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DESIGN EFFICIENCY AND OPTIMAL VALUES OF REPLICATED CENTRAL COMPOSITE DESIGNS WITH FULL FACTORIAL PORTIONS

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ABSTRACT: *Efficiency and optimal properties of four varieties of Central Composite Design, namely, SCCD, RCCD, OCCD and FCCD and having r_f replicates of the full factorial portion, r_α replicates of the axial portion and r_c replicates of the center portion are studied in four to six design variables. Optimal combination, $[r_f: r_\alpha: r_c]$ of design points associated with the three portions of each central composite design is presented. For SCCD, the optimal combinations resulting in A- and D- efficient designs generally put emphasis on replicating the center portion of the SCCD. However, replicating the center and axial portions allows for G-optimal and efficient designs. For RCCD, the optimal combinations resulting in A- and D- efficient designs generally put emphasis on replicating the factorial and center portions of the RCCD. However, replicating the center and axial portions allows for G-optimal and efficient designs. For OCCD, the optimal combinations resulting in A- optimal and efficient designs generally put emphasis on replicating the axial and center portion of the OCCD. The optimal combinations resulting in G- optimal and efficient designs generally put emphasis on replicating the factorial and axial portions of the OCCD. To achieve designs that are D-optimal and D-efficient, the optimal combination of design points generally put emphasis on replicating the center portion of the OCCD. For FCCD, the optimal combinations of design points resulting in A-efficient designs put emphasis on replicating the axial portion of the FCCD. The optimal combinations resulting in G- optimal and efficient designs as well as G-optimal and efficient designs generally put emphasis on replicating the factorial and axial portions of the FCCD. It is interesting to note that for FCCD in five design variables, any r^{th} complete replicate of the distinct design points of the combination $[r_f: r_\alpha: r_c]$ resulted in a D-efficient design. Many super-efficient designs having efficiency values greater than 1.0 emerged under the D-criterion. Unfortunately, these designs did not perform very well under A- and G-criteria, having some efficiency values much below 0.5 or just about 0.6.*

KEYWORDS: SCCD, RCCD, OCCD, FCCD, Optimal Values, Design Efficiency

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INTRODUCTION

Full factorial experiment, refers to experiment with two or more factors, and each factor with discrete possible values or levels. The choice of the level of factors is determined by the experimenter to suit the type(s) of research he or she undertakes. The effect of each factor as well as the interaction effects between factors is carefully studied to ascertain how they influence the dependent or the response variable. Conventionally, the design is coded at low and high levels for each continuous factor as -1 and +1 respectively. The required number of experiments given k factors, is given by

$$N = \mathbf{2^{K-m} + 2k + n_C} \tag{1.1}$$

for a non-replicated design

and

$$N = \mathbf{2^{K-m} r_f + 2k r_\alpha + n_{C_r}} \tag{1.2}$$

for replicated design

where $\boldsymbol{r_f}$ the number of replications of the factorial point is $\boldsymbol{r_\alpha}$ is the number of replications of the axial point and $\boldsymbol{n_{rc}}$ is the number of replications of the center points

The factorial design is considered a first order model design which consists of constant, linear and interaction terms, and takes the form of

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k-1} \sum_{i<j=2}^{k} \beta_{ij} x_i x_j + \varepsilon \tag{1.3}$$

Where y is measured response; x_i = the coded independent variables; i =1, 2,…, k; β's are unknown parameters and ε is the random error with mean zero and variance σ^2

When there are no replications in a two level factorial design which involves only continuous factor, the error sum of squares cannot be estimated because there is no degree of freedom available; hence, the model coefficients cannot be statistically tested. In this situation, replication becomes unavoidable. It is necessary to consider a second-order model for the purpose of obtaining a more precise estimate of experimental error and for modeling curvature. Potentially, a Second-Order polynomial model contains all the terms of the First-Order model, all quadratic terms and all cross product terms. It is expressed in the form.

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j + \sum_{k=1}^{k} \beta_{ii} {x_i}^2 + \varepsilon \tag{1.4}$$

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where y is measured response; x_i = the coded independent variables; i =1, 2,…, k; β's are unknown parameters and ε is the random error with mean zero and variance σ^2

In this research work, our focus is on one of the most flexible and widely used second-order design - the Central Composite Design. According to Box and Wilson (1951), Central Composite Design is the most popular class of second order designs. They are replications of 2^k factorial points or 2^{k-p} fractional factorial points of resolution V with replications of the cube, the axial and the center portions. The various replications are denoted in this paper by r_f, r_α and n_{rc} respectively.The CCD is constructed by first creating a 2^k factorial or the 2^{k-p}fraction designs and then appending a set of extra runs referred to as axial or star point. The axial points supply the extra levels required to fit a second order model in all factors. The $2^k$the 2k-p portion of the design allows us to fit first order terms and interactions, the axial portion allows us to fit quadratic terms in the factors. So, typically, one first runs a factorial portion with center point, this lets us estimate main effects and two-way interactions, and the center point lets us test for curvature. Replication of the center runs allows us to generate a pure error.

In order to obtain a better estimate of all linear and product term coefficient, squared term coefficient and also to estimate pure error, it is necessary to repeat the cube, the axial and the center portions a few or several times, this is referred to as replication in Central Composite Design. Replication in the Central Composite Design may take the form of Complete or Partial replications.

In this work, equal and partial replications of the cube, the star and the center points are employed. Equal replication implies replicating the cube point, the axial point and the center point equally while partial replication involves replicating either the cube point or the axial point or the center point.

Our focus is to locate the optimum combination of replications of the different portions, i.e., the cube, the axial and the center portions that yield optimal designs. To achieve this, a total of twenty seven (27) equal and partial replicated random experiments for k = 4, 5 and 6 for the various portions of the varying CCD's, namely, Spherical Central Composite Design (SCCD), Rotatable Central Composite Design (RCCD), Orthogonal Central Composite Design (OCCD) and Face Centered Central Composite Design (FCCD).

An appropriate experimental design is based on finding the best optimality criterion in which larger efficiency value implies a better design; Boonorm and Borkowski (2012).

