are always imaginary, r_j ; real when the frequency ω is less than $\sqrt{\eta_2}$ and imaginary when the frequency is greater than $\sqrt{\eta_2}$. We call this cutoff frequency or the critical frequency ω_c . Therefore, we must consider two cases when obtaining spatial solutions: (i.e. $\omega \leq \omega_c$ and $\omega > \omega_c$). By using the four eigenvalues given by Equation (11), the spatial solution can be written as follows:

(a) When $0 < \omega \leq \omega_c$

$$\mathbf{V}(x) \equiv \begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} = \sum_{i=1}^4 d_i \mathbf{v}_i e^{r_i x} = d_1 \mathbf{u}_1 e^{bx} + d_2 \mathbf{u}_2 e^{-bx} + d_3 \mathbf{u}_3 e^{iax} + d_4 \mathbf{u}_4 e^{-iax} \quad (15)$$

(b) When $\omega > \omega_c$

$$\mathbf{V}(x) \equiv \begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} = \sum_{i=1}^4 \bar{d}_i \mathbf{v}_i e^{r_i x} = d_1 \mathbf{u}_1 e^{\bar{b}x} + d_2 \mathbf{u}_2 e^{-\bar{b}x} + d_3 \mathbf{u}_3 e^{iax} + d_4 \mathbf{u}_4 e^{-iax} \quad (16)$$

where

$$a = \sqrt{\frac{1}{2r_0\beta_1} \left[(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3) + \sqrt{(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3)^2 - 4r_0\beta_1\beta_2\beta_3} \right]} \quad (17)$$

$$b = \sqrt{\frac{1}{2r_0\beta_1} \left[-(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3) + \sqrt{(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3)^2 - 4r_0\beta_1\beta_2\beta_3} \right]} \quad (18)$$

$$c = \sqrt{\frac{1}{2r_0\beta_1} \left[(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3) + \sqrt{(\eta_1\eta_2 - \beta_1\beta_2 - r_0\beta_3)^2 - 4r_0\beta_1\beta_2\beta_3} \right]} \quad (19)$$

By using the results, Equations (15) and (16) can be written in terms of sinusoidal and hyperbolic function as follows:

(a) When $0 < \omega \leq \omega_c$

$$\begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} = A_1 \begin{Bmatrix} \sin(ax) \\ g_a \cos(ax) \end{Bmatrix} + A_2 \begin{Bmatrix} \cos(ax) \\ -g_a \sin(ax) \end{Bmatrix} + A_3 \begin{Bmatrix} \sinh(ax) \\ g_a \cosh(ax) \end{Bmatrix} + A_4 \begin{Bmatrix} \cosh(bx) \\ g_a \sin(bx) \end{Bmatrix} \quad (20)$$

(b) When $\omega > \omega_c$

$$\begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} = A_1 \begin{Bmatrix} \sin(ax) \\ g_a \cos(ax) \end{Bmatrix} + A_2 \begin{Bmatrix} \cos(ax) \\ -g_a \sin(ax) \end{Bmatrix} + A_3 \begin{Bmatrix} \sinh(ax) \\ g_a \cosh(ax) \end{Bmatrix} + A_4 \begin{Bmatrix} \cosh(bx) \\ g_b \sinh(bx) \end{Bmatrix} \quad (21)$$

where

$$g_a = \frac{1}{a\eta_1} (\beta_1 a^2 + K_1 - \omega^2), \quad g_b = \frac{1}{b\eta_1} (\beta_1 b^2 - K_1 + \omega^2), \quad g_{\bar{b}} = \frac{1}{\bar{b}\eta_1} (\beta_1 \bar{b}^2 + K_1 - \omega^2) \quad (22)$$

The present spatial solutions (20) and (21) are now the expressions for three frequency ranges $0 < \omega \leq \omega_c$ and $\omega > \omega_c$.

Natural Frequency and Mode Shape

Analytical closed forms of natural frequencies and mode shapes are obtained for specific boundary condition; our present study is limited to simply supported conditions given by

$$u(0, t) = u(L, t) = 0; \quad EI \frac{\partial \phi(L, t)}{\partial x} = 0 \quad (23)$$

We consider just one frequency range as follows:

3.2.1 When $0 < \omega < \omega_c$

Substituting Equation (20) into Equation (23) yields a matrix equation

$$\begin{Bmatrix} U(0) \\ \Phi'(0) \\ U(L) \\ \Phi'(L) \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -ag_a & 0 & bg_b \\ \sin(aL) & \cos(aL) & \sinh(bL) & \cosh(bL) \\ -ag_a \sin(aL) & -ag_a \cos(aL) & bg_b \sinh(bL) & bg_b \cosh(bL) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (24)$$

For existence of a non-trivial solution of A_i , ($i=1, 2, 3, 4$) in Equation (24), the determinant of the matrix of eigenvalue problem must vanish. Thus, we obtain

$$A_2 = A_4 = 0 \quad \text{and} \quad (ag_a + bg_b) \sin(aL) \sinh(bL) \quad (25)$$

Since $(ag_a + bg_b) \neq 0$ and $\sinh(bL) \neq 0$, then if $0 < \omega < \omega_c$ we have

$$\sin(a_n L) = 0 \quad (26)$$

from which we obtain

$$a_n = \frac{n\pi}{L} (n=1, 2, 3, \dots, n_a) \quad (27)$$

Therefore, substituting Equation (27) into Equations (17) - (19) yields natural frequencies ω_n given as

$$\omega_{a(n)} = \sqrt{Z(n) - \sqrt{Z(n)^2 - 4R(n)}}_c \quad (n=1, 2, 3, \dots, n_a) \quad (28)$$

where

$$Z(n) = \frac{1}{2} \left[(\eta_2 + K_1) - \left(\frac{n\pi}{L} \right)^2 (r_0 + \beta_1) \right] R(n) = \frac{1}{4} \left(\frac{n\pi}{L} \right)^2 \left[r_0 \beta_1 \left(\frac{n\pi}{L} \right)^2 + \eta_1 \eta_2 - \eta_2 \beta_1 - r_0 K_1 \right] \quad (29)$$