The adoption of an appropriate experimental design capable of representing the response surface design greatly influences the efficiency of the experimental design Francis and Lilian (2018).

Many researchers have carried out different works on replication of Central Composite Designs. Chigbu and Ukaegbu (2017) examined earlier and later studies on the partial replication of the response surface central composite designs (CCDs). The results showed that the optimum performance of the replicated variations of the CCD depends on the axial distance, α, and also the cuboidal or spherical design region when the factorial and axial points are replicated in both design regions studied, no particular replicated variation of the CCD is consistently optimum. They concluded that in most cases replicating the axial points, improves the designs.

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Ibanga, (2013) compared some variations of experimental points of central composite designs in the presence of complete replication under rotatable and orthogonal design restrictions using the A-, D- and E- optimality criteria The efficiency values obtained suggest that replicated cubes plus replicated star points are better than partial replication of cube and star points under the design restrictions of rotatability and orthogonally.

Iwundu (2015) carefully examined the optimal partially replicated cube, star and center runs in Face centered central composite designs. The cube points were replicated while holding the star points and center points were not replicated, the star points were replicated while the cube points and star points were not replicated and the center points were replicated while holding the cube points and the star points fixed. The efficiencies of the designs were assessed using the D and G optimality criteria. The results showed that with the Face centered central composite design replicating the cube portion twice with fixed star and center points performed better than other variations under D and G-optimality criteria. It was also observed that replicating the cube points was more efficient than replicating the center points, and as such, emphasis should shift from replication of only center points, as non-center points performed better.

Two variations of central composite designs under the orthogonal and rotatable restriction using the D optimality criterion were compared by Chigbu and Ohaegbulem (2011) and they concluded that the replicated cube plus one star variation is better than the replicated star plus one cube variation under both restrictions.

Nduka and Chigbu (2014) compared two variation of N point orthogonal and rotatable central

Composite design based on Schur's ordering of design which says

$$\sum_{i=1}^{k} \lambda_i(\xi) \geq \sum_{i=1}^{k} \lambda_i(\eta); \ (k = 1,2,\dots,p) \tag{1.4}$$

as well as the D optimality and A optimality criteria. They came to the conclusion that by Schur's ordering of designs; a fixed axial point with replicated cube point is better than a fixed cube point plus replicated axial point. It was further demonstrated that the result remained the same for both A-optimality and D-Optimality.

Francis and Lilian (2018) investigated the effect of replicating the cube point, the axial point or center point and the results suggest that replication affects the different criteria in different dimensions because what improves one criterion positively may have negative impact on different criterion. It was further suggested that experimenters should be willing to sacrifice design efficiency to gain a pure error degree of freedom for lack of fit in the case of a decrease in efficiency of the replicated star or cube portion.

Iwundu (2017) studied the effects of addition of n_C center points on the optimality of Box-Behnken and Box-Wilson second-order designs. Relationships were seen to exist between optimal design properties and varying size of the designs by the addition of center points, the relationships between the Box-Behnken designs and the central composite design defined at α

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= $\sqrt{k}$ and α = $f^{\frac{1}{4}}$ are very strong and variations seem to exist with central composite designs defined at α = 1

METHODOLOGY

In this work, equal replication and partial replication for the various central composite designs; Spherical, Rotatable, Orthogonal, and Face centered central composite designs were employed to determine at what level of combination of the replication of the factorial point, the axial point and the center point, an optimum can be attained using the A, D and G optimality criteria. This was computed using some statistical Softwares, namely, MINITAB, DESIGN

EXPERT, MATLAB and JMP. This study covers the Central Composite Design with k = 4, 5 and 6 design variables.

Model and Design

The Central composite design (CCD) emanated from the response surface designs and is the most popular and commonly used classes of experimental design for fitting a second-order response surface model and is given as

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j + \sum_{k=1}^{k} \beta_{ii} {x_i}^2 + \varepsilon \qquad (2.1)$$

where y is measured response; x_i = the coded independent variables; i =1, 2,…, k; β's are unknown parameters and ε is the random error with mean zero and variance σ^2

The Central Composite Design is made up of the cube portion, the axial or star portion and the center runs. The four varieties of the Central Composite Design discussed in this research paper are: Spherical Central Composite Design (SCCD), the Rotatable Central Composite Design (RCCD), the Orthogonal Central Composite Design (OCCD) and the Face Centered Central Composite Design (FCCD).

The Spherical Central Composite design is a design that puts all the factorial and axial design point on the surface of a sphere of radius $\sqrt{k}$, and one of the major features is rotatability.

Rotatable Central Composite Design is a design that has the same prediction variance for any defined point equidistant from the design center. The alpha value is calculated as

$$\alpha = F^{\frac{1}{4}}$$

where F is the number of factorial or fractional factorial points.

Orthogonal Central Composite Design plays an important role as a second-order design. A design is said to be orthogonal if the effect of any factor balances out (totals up to zero) across the effects of other factors or if the off-diagonal elements of the information matrix $X'X$ are zero. As in Khuri (1996), the condition for making a CCD to be orthogonal is by setting

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$$\boldsymbol{\alpha} = \left[\frac{\sqrt{\boldsymbol{Nf}} - \boldsymbol{f}}{\mathbf{2}}\right]^{\frac{1}{2}}$$

The Face Centered Cube Design shall also be explored. This type of design places the axial point at the center of each face of the factorial space. Setting α = 1 makes the CCD Face Centered Cube Design. For a given second-order model, an N x p model matrix shall be formed using the design and the model. For example, the model matrix associated with central composite design with no replications and in k design variables for axial distance α and design size, N is represented in Algebraic form as

X =

$$\begin{vmatrix}
1 & x_{11} & x_{12} & \cdots & x_{1k} & x_{11}^2 & x_{12}^2 & \cdots & x_{1k}^2 & x_{11}x_{12} & x_{11}x_{13} & \cdots & x_{1(k-1)}x_{1k} \\
1 & x_{21} & x_{22} & \cdots & x_{2k} & x_{21}^2 & x_{22}^2 & \cdots & x_{2k}^2 & x_{21}x_{22} & x_{21}x_{23} & \cdots & x_{2(k-1)}x_{2k} \\
1 & x_{31} & x_{32} & \cdots & x_{3k} & x_{31}^2 & x_{32}^2 & \cdots & x_{3k}^2 & x_{31}x_{32} & x_{31}x_{33} & \cdots & x_{3(k-1)}x_{3k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_{n1} & x_{n2} & \cdots & x_{nk} & x_{n1}^2 & x_{n2}^2 & \cdots & x_{nk}^2 & x_{n1}x_{n2} & x_{n1}x_{n3} & \cdots & x_{n(k-1)}x_{nk} \\
1 & -\alpha & 0 & \cdots & 0 & \alpha^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & \alpha & 0 & \cdots & 0 & \alpha^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & -\alpha & \cdots & 0 & 0 & \alpha^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & \alpha & \cdots & 0 & 0 & \alpha^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & -\alpha & 0 & 0 & \cdots & \alpha^2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \alpha & \vdots & \vdots & \vdots & \alpha^2 & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{vmatrix}$$

The Information Matrix denoted by J in this research work is given as

$$J = X^T X$$

The inverse of the information matrix denoted by Z in this research work is written as

$$Z = (X^T X)^{-1}$$

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The Normalized Information Matrix otherwise known as the Moment Matrix denoted by M is given as

$$M = \frac{X^T X}{N}$$

where N the total number of runs is used as a penalty for the larger design.

Optimality Criteria

In studying the design efficiencies of replicated central composite designs with full factorial portions, the A, D and G design efficiencies shall be employed to assess the quality of the designs.

According to Chernoff (1953), the A-Optimality criterion seeks to minimize the trace of the inverse of the information matrix ($X'X$). A-efficiency is directly related to minimizing the individual variances of the model parameters; it provides a way of comparing designs across different sample sizes.

A-Optimality

This criterion introduced by Chemoff (1953) seeks to minimize the trace of the inverse of the information matrix $X^T X$, that is

$$A_{opt} = \{\min [\text{trace } (X^T X)^{-1}]\} \tag{2.1}$$

Where X is the design matrix and trace is the sum of the design elements of the matrix. The A-efficiency is given as

$$A_{eff} = \frac{100p}{trace[N(X'X)^{-1}]} \tag{2.2}$$

D-Optimality

The D-optimality criterion was the first alphabetical optimality criterion established, according to Wald (1943), this is based on the determinant of $X^T X$ which is inversely proportional to the square of the volume of the confidence region on the regression coefficients. It indicates how well the set of coefficients are estimated. Therefore, a smaller $|X^T X|$ or equivalently, a larger $|(X^T X)^{-1}|$ implies poorer estimation of the regression coefficient in the model. The goal of D-optimality is to maximize $|X^T X|$ or equivalently minimize $|(X^T X)^{-1}|$where X is the design matrix. Mathematically,

$$D_{opt} = [\max| (X^T X)|] \text{ or } [\min| (X^T X)^{-1}|] \tag{2.3}$$

The D-efficiency according to Crosier (1993) is the pth root of the ratio of $\frac{det\,(X'X)}{N^P}$ to maximum possible value of $\frac{det\,(X'X)}{N^P}$ for any design defined on the same region. The D-efficiency is

$$D_{eff} = 100\frac{|X'X|^{\frac{1}{p}}}{N} \tag{2.4}$$

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G-Optimality

The aim of G-optimality criterion; is to minimize the maximum prediction variance in the design region. Hence,

$$D_{opt} = min[N\hat{\sigma}^2_{max}] \tag{2.5}$$

The G-efficiency of a design is defined as

$$G = \frac{P}{V(x)_{max}} \tag{2.6}$$

Where p is the number of parameters in the model and V(x)max is maximum scaled variance of prediction. The variance of the function at x according to Myer (1966) is

$$V(\hat{y}(x)) = \frac{V(x)\sigma^2}{N} \tag{2.7}$$

where $V(x) = N\underline{\underline{x}}\,(X'X)^{-1}\underline{\underline{x}}$ is the scaled prediction variance for any point $\underline{x}$ in the design region.

Thus

$Var\,(\hat{y}(x)) = \underline{\underline{x}}'M^{-1}\underline{\underline{x}}.$

The vector $\underline{\underline{x}}$ is the row vector of the design matrix X, associated with the design point $\underline{x}$.

G-efficiency thus examines the maximum value of $V(\underline{x}) = \frac{NVar\,(\hat{y}(x))}{\sigma^2}$ within the design region with respect to its theoretical minimum variance p. Therefore, A G-optimality and the corresponding G-efficiency emphasize the use of designs for which the maximum $\frac{Nvar[\hat{y}(x)]}{\sigma^2}$ in the region of the design is not too large.

RESULTS

In this research work, the optimal A, D and G-optimality values were obtained for equal and partial replication of cube, axial and center points using selected varieties of the Central Composite Designs; the SCCD, RCCD, OCCD and FCCD, for factor k = 4, 5 and 6. Various combinations of replications of the cube, the axial and the center portions were employed to track the exact points where optimal values occur as can be seen in the tables below.