Next, we obtain the mode shapes corresponding to the natural frequencies

$\omega_a(n) (n=1, 2, \dots, n_a)$ by determining the values of A_1 and A_3 from Equation (24) in the following forms:

$$A_1(n) \neq 0; A_3(n) = 0 \quad (30)$$

Thus, the n th mode shape corresponding to $\omega_a(n)$ from equation (21) is written as

$$\mathbf{V}_{a(n)}(x) \equiv \begin{Bmatrix} U(x) \\ \Phi(x) \end{Bmatrix} = A_{a(n)} = \begin{Bmatrix} \sin \frac{n\pi x}{L} \\ g_{a(n)} \cos \frac{n\pi x}{L} \end{Bmatrix} \quad (n=1, 2, \dots, n_a) \quad (31)$$

where

$$g_{a(n)} = \frac{L}{n\pi\eta_1} \left[\beta_1 \left(\frac{n\pi x}{L} \right)^2 + K_1 - \omega_{a(n)}^2 \right] \quad (32)$$

Thus, in this study, we need to consider the following type of mode shape for the transverse vibration of the simply supported non-uniform prestressed thick beam on an elastic foundation.

$$\mathbf{V}_{a(n)}(x) = A_{a(n)} = \begin{Bmatrix} \sin \frac{n\pi x}{L} \\ g_{a(n)} \cos \frac{n\pi x}{L} \end{Bmatrix} \quad (\text{mode shape for } \omega_{a(n)}) \quad (33)$$

The Orthogonality Conditions for the Model

In order to obtain the forced response of a thick beam, we set the method of eigenfunctions expansion. Therefore, the Orthogonality conditions of the eigenfunctions have to be established for the beam model discussed so far. Thus, the spatial equations of the homogeneous problem (9) can be written in the form

$$\mathbf{K}(V_n) = \omega_n^2 \mathbf{M}(V_n) \quad (34)$$

where V_n denotes the n th eigenfunctions for the vector $[U_n \ \Phi_n]^T$ of the beam model and corresponding to the natural frequency ω_n^2 uniquely within an arbitrary constant. The operators \mathbf{K} and \mathbf{M} are self-adjoint (with corresponding boundary conditions) if

$$\int_0^L [V_n^T K(V_m) - V_m^T K(V_n)] dx = 0 \quad (35)$$

and

$$\int_0^L [V_n^T M(V_m) - V_m^T M(V_n)] dx = 0 \quad (36)$$

Since the condition in Equation (36) is automatically satisfied for the model, then using Equation (34), we can write the condition in Equation (35) as

$$(\omega_n^2 - \omega_m^2) \int_0^L V_n^T M(V_m) dx = 0 \quad (37)$$

However, eigenvalues are unique to the eigenfunctions, $\omega_m^2 \neq \omega_n^2$ for $(m \neq n)$. In order for the above equation to be zero, the integral has to be zero, i.e.

$$\int_0^L V_n^T M(V_m) dx = 0 \quad (\text{for } m \neq n) \quad (38)$$

This is the Orthogonality condition for the eigenfunctions. When $m = n$, we normalize the eigenfunctions by setting the integral equal to one

$$\int_0^L V_n^T M(V_m) dx = 1 \quad (\text{for } n = 1, 2, 3, \dots) \quad (39)$$

Combining Equations (38) and (39), we can write

$$\int_0^L V_n^T M(V_m) dx = \delta_{mn} \quad (40)$$

where δ_{mn} is the Kronecker delta.

For this model, the corresponding boundary conditions for the self-adjoint operator \mathbf{K} are found to be

$$\varphi GA \left[U_n \left(\frac{dU_m}{dx} - \Phi_m \right) - U_m \left(\frac{dU_n}{dx} - \Phi_n \right) \right] \Big|_0^L + \left[\Phi_n \frac{d\Phi_m}{dx} - \Phi_m \frac{d\Phi_n}{dx} \right] \Big|_0^L \quad (41)$$

Then, substituting Equation (33) into Equation (39), we derive the coefficient of normal mode shape as

$$A_{a(n)} = \sqrt{\frac{2}{L(1 + g_{a(n)}^2)}} \quad (42)$$

Modal Analysis of Forced Vibration of the Model

The forced vibration of equation (1) can be represented by using the mode summation given as

$$\mathbf{v}(x, t) = \sum_{n=1}^{\infty} V_{a(n)}(x) y_{a(n)}(t) \quad (43)$$

where $\mathbf{v}(x, t)$ stands for $[u(x, t) \ \phi(x, t)]^T$ in this model while $y_{a(n)}(t)$ is the generalized time-dependent coordinates to be determined in order to satisfy initial boundary conditions. Therefore, substituting Equation (43) into Equation (1) and applying the Orthogonality conditions of the normal mode shapes yield the following equation:

$$\mathbf{M} \mathbf{V}_{a(n)} \frac{d^2 y_{a(n)}(x, t)}{dt^2} + \mathbf{K} \mathbf{V}_{a(n)} y_{a(n)} = f_{a(n)} \quad (44)$$

where the generalized force is defined as

$$f_{a(n)} = \int_0^L V_{a(n)}(x) f(x, t) dx \quad (45)$$

where $f(x, t)$ is the distributed harmonic load parameter. For this problem, the distributed load moving on the beam has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behavior of the dynamical system. Thus, the distributed load $F(x, t)$ takes the form

$$F(x, t) = \cos(\alpha t) M_0 g H \left[x - ct \left[1 - \frac{1}{g} \left(\frac{\partial^2}{\partial t^2} + 2c \frac{\partial^2}{\partial x \partial t} + c^2 \frac{\partial^2}{\partial x^2} \right) \right] u(x, t) \right] \quad (46)$$

where P_0 is the magnitude of the transverse distributed force, g is the acceleration due to gravity, c is the initial velocity, $H[x-ct]$ is the Heaviside function which is a typical engineering function made to measure engineering applications involving functions that are either “on” or “off”. Therefore, incorporating Equations (45) and (46) into Equation (44), and thereafter evaluating the integrals with the use of Fourier sine series representation of the Heaviside function yields the following equations:

$$\begin{aligned} & \frac{d^2 y_{a(n)}}{dt^2} + \omega_{(n)}^2 y_{a(n)}(t) + \varepsilon_a \cos(\omega t) \left\{ \left[\frac{L}{8} + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_{6a}(m, n) \right. \right. \\ & \left. \left. - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{6b}(m, n) \right] \ddot{y}_{a(n)}(t) + 2c \left(\frac{n\pi}{L} \right) \left[\frac{I_{7a}(mn)}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_{7b}(m, n) \right. \right. \\ & \left. \left. - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{7c}(m, n) \right] \dot{y}_{a(n)}(t) - c^2 \left(\frac{n\pi}{L} \right)^2 \left[\frac{L}{8} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_{6a}(m, n) \right. \right. \\ & \left. \left. + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{6a}(m, n) \right] y_{a(n)}(t) \right\} = P_{a(n)} [\cos \theta_n t - R_n] \end{aligned} \quad (47)$$

where

$$\begin{aligned} \omega_{(n)}^2 &= \left[\frac{n^2 \pi^2}{L} \left(\frac{L}{2} \beta_1 + K_N I_1 \right) + \frac{1}{2} \left(r_0 \left(\frac{n\pi}{L} \right)^2 + \eta_2 - \eta_2 \left(\frac{n\pi}{g_{a(n)} L} \right) \right) - \left(\frac{n\pi^3}{L^3} \right) K_N I_2 \right. \\ & \quad \left. + 0.5 K_1 - \eta_1 \left(\frac{n\pi}{2L g_{a(n)}} \right) \right]; \quad I_1 = \int_0^L \sin \frac{\pi x}{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \\ I_2 &= \int_0^L \cos \frac{\pi x}{L} \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx; \quad \varepsilon_a = \frac{M_0}{\rho A L}; \quad P_{a(n)} = \frac{M_0 g \cos(\omega t)}{A_{a(n)} \rho A n \pi} \\ \theta_n &= \frac{n\pi c}{L}; \quad R_n = -(-1)^n; \quad I_{6a} = \int_0^L \sin(2n+1) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \\ I_{6b} &= \int_0^L \cos(2n+1) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx; \quad I_{6a} = \int_0^L \cos(2n+1) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \end{aligned} \quad (48)$$

Equation (47) now represents the transformed governing equations for the simply supported non-uniform prestressed Timoshenko beam resting on an elastic foundation under distributed harmonic moving load. By considering Equation (47), we can derive the responses of the beam for two cases:

Case I: When the beam is subjected to a load of negligible inertia, i.e. by setting $\varepsilon_a = 0$. This is termed **moving force** problem.

Case II: When the beam is subjected to a load of non-negligible inertia, i.e. when ε_a is greater than zero. This is termed **moving mass** problem.



Modal Analysis of Forced Vibration of the Model when the Beam is Traversed by Moving Force

In this section, an approximate model of the differential equation describing the response of a uniform Timoshenko beam resting on Pasternak foundation and under the action of a moving distributed force is obtained by neglecting inertia terms, that is $\varepsilon_a = 0$. equation (47) is now of the form

$$\ddot{y}_{a(n)}(t) + \omega_{(n)}^2 y_{a(n)} = W_a(t) \quad (49)$$

where

$$W_a(t) = P_{a(n)} [\cos \theta_n t + R_n] \quad (50)$$

Therefore, solving equation (49) using the method of Laplace transforms in conjunction with the convolution theory for unknown generalized coordinates $y_{a(n)}$ and then substituted the results into Equation (43) to obtain the vibration responses as follows:

$$v(x, t) = \sum_{n=1}^{\infty} \frac{V_{a(n)}(x)}{\omega_{(n)}} \left[\int_0^t W_a(t) \sin \omega_{(n)}(t - \tau) d\tau + y_{a(n)}(0) \cos \omega_{(n)} t + \frac{1}{\omega_{(n)}} \dot{y}_{a(n)}(0) \right] \quad (51)$$

Thus, Equation (51) represents the solution to forced vibration of non-uniform prestressed Timoshenko beams resting on an elastic foundation under distributed harmonic moving load for moving force case.

Modal Analysis of Forced Vibration of the Model when the Beam is Traversed by Moving Mass

In this section, the solution of the entire Equation (47) is sought when no terms of the couple differential equation is neglected. Evidently, an exact solution to this equation is not possible. All conventional methods break down; even the popular Struble's technique cannot handle it because of the variability of the magnitude of the moving harmonic load; hence, we resort to Runge-Kutta method. Thus, Equation (47) takes the form

$$\ddot{y}_{a(n)}(t) + \frac{2\varepsilon_a c Q_2(m, n)}{1 + \varepsilon_a Q_1(m, n)} \dot{y}_{a(n)} t + \frac{\omega_{a(n)}^2 + \varepsilon_a c^2 Q_2(m, n)}{1 + \varepsilon_a Q_1(m, n)} y_{a(n)} t = \frac{P_{a(n)} [\cos \theta_k t + R_n]}{1 + \varepsilon_a Q_1(m, n)} \quad (52)$$

where

$$Q_1(m, n) = \left[\frac{L}{8} + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_{6a}(m, n) - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{6b}(m, n) \right]$$

$$Q_2(m, n) = \left(\frac{n\pi}{L} \right) \left[\frac{I_{7a}(m, n)}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_{7a}(m, n) - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{7c}(m, n) \right]$$

$$Q_3(m, n) = \left(\frac{n\pi}{L} \right)^2 Q_1(m, n) \quad (53)$$

Therefore, solving Equation (52) using Runge-Kutta method for unknown generalized coordinates $y_{a(n)}$ and then substituting the results into Equation (43) yields

$$\mathbf{v}(x, t) = \sum_{n=1}^{\infty} V_{a(n)}(x) y_{a(n)}(t) \quad (54)$$

Thus, Equation (51) represents the solution to forced vibration of non-uniform prestressed Timoshenko beams resting on an elastic foundation, under distributed harmonic moving load for moving mass case.