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Table 1 Optimality properties for SCCD with Full Factorial Replicates in k = 4 variables

Design type: SCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								$\max\left\|\left(\frac{X'X}{N}\right)\right\|$	$min\left\|\left(\frac{X'X}{N}\right)^{-1}$				
1	25	15	2	1	1	1	47.40	0.0188	53.188	25.00	31.65	76.73	60.00
2	26	15	2	1	1	2	33.05	0.0209	47.894	15.17	45.40	77.27	98.90
3	27	15	2	1	1	3	28.70	0.0178	56.241	15.75	52.29	76.44	95.24
4	33	15	2	1	2	1	59.67	0.0098	101.541	33.00	25.19	73.49	45.46
5	34	15	2	1	2	2	40.12	0.0126	79.448	18.06	37.40	74.70	83.05
6	35	15	2	1	2	3	33.99	0.0122	81.814	18.59	44.12	74.56	80.67
7	41	15	2	1	3	1	72.28	0.0039	255.779	41.00	20.76	69.10	36.59
8	42	15	2	1	3	2	47.80	0.0054	183.576	21.00	31.40	70.65	71.43
9	43	15	2	1	3	3	39.95	0.0057	174.186	21.50	37.54	70.89	69.77
10	41	15	2	2	1	1	67.16	0.0093	107.907	36.59	22.34	73.19	36.59
11	42	15	2	2	1	2	42.55	0.0129	77.446	21.00	35.27	74.83	71.43
12	43	15	2	2	1	3	34.57	0.0136	73.485	21.50	43.38	75.09	69.77
13	49	15	2	2	2	1	77.08	0.0127	78.566	49.00	19.46	74.76	30.61
14	50	15	2	2	2	2	47.25	0.0188	53.188	25.00	31.65	76.73	60.00
15	51	15	2	2	2	3	37.74	0.0210	47.723	17.00	39.77	77.28	88.24
16	57	15	2	2	3	1	88.07	0.0096	104.090	57.00	17.04	73.37	26.32
17	58	15	2	2	3	2	53.36	0.0148	67.558	29.00	28.12	75.52	51.72
18	59	15	2	2	3	3	41.95	0.0172	58.203	19.67	35.74	76.27	76.27
19	57	15	2	3	1	1	88.07	0.0041	246.732	57.00	17.04	69.27	26.32
20	58	15	2	3	1	2	53.36	0.0062	160.137	29.00	28.12	71.29	51.72
21	59	15	2	3	1	3	41.95	0.0072	137.963	27.39	35.74	72.00	54.76
22	65	15	2	3	2	1	96.72	0.0088	113.437	65.00	15.51	72.95	23.08
23	66	15	2	3	2	2	56.96	0.0140	71.315	33.00	26.33	75.24	45.46
24	67	15	2	3	2	3	43.89	0.0168	59.574	51.72	34.18	76.15	51.72
25	73	15	2	3	3	1	107.0	0.0151	66.029	73.00	14.02	73.26	20.55
26	74	15	2	3	3	2	62.16	0.0153	65.232	37.00	24.12	75.69	40.54
27	75	15	2	3	3	3	47.40	0.0188	53.188	25.00	31.61	76.73	60.00

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Table 2 Optimality properties for SCCD with Full Factorial Replicates in k = 5 variables

Design type: SCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								max$\left\lvert\left(\frac{X'X}{N}\right)\right\rvert$	$min\left\lvert\left(\frac{X'X}{N}\right)^{-1}\right.$				
1	43	21	2.236	1	1	1	73.79	0.0096	104.00	43.00	28.45	80.16	48.84
2	44	21	2.236	1	1	2	49.10	0.0119	84.28	23.89	42.75	80.96	87.92
3	45	21	2.236	1	1	3	41.22	0.0111	90.06	24.43	50.93	80.71	85.97
4	53	21	2.236	1	2	1	87.61	0.0069	145.67	53.00	23.98	78.88	39.62
5	54	21	2.236	1	2	2	56.86	0.0093	107.85	27.00	36.95	80.02	77.78
6	55	21	2.236	1	2	3	46.92	0.0095	105.90	23.53	44.79	80.10	89.23
7	63	21	2.236	1	3	1	102.3	0.0026	377.60	63.00	20.54	75.38	33.33
8	64	21	2.236	1	3	2	65.47	0.0038	262.80	32.00	32.07	76.70	65.63
9	65	21	2.236	1	3	3	53.50	0.0041	242.63	26.60	39.25	76.99	78.94
10	75	21	2.236	2	1	1	113.0	0.0025	402.04	75.00	18.59	75.16	28.00
11	76	21	2.236	2	1	2	68.93	0.0038	265.48	38.00	30.48	76.66	55.26
12	77	21	2.236	2	1	3	54.44	0.0043	232.90	37.04	38.60	77.14	56.69
13	85	21	2.236	2	2	1	123.9	0.0061	162.70	85.00	16.94	78.47	24.71
14	86	21	2.236	2	2	2	73.79	0.0096	104.00	43.00	28.46	80.16	48.84
15	87	21	2.236	2	2	3	57.25	0.0113	88.38	29.00	36.68	80.78	72.41
16	95	21	2.236	2	3	1	136.6	0.0059	169.18	95.00	15.37	78.32	22.11
17	96	21	2.236	2	3	2	80.45	0.0095	105.39	48.00	26.10	80.11	43.75
18	97	21	2.236	2	3	3	61.91	0.0114	87.35	32.33	33.92	80.83	64.95
19	107	21	2.236	3	1	1	153.3	0.0007	1405	107.0	13.69	70.81	19.63
20	108	21	2.236	3	1	2	89.96	0.0012	854.05	54.00	23.34	72.51	38.89
21	109	21	2.236	3	1	3	69.02	0.0014	690.96	36.33	30.43	73.25	49.77
22	117	21	2.236	3	2	1	162.5	0.0030	333.81	117.0	12.92	75.83	17.95
23	118	21	2.236	3	2	2	93.10	0.0050	199.57	59.00	22.56	77.71	35.59
24	119	21	2.236	3	2	3	70.09	0.0063	158.84	39.67	29.96	78.56	52.94
25	127	21	2.236	3	3	1	174.2	0.0045	224.72	127.0	12.05	77.27	16.54
26	128	21	2.236	3	3	2	98.82	0.0075	132.47	64.00	21.25	79.24	32.81
27	129	21	2.236	3	3	3	73.79	0.0096	104.00	43.00	28.46	80.16	48.84

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Table 3 Optimality properties for SCCD with Full Factorial Replicates in k = 6 variables

Design type: SCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$: max$\left\vert\left(\frac{X'X}{N}\right)\right\vert$	$Dopt$: $min\left\vert\left(\frac{X'X}{N}\right)^{-1}\right.$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	77	28	2.450	1	1	1	119.5	0.0032	315.04	77.00	23.44	81.43	36.36
2	78	28	2.450	1	1	2	75.52	0.0044	226.07	39.69	37.07	82.40	70.55
3	79	28	2.450	1	1	3	61.13	0.0046	215.31	40.20	45.80	82.54	69.66
4	89	28	2.450	1	2	1	134.0	0.0049	203.50	89.00	20.89	82.71	31.46
5	90	28	2.450	1	2	2	83.03	0.0072	139.10	45.00	33.73	83.84	62.22
6	91	28	2.450	1	2	3	66.25	0.0079	126.39	30.33	42.26	84.13	92.31
7	101	28	2.450	1	3	1	150.0	0.0026	378.00	101.0	18.66	80.90	27.73
8	102	28	2.450	1	3	2	92.06	0.0040	249.04	51.00	30.42	82.11	54.90
9	103	28	2.450	1	3	3	72.93	0.0046	218.18	34.33	38.39	82.50	81.55
10	141	28	2.450	2	1	1	197.0	0.0003	3035.00	141.0	14.21	75.10	19.86
11	142	28	2.450	2	1	2	115.6	0.00054	1850.00	71.00	24.22	76.44	39.44
12	143	28	2.450	2	1	3	88.50	0.00067	11501.00	66.74	31.60	77.01	41.96
13	153	28	2.450	2	2	1	208.0	0.0019	525.01	153.0	13.47	79.96	18.30
14	154	28	2.450	2	2	2	119.5	0.0032	315.04	77.00	23.44	81.43	36.37
15	155	28	2.450	2	2	3	90.11	0.0040	251.75	51.67	31.07	82.08	54.19
16	165	28	2.450	2	3	1	221.9	0.0030	336.38	165.0	12.62	81.24	16.97
17	166	28	2.450	2	3	2	126.4	0.0050	199.20	83.00	22.16	82.77	33.74
18	167	28	2.450	2	3	3	94.66	0.0064	157.12	55.67	29.58	83.48	50.30
19	205	28	2.450	3	1	1	275.6	0.000057	17673.00	205.0	10.16	70.52	13.66
20	206	28	2.450	3	1	2	156.8	0.000099	10126.00	103.0	17.86	71.94	27.19
21	207	28	2.450	3	1	3	117.3	0.00013.00	7731.00	93.00	23.87	72.63	29.99
22	217	28	2.450	3	2	1	283.9	0.00055	1820.00	217.0	9.864	76.48	12.90
23	218	28	2.450	3	2	2	158.0	0.00097	1035.00	109.0	17.72	78.04	25.69
24	219	28	2.450	3	2	3	116.1	0.0013	784.40	73.00	24.11	78.82	38.36
25	229	28	2.450	3	3	1	296.6	0.0013	740.87	229.0	9.442	78.98	12.23
26	230	28	2.450	3	3	2	163.7	0.0024	418.50	115.0	17.11	80.61	24.35
27	231	28	2.450	3	3	3	119.5	0.0032	315.04	77.00	23.44	81.43	36.36

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Table 4 Optimality properties for RCCD with Full Factorial Replicates in k = 4 variables

Design type: RCCD	**N**	**P**	**α**	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								max$\lvert\left(\frac{X'X}{N}\right)\rvert$	$min\lvert\left(\frac{X'X}{N}\right)^{-1}$				
1	25	15	2.000	1	1	1	47.40	0.0188	53.188	25.00	31.65	76.73	60.00
2	26	15	2.000	1	1	2	33.05	0.0209	47.894	15.17	45.40	77.27	98.90
3	27	15	2.000	1	1	3	28.70	0.0178	56.241	15.75	52.29	76.44	95.24
4	33	15	1.682	1	2	1	43.47	9.40×10^{-4}	1.064×10^{3}	18.91	34.52	62.83	79.32
5	34	15	1.682	1	2	2	36.72	9.144×10^{-4}	1.094×10^{3}	19.37	40.86	62.71	77.45
6	35	15	1.682	1	2	3	33.73	7.951×10^{-4}	1.258×10^{3}	19.88	44.47	62.13	75.46
7	41	15	1.520	1	3	1	39.89	1.043×10^{-4}	9.587×10^{3}	23.21	37.62	54.26	64.63
8	42	15	1.520	1	3	2	37.75	9.153×10^{-5}	1.093×10^{4}	23.64	39.74	53.79	63.45
9	43	15	1.520	1	3	3	36.55	7.756×10^{-5}	1.289×10^{4}	24.12	41.05	53.20	62.20
10	41	15	2.378	2	1	1	39.40	0.2334	4.2850	24.07	38.09	90.78	62.32
11	42	15	2.378	2	1	2	31.30	0.2476	4.0393	24.08	47.95	91.13	62.28
12	43	15	2.378	2	1	3	27.50	0.2337	4.2799	24.37	54.56	90.78	61.55
13	49	15	2.000	2	2	1	77.08	0.0127	78.566	49.00	19.46	74.76	30.61
14	50	15	2.000	2	2	2	47.25	0.0188	53.188	25.00	31.65	76.73	60.00
15	51	15	2.000	2	2	3	37.74	0.0210	47.723	17.00	39.77	77.28	88.24
16	57	15	1.807	2	3	1	65.79	0.0027	370.015	37.04	22.79	67.42	40.50
17	58	15	1.807	2	3	2	47.83	0.0034	291.203	22.85	31.36	68.51	65.66
18	59	15	1.807	2	3	3	40.19	0.0037	270.013	16.96	37.32	68.86	88.45
19	57	15	2.632	3	1	1	30.30	0.8071	1.2390	33.65	49.52	98.59	44.57
20	58	15	2.632	3	1	2	27.16	0.7830	1.2772	33.51	55.24	98.39	44.77
21	59	15	2.632	3	1	3	25.17	0.7306	1.3687	33.59	59.60	97.94	44.66
22	65	15	2.213	3	2	1	65.84	0.0639	15.6610	42.24	22.81	83.27	35.51
23	66	15	2.213	3	2	2	46.20	0.0838	11.9296	26.00	32.49	84.79	57.70
24	67	15	2.213	3	2	3	37.79	0.0933	10.7219	19.24	37.71	85.39	77.97
25	73	15	2.000	3	3	1	107.0	0.0151	66.029	73.00	14.02	73.26	20.55
26	74	15	2.000	3	3	2	62.16	0.0153	65.232	37.00	24.12	75.69	40.54
27	75	15	2.000	3	3	3	47.40	0.0188	53.188	25.00	31.61	76.73	60.00