NUMERICAL INVESTIGATION

In order to investigate the dynamic response of the present study, we reconsidered the uniform simply supported Timoshenko beam that was previously employed by Esmalizedeh and Ghorashi [7]. The geometric and material properties data of the beam are as follows: length $L = 27.374m$, area moment of inertia $I = 5.71 \times 10^{-7} m^4$, cross-sectional area $A = 3.3183 \times 10^{-5}$, Young modulus $E = 2.02 \times 10^{11} Nm^2$, shear modulus $G = 7.7 \times 10^{10} Nm^{-2}$, mass density $\rho = 15267 kg/m^3$ and shear correction factor taken as $\varphi = 0.7$. For the analyses of forced vibrations, we assumed that the mass $M_0 = 454.08kg$ and $g = 9.8ms^{-2}$. We assumed that the beam has null initial conditions.

Model Verification

In this subsection, we aim to verify the accuracy of the present method **MA**. Thus, the eigenfrequency and the dynamic response of the simply supported (SS) beam is computed and compared to the existing literature. Following the work of Tachyum Kim et al. (2017), we compare the natural frequencies and mode shape of the present work.

Table 1: Frequency parameter of SS thick beam on an elastic foundation at various values of the Mode number (n)

Mode (n) ($K_0 = 0, N_0 = 0$)	g_a	$\omega_{a(n)}$	Tachyum g_a (2017)	$\omega_{a(n)}$ 2017
1	0.70	6.29	0.72	6.29
2	1.21	25.19	1.44	25.14
3	1.51	56.84	2.15	56.41
4	1.93	101.43	2.86	99.92
5	3.28	159.28	3.53	155.39
Mode (n) ($K_0 = 0, N_0 = 400$)	g_a	$\omega_{a(n)}$	Tachyum g_a (2017)	$\omega_{a(n)}$ 2017
1	0.70	5.40	0.72	6.29
2	1.21	24.35	1.44	25.14
3	1.51	56.01	2.15	56.41
4	1.93	100.61	2.86	99.92
5	3.28	158.46	3.53	155.39

Mode ⁽ⁿ⁾ ($K_0 = 400$, $N_0 = 0$)	g_a	$\omega_{a(n)}$	Tachyum g_a (2017)	$\omega_{a(n)}$ 2017
1	0.70	6.23	0.72	6.29
2	1.21	25.14	1.44	25.14
3	1.51	56.79	2.15	56.41
4	1.93	101.38	2.86	99.92
5	3.28	159.23	3.53	155.39

It is seen from Table 1 that regardless of the values of axial force and foundation stiffness, the natural frequencies $\omega_{a(n)}$ and the corresponding mode shape parameter g_a of the simply supported beam on the elastic foundation in the present work is in good agreement with that reported by [23], who computed the frequency parameter without considering both axial force and foundation stiffness.

Figure 1 shows the dynamic responses predicted at $x/L = 0.25$, when the Timoshenko beam is subjected to a distributed force at its middle point $x/L = 0.5$. The responses are the transverse displacement $u(0.25L, t)$, the total slope $u'(0.25L, t)$, the slope due to bending $\phi(0.25L, t)$ and the shear angle due to transverse shear force $\Psi(0.25L, t) = u'(0.25L, t) - \phi(0.25L, t)$. Figures (1-4) show the deformed shape at three different values of axial force ($N_0 = 0, 4 \times 10^5, 4 \times 10^8$) for the dynamic responses. As seen from the figures, the dynamic deflection of the beam decreases as the value of axial force increases. Figures (5-8) show the effect of the axial force on the dynamic response of the prestressed simply supported Timoshenko beam resting on a Winkler foundation at constant velocity $c = 0.25v_c$, (where v_c is the lowest critical speed given as $v_c = 2\pi f_1 L$ and f_1 is the first natural frequency in Hz.) and different excitation frequencies $\alpha = 0 \text{ rad/s}$ and $\alpha = 60 \text{ rad/s}$. As seen from the figures, the dynamic deflection of the beams decreases with an increase in the values of axial force, regardless of the values of excitation frequency.

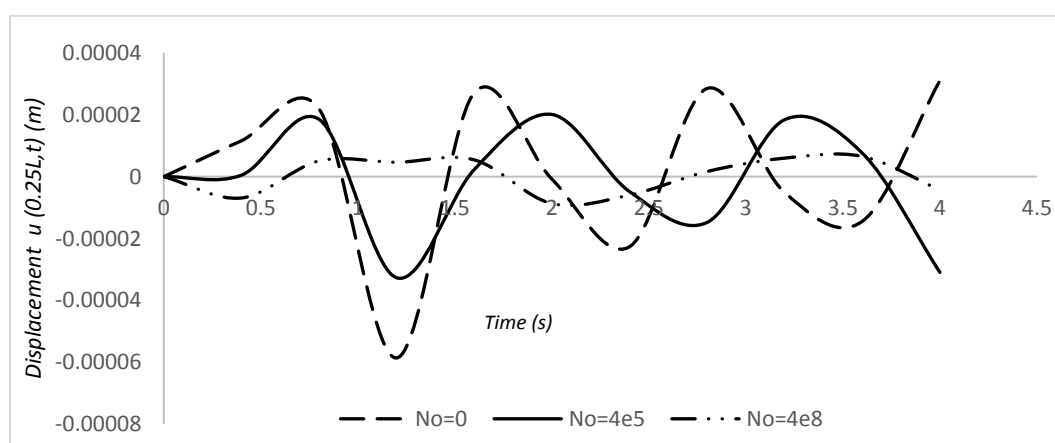


Figure 1: Dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when $N_0 = 4 \times 10^4$, $K_0 = 4 \times 10^3$ and excitation frequency $\alpha = 6.283$.

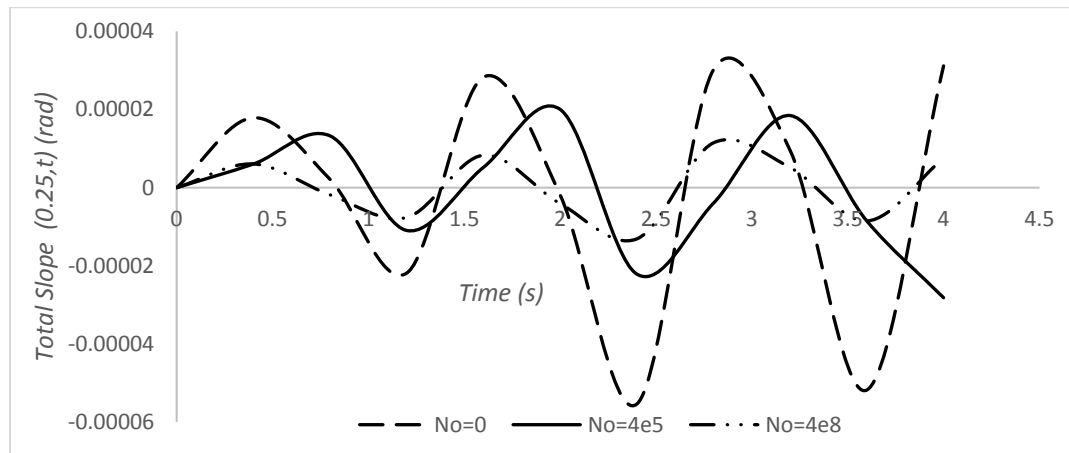


Figure 2: Dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when $No = 4 \times 10^4$, $Ko = 4 \times 10^3$ and excitation frequency $\alpha = 6.283$ for total slope u'

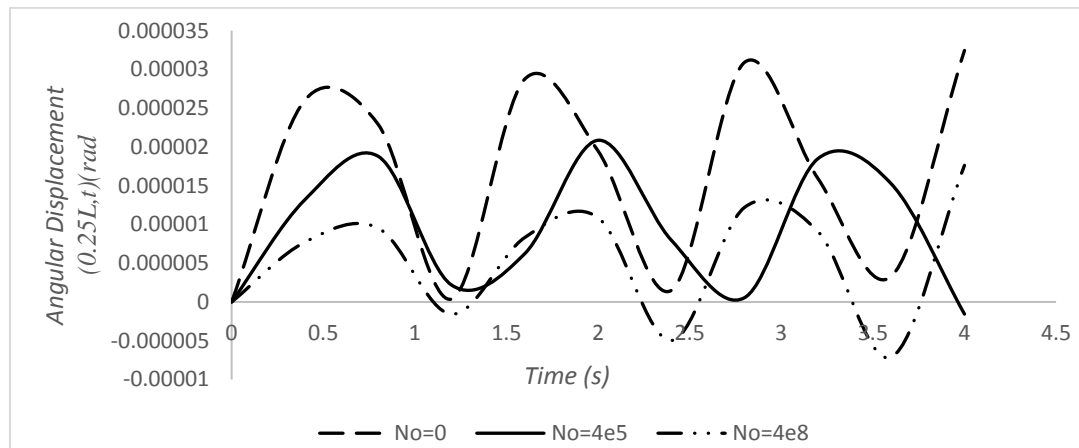


Figure 3: Dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when $No = 4 \times 10^4$, $Ko = 4 \times 10^3$ and excitation frequency $\alpha = 6.283$ for slope due to bending (ϕ)

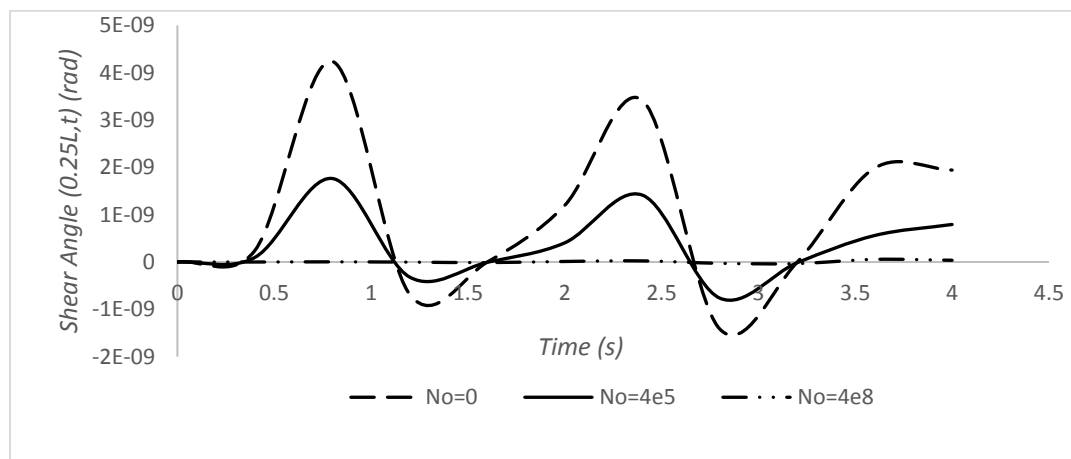


Figure 4: Dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when $No = 4 \times 10^4$, $Ko = 4 \times 10^3$ and excitation frequency $\alpha = 6.283$ for shear angle (Ψ)

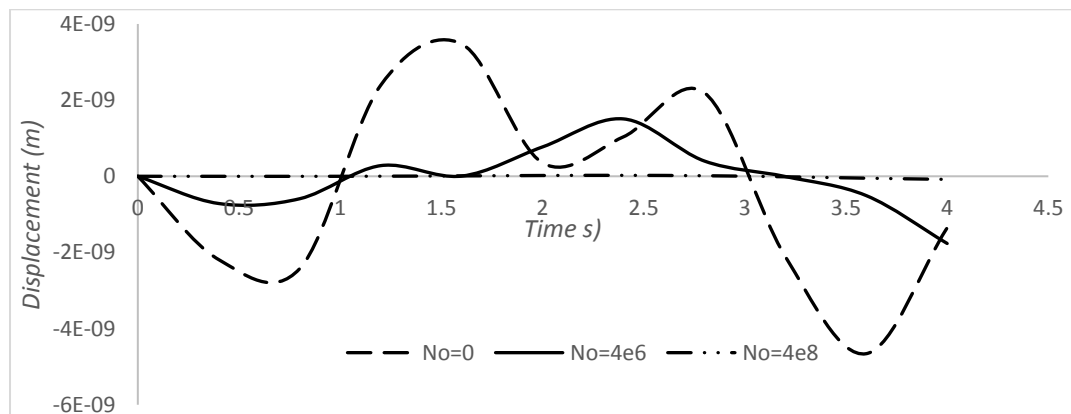


Figure 5: Effect of axial force No on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed force applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 8$