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Table 5 Optimality properties for RCCD with Full Factorial Replicates in k = 5 variables

Design type: RCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$ max$\left|\left(\frac{X'X}{N}\right)\right|$	$Dopt$ $min\left|\left(\frac{X'X}{N}\right)^{-1}\right|$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	43	21	2.378	1	1	1	66.72	0.0361	27.7216	38.28	31.48	85.37	54.86
2	44	21	2.378	1	1	2	46.38	0.0421	23.7663	24.79	45.29	86.00	84.73
3	45	21	2.378	1	1	3	39.37	0.0386	25.9008	25.90	53.34	85.65	83.04
4	53	21	2.000	1	2	1	67.10	0.0010	956.8079	33.73	31.30	72.12	62.26
5	54	21	2.000	1	2	2	51.94	0.0012	865.8060	23.81	40.43	72.46	88.19
6	55	21	2.000	1	2	3	45.55	0.0011	916.4328	24.22	46.12	72.27	86.70
7	63	21	1.807	1	3	1	52.28	9.041×10^{-5}	1.106×10^{4}	27.60	40.16	64.19	76.08
8	64	21	1.807	1	3	2	48.17	8.310×10^{-5}	1.203×10^{4}	27.95	43.59	63.93	75.13
9	65	21	1.807	1	3	3	45.71	7.311×10^{-5}	1.368×10^{4}	28.33	45.94	63.55	74.13
10	75	21	2.828	2	1	1	43.29	0.6795	1.4716	43.50	48.54	98.19	48.28
11	76	21	2.828	2	1	2	38.46	0.6588	1.5180	43.23	54.62	98.05	48.58
12	77	21	2.828	2	1	3	35.45	0.6102	1.6388	43.24	59.24	97.69	48.57
13	85	21	2.378	2	2	1	102.1	0.0255	39.2542	68.19	20.59	83.98	30.80
14	86	21	2.378	2	2	2	66.75	0.0359	27.8309	38.28	31.48	85.37	54.86
15	87	21	2.378	2	2	3	53.35	0.0407	24.5456	26.80	39.37	85.88	78.36
16	95	21	2.149	2	3	1	125.4	0.0030	336.9478	84.47	16.75	75.80	24.86
17	96	21	2.149	2	3	2	78.09	0.0045	222.2533	45.18	26.89	77.31	46.48
18	97	21	2.149	2	3	3	61.24	0.0053	187.8732	31.04	34.28	77.93	67.65
19	107	21	3.130	3	1	1	34.38	2.2902	0.4366	61.56	61.10	104.03	34.11
20	108	21	3.130	3	1	2	32.79	2.1325	0.4689	61.36	64.05	103.68	34.23
21	109	21	3.130	3	1	3	31.57	1.9621	0.5096	61.30	66.53	103.27	34.26
22	117	21	2.632	3	2	1	63.34	0.2030	4.9255	37.44	31.17	92.70	56.09
23	118	21	2.632	3	2	2	53.17	0.2242	4.4609	34.07	39.51	93.14	61.65
24	119	21	2.632	3	2	3	46.63	0.2785	3.5902	33.93	44.67	93.31	61.90
25	127	21	2.378	3	3	1	131.2	0.0203	49.3593	92.72	16.03	83.07	22.65
26	128	21	2.378	3	3	2	85.29	0.0297	33.6172	54.01	24.64	84.61	38.88
27	129	21	2.378	3	3	3	66.75	0.0359	27.8309	38.28	31.48	85.37	54.86

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Table 6 Optimality properties for RCCD with Full Factorial Replicates in k = 6 variables

Design type: RCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$ max$\left\|\left(\frac{X'X}{N}\right)\right\|$	$Dopt$ $min\left\|\left(\frac{X'X}{N}\right)^{-1}\right.$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	77	28	2.828	1	1	1	71.17	0.1713	5.8368	43.79	39.37	93.91	63.94
2	78	28	2.828	1	1	2	57.16	0.1791	5.5825	43.55	49.00	94.06	64.30
3	79	28	2.828	1	1	3	50.34	0.1672	5.9803	43.70	55.64	93.83	64.08
4	89	28	2.378	1	2	1	129.1	0.0025	394.1553	84.05	21.68	80.79	33.32
5	90	28	2.378	1	2	2	82.23	0.0036	277.2613	43.71	34.05	81.81	64.06
6	91	28	2.378	1	2	3	66.22	0.0039	254.3159	29.75	42.28	82.06	94.12
7	101	28	2.149	1	3	1	83.95	2.382×10^{-4}	4.199×10^{3}	41.54	33.35	74.24	67.40
8	102	28	2.149	1	3	2	70.21	2.551×10^{-4}	3.921×10^{3}	31.94	39.88	74.42	87.66
9	103	28	2.149	1	3	3	62.82	2.507×10^{-4}	3.990×10^{3}	32.22	44.57	74.37	86.91
10	141	28	3.364	2	1	1	44.78	2.8628	0.3493	79.80	62.50	103.82	35.00
11	142	28	3.364	2	1	2	42.91	2.6400	0.3788	79.71	65.24	103.52	35.13
12	143	28	3.364	2	1	3	41.44	2.4083	0.4152	79.58	67.55	103.18	35.19
13	153	28	2.828	2	2	1	85.37	0.1541	6.4872	51.00	32.84	93.56	54.90
14	154	28	2.828	2	2	2	71.17	0.1713	5.8368	43.79	39.37	93.91	63.94
15	155	28	2.828	2	2	3	62.73	0.1787	5.5960	43.59	44.66	94.05	64.23
16	165	28	2.556	2	3	1	186.4	0.0099	101.3382	136.1	15.01	84.79	20.57
17	166	28	2.556	2	3	2	115.9	0.0152	65.7891	75.03	24.17	86.11	37.32
18	167	28	2.556	2	3	3	89.27	0.0187	53.6128	51.99	31.36	86.74	53.86
19	205	28	3.722	3	1	1	38.44	7.8319	0.1277	114.7	72.87	107.64	24.42
20	206	28	3.722	3	1	2	37.81	7.2505	0.1379	114.5	74.06	107.3	24.45
21	207	28	3.722	3	1	3	37.27	6.6946	0.1494	114.5	75.14	107.0	24.47
22	217	28	3.130	3	2	1	55.58	1.0821	0.9242	62.62	50.39	100.3	44.71
23	218	28	3.130	3	2	2	52.56	1.0673	0.9370	62.26	53.29	100.2	44.97
24	219	28	3.130	3	2	3	50.16	1.0409	0.9607	62.02	55.84	100.2	45.15
25	229	28	2.828	3	3	1	92.50	0.1456	6.8671	57.25	30.31	93.37	48.91
26	230	28	2.828	3	3	2	79.68	0.1612	6.2044	46.00	35.18	93.71	60.87
27	231	28	2.828	3	3	3	71.17	0.1713	5.8368	43.79	39.37	93.91	63.94