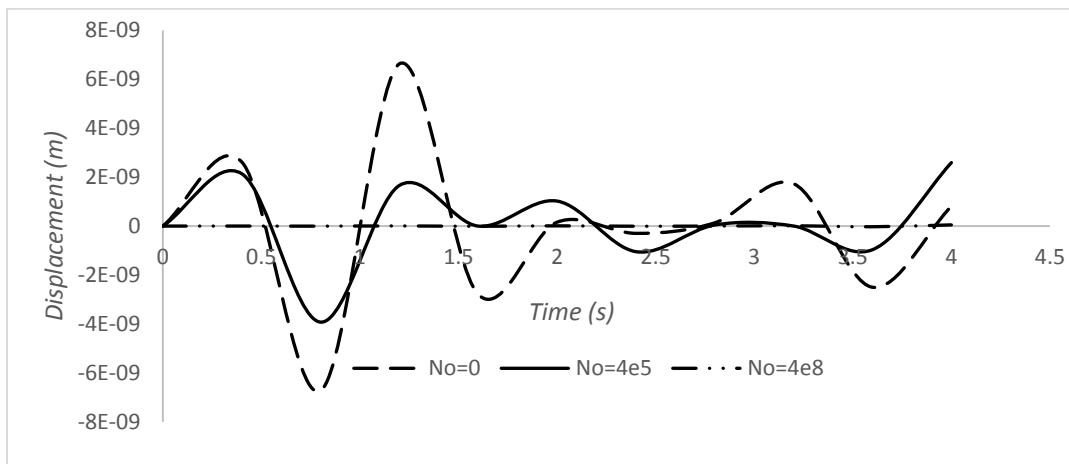


Figure 6: Effect of axial force No on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed force applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 32$

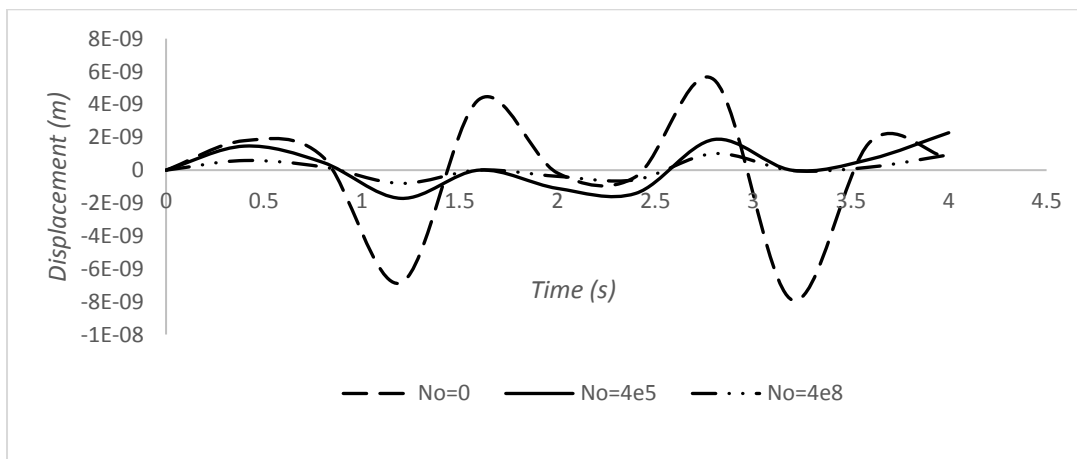


Figure 7: Effect of axial force No on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 8$

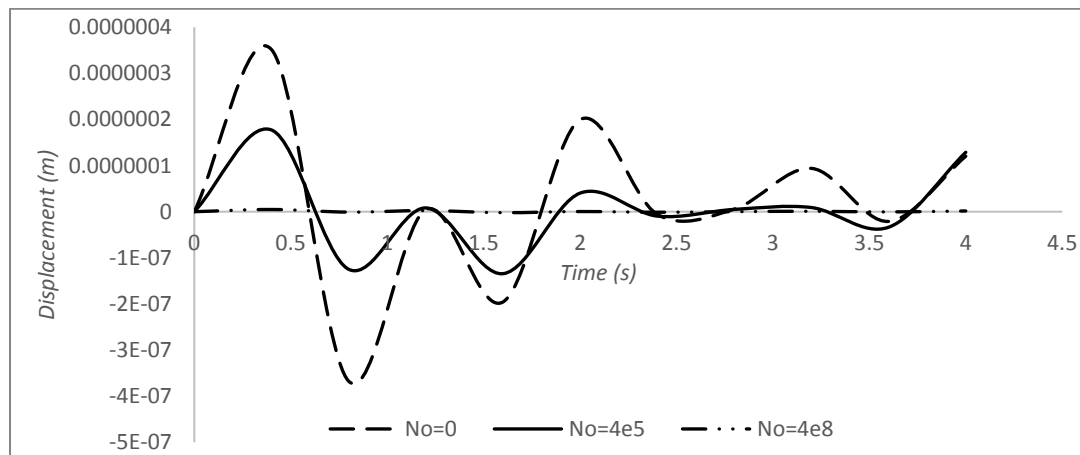


Figure 8: Effect of axial force No on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 32$

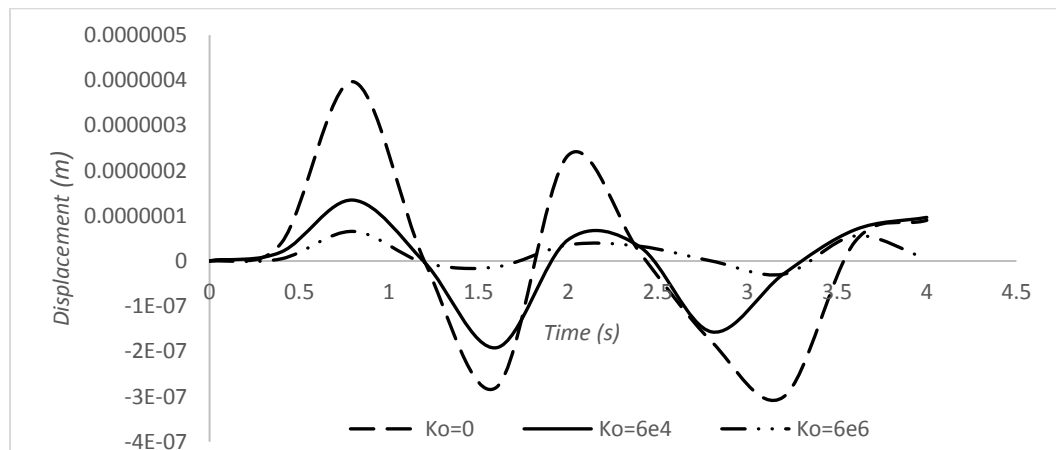


Figure 9: Effect of foundation stiffness Ko on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed force applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 8$

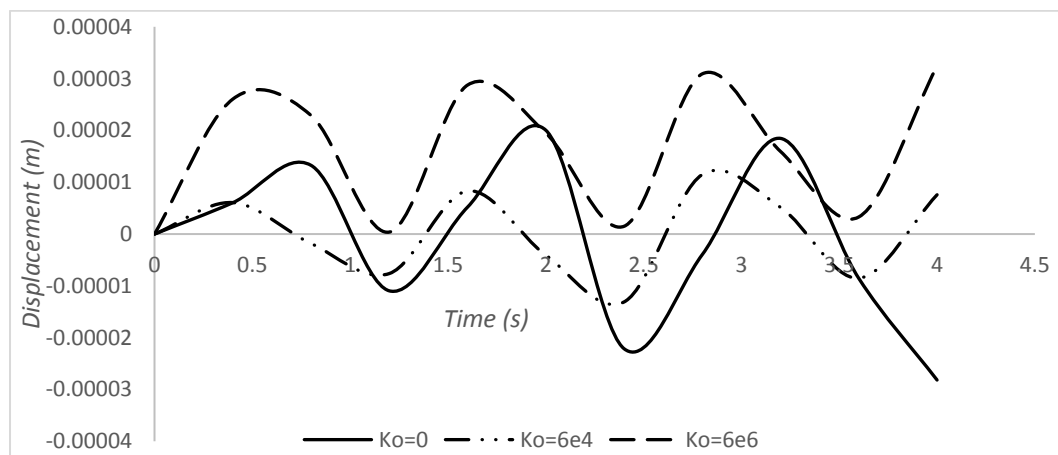


Figure 10: Effect of foundation stiffness K_o on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed force applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 32$

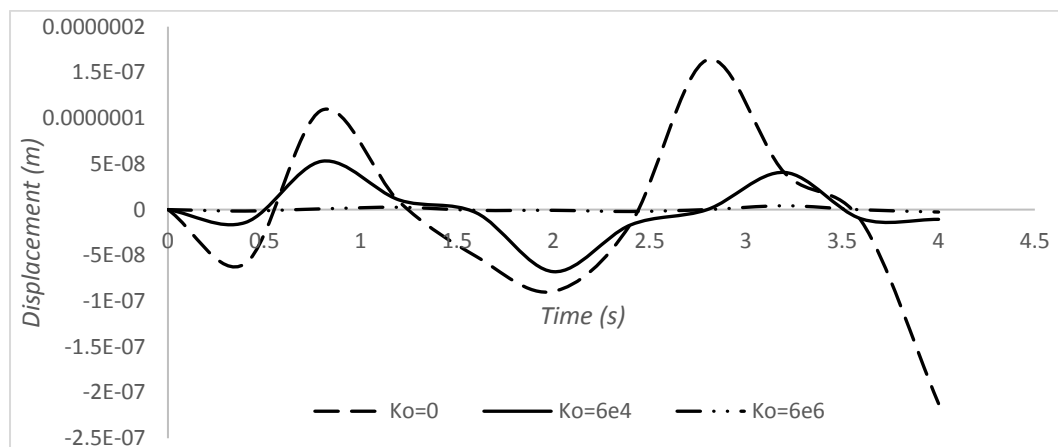


Figure 11: Effect of foundation stiffness K_o on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 8$

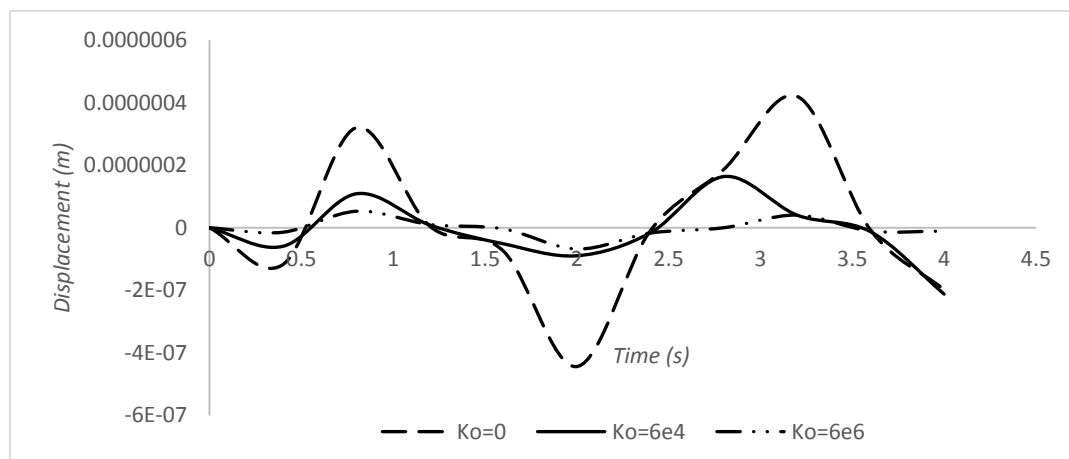


Figure 12: Effect of foundation stiffness K_o on the dynamic response at $x/L = 0.25$ of a simply supported beam subjected to a distributed mass applied at $x/L = 0.5$ for the case of constant velocity $c = 0.25v_c$ when the excitation frequency $\alpha = 32$