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Table 7 Optimality properties for OCCD with Full Factorial Replicates in k = 4 variables

Design type: OCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$ $\max\|\left(\frac{X'X}{N}\right)\|$	$min\|\left(\frac{X'X}{N}\right)^{-1}$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	25	15	1.414	1	1	1	35.88	2.951×10^{-4}	3.388×10^{3}	15.88	41.81	58.17	94.49
2	26	15	1.483	1	1	2	33.23	3.937×10^{-4}	2.540×10^{3}	16.35	45.13	59.28	91.75
3	27	15	1.547	1	1	3	31.35	4.924×10^{-4}	2.031×10^{3}	16.82	47.85	60.18	89.17
4	33	15	1.321	1	2	1	35.22	5.666×10^{-5}	1.765×10^{4}	20.12	42.59	52.10	74.55
5	34	15	1.353	1	2	2	34.50	5.812×10^{-5}	1.720×10^{4}	20.58	43.48	52.20	72.88
6	35	15	1.384	1	2	3	33.94	5.899×10^{-5}	1.695×10^{4}	21.04	44.21	52.25	71.29
7	41	15	1.266	1	3	1	37.58	1.074×10^{-5}	9.315×10^{4}	24.28	39.91	46.63	61.78
8	42	15	1.286	1	3	2	37.37	1.034×10^{-5}	9.673×10^{4}	24.73	40.14	46.52	60.65
9	43	15	1.306	1	3	3	37.21	9.973×10^{-6}	1.003×10^{5}	25.18	40.31	46.40	59.57
10	41	15	1.453	2	1	1	45.99	3.075×10^{-4}	3.252×10^{3}	21.09	32.62	58.32	71.11
11	42	15	1.527	2	1	2	40.71	5.094×10^{-4}	1.963×10^{3}	20.72	36.84	60.31	72.38
12	43	15	1.596	2	1	3	36.82	7.803×10^{-4}	1.282×10^{3}	20.75	40.74	62.06	72.28
13	49	15	1.378	2	2	1	37.59	2.486×10^{-4}	4.023×10^{3}	15.64	39.91	57.51	95.93
14	50	15	1.414	2	2	2	35.88	2.947×10^{-4}	3.394×10^{3}	15.88	41.81	58.17	94.49
15	51	15	1.449	2	2	3	34.44	3.433×10^{-4}	2.913×10^{3}	16.11	43.55	5876	93.10
16	57	15	1.336	2	3	1	35.65	1.248×10^{-4}	8.015×10^{3}	17.78	42.07	54.92	84.38
17	58	15	1.359	2	3	2	34.85	1.331×10^{-4}	7.513×10^{3}	18.01	43.04	55.16	83.29
18	59	15	1.382	2	3	3	34.15	1.416×10^{-4}	7.062×10^{3}	18.24	43.93	55.37	82.22
19	57	15	1.467	3	1	1	57.76	1.767×10^{-4}	5.659×10^{3}	28.26	25.97	56.22	53.08
20	58	15	1.543	3	1	2	60.18	2.222×10^{-4}	4.500×10^{3}	27.39	29.99	58.51	54.77
21	59	15	1.615	3	1	3	64.39	2.647×10^{-4}	3777×10^{3}	27.10	33.89	60.58	55.35
22	65	15	1.402	3	2	1	43.07	2.820×10^{-4}	3.546×10^{3}	17.76	34.83	57.98	84.47
23	66	15	1.439	3	2	2	40.51	3.572×10^{-4}	2.800×10^{3}	17.53	37.03	58.92	85.57
24	67	15	1.476	3	2	3	38.35	4.468×10^{-4}	2.238×10^{3}	17.42	39.11	59.78	86.09
25	73	15	1.366	3	3	1	38.23	2.343×10^{-4}	4.268×10^{3}	15.56	39.24	57.28	96.41
26	74	15	1.390	3	3	2	36.98	2.634×10^{-4}	3.797×10^{3}	15.72	40.56	57.74	95.44
27	75	15	1.414	3	3	3	40.80	1.304×10^{-4}	7.671×10^{3}	15.88	41.81	58.17	94.49

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Table 8 Optimality properties for OCCD with Full Factorial Replicates in k = 5 variables