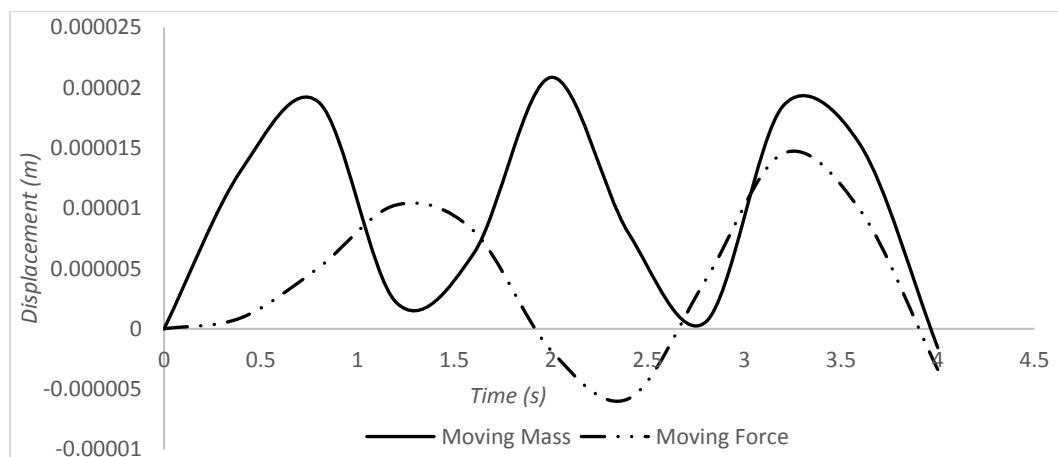


Figure 13: Comparison of the moving force and moving mass cases of transverse displacement of

the thick beam resting on non-uniform Vlasov foundation subjected to harmonic moving

loads for fixed values of $N_0 = 4000$ and $K_0 = 4000$ when the excitation frequency $\alpha = 8$

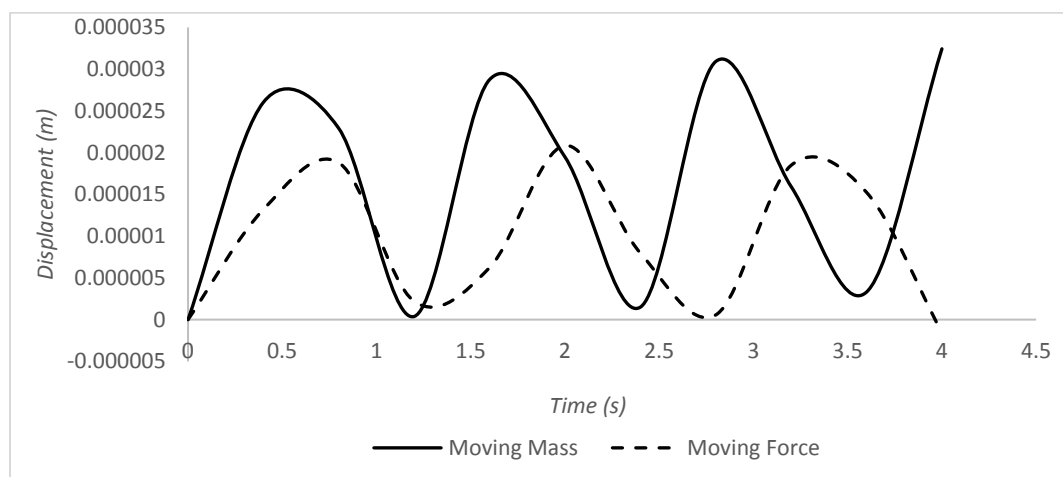


Figure 14: Comparison of the moving force and moving mass cases of transverse displacement of

the thick beam resting on non-uniform Vlasov foundation subjected to harmonic moving

loads for fixed values of $N_0 = 4000$ and $K_0 = 4000$ when the excitation frequency $\alpha = 32$

Figures 9-12 show the effects of foundation stiffness on the dynamic response of the simply supported Timoshenko beam resting on a Winkler foundation for a moving mass case. As seen from the figures, the dynamic deflection of the beam decreases with an increase in the value of excitation frequency. The deflections of the beam are much more pronounced in the moving mass cases than in the moving force cases.

Finally, Figures 13 and 14 depict the comparison of the transverse displacement of moving distributed force and the moving distributed mass cases for simply supported beam traversed by harmonic moving distributed load for $G_0 = 4000$ and $K_0 = 4000$ at two different excitation frequencies. Clearly, the response amplitude of moving distributed mass is greater than that of moving distributed force; problems for the two excitation frequencies must be considered when heavy loads are involved. This shows that a moving distributed force solution is not always upper bound to the solution of a moving distributed mass problem.

CONCLUSION

The problem of dynamic response of non-prestressed Timoshenko beams resting on an elastic foundation subjected to harmonic moving loads using modal analysis (MA) was examined. The dynamic response of the simply supported beams for moving force and moving mass cases has been computed at different values of axial force, foundation stiffness and excitation frequency. The analyses exhibited the following features:

- (i) The deformed shapes of Timoshenko beams strongly depend on the speeds of the moving load. There is a critical speed at which the dynamic system reaches a pick value, and this speed called critical speed is affected by the foundation stiffness and the excitation frequency.



- (ii) The effects of the moving velocity depend on the excitation frequency and this affects the dynamic deflection of the beam.
- (iii) A set of natural frequencies and mode shapes are presented in closed forms for frequency range $0 < \omega \leq \omega_c$ where ω_c is the cutoff frequency.
- (iv) As the value of axial force and foundation stiffness increases, the transverse deflection of the beam model decreases. This strongly depends on the excitation frequency.

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