Design type: OCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$ $\max\left\|\left(\frac{X'X}{N}\right)\right\|$	$Dopt$ $min\left\|\left(\frac{X'X}{N}\right)^{-1}\right.$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	43	21	1.596	1	1	1	49.13	6.230×10^{-5}	1.605×10^{4}	23.22	42.74	63.06	90.44
2	44	21	1.662	1	1	2	45.52	9.365×10^{-5}	1.068×10^{4}	23.14	46.13	64.29	90.75
3	45	21	1.724	1	1	3	44.77	1.317×10^{-4}	7.595×10^{3}	23.31	49.11	65.36	90.09
4	53	21	1.515	1	2	1	44.16	1.813×10^{-5}	5.517×10^{4}	24.62	47.56	59.46	85.29
5	54	21	1.547	1	2	2	43.15	1.983×10^{-5}	5.044×10^{4}	24.99	48.66	59.70	84.02
6	55	21	1.577	1	2	3	42.31	2.115×10^{-5}	4.728×10^{4}	25.37	49.64	59.91	82.78
7	63	21	1.466	1	3	1	44.83	3.465×10^{-6}	2.886×10^{5}	28.64	46.84	54.96	73.32
8	64	21	1.486	1	3	2	44.47	3.449×10^{-6}	2.900×10^{5}	29.01	47.23	54.95	72.39
9	65	21	1.506	1	3	3	44.16	3.432×10^{-6}	2.914×10^{5}	29.38	47.56	54.92	71.49
10	75	21	1.625	2	1	1	71.65	2.112×10^{-5}	4.734×10^{4}	38.07	30.90	60.96	55.17
11	76	21	1.695	2	1	2	60.78	5.634×10^{-5}	1.775×10^{4}	37.25	34.55	62.74	56.38
12	77	21	1.761	2	1	3	55.20	9.628×10^{-5}	1.039×10^{4}	36.91	38.04	64.37	56.90
13	85	21	1.562	2	2	1	51.35	4.965×10^{-5}	2.014×10^{4}	23.39	40.90	62.38	89.79
14	86	21	1.596	2	2	2	49.13	6.230×10^{-5}	1.605×10^{4}	23.22	42.74	63.06	90.44
15	87	21	1.629	2	2	3	47.21	7.671×10^{-5}	1.304×10^{4}	23.14	44.49	63.70	90.74
16	95	21	1.526	2	3	1	46.36	3.382×10^{-5}	2.957×10^{4}	22.40	45.30	61.25	93.73
17	96	21	1.548	2	3	2	45.33	3.751×10^{-5}	2.666×10^{4}	22.59	46.33	61.56	92.95
18	97	21	1.570	2	3	3	44.39	4.146×10^{-5}	2.412×10^{4}	22.78	47.31	61.85	92.18
19	107	21	1.636	3	1	1	88.33	1.108×10^{-5}	9.028×10^{4}	49.84	20.51	56.57	42.13
20	108	21	1.706	3	1	2	77.71	2.182×10^{-5}	4.584×10^{4}	51.43	27.02	59.99	40.83
21	109	21	1.774	3	1	3	69.44	4.051×10^{-5}	2.469×10^{4}	53.34	21.93	59.97	39.37
22	117	21	1.580	3	2	1	61.47	3.698×10^{-5}	2.704×10^{4}	31.04	34.16	61.50	67.65
23	118	21	1.615	3	2	2	58.16	4.946×10^{-5}	2.022×10^{4}	30.63	36.11	62.37	68.57
24	119	21	1.650	3	2	3	55.27	6.539×10^{-5}	1.529×10^{4}	30.35	38.00	63.19	69.20
25	127	21	1.550	3	3	1	52.17	4.565×10^{-5}	2.191×10^{4}	23.47	40.26	62.13	89.48
26	128	21	1.573	3	3	2	50.58	5.345×10^{-5}	1.871×10^{4}	23.32	41.53	62.61	90.05
27	129	21	1.596	3	3	3	49.13	6.230×10^{-5}	1.605×10^{4}	23.22	42.74	63.06	90.44

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Table 9 Optimality properties for SCCD with Full Factorial Replicates in k = 6 variables

Design type: OCCD	**N**	**P**	**α**	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								max$\left|\left(\frac{X'X}{N}\right)\right|$	$min\left|\left(\frac{X'X}{N}\right)^{-}\right|$				
1	77	28	1.761	1	1	1	69.65	8.664 X 10^{-6}	1.154 X 10^{5}	40.24	40.20	65.95	69.59
2	78	28	1.824	1	1	2	64.39	1.485 X 10^{-5}	6.733 X 10^{4}	39.78	43.49	67.23	70.39
3	79	28	1.885	1	1	3	60.17	2.408 X 10^{-5}	4.152 X 10^{4}	39.64	46.54	68.40	70.63
4	89	28	1.694	1	2	1	56.84	6.784 X 10^{-6}	1.474 X 10^{5}	29.31	49.26	65.36	95.53
5	90	28	1.724	1	2	2	55.33	7.929 X 10^{-6}	1.261 X 10^{5}	29.59	50.60	65.75	94.65
6	91	28	1.755	1	2	3	54.01	9.308 X 10^{-6}	1.074 X 10^{5}	29.86	51.84	66.11	93.78
7	101	28	1.653	1	3	1	54.30	2.067 X 10^{-6}	4.839 X 10^{5}	32.77	51.57	62.66	85.44
8	102	28	1.673	1	3	2	53.67	2.177 X 10^{-6}	4.594 X 10^{5}	33.04	52.18	62.78	84.74
9	103	28	1.693	1	3	3	53.09	2.288 X 10^{-6}	4.370 X 10^{5}	33.32	52.74	62.88	84.05
10	141	28	1.781	2	1	1	104.1	1.480 X 10^{-6}	6.755 X 10^{5}	70.65	26.91	61.91	39.63
11	142	28	1.846	2	1	2	93.65	2.959 X 10^{-6}	3.379 X 10^{5}	69.08	29.90	63.48	40.53
12	143	28	1.910	2	1	3	85.25	5.650 X 10^{-6}	1.770 X 10^{5}	68.13	32.84	64.94	41.10
13	153	28	1.728	2	2	1	72.77	6.464 X 10^{-6}	1.547 X 10^{5}	40.62	38.48	65.26	68.94
14	154	28	1.761	2	2	2	69.65	8.696 X 10^{-6}	1.150 X 10^{5}	40.24	40.20	65.95	69.59
15	155	28	1.793	2	2	3	66.90	1.147 X 10^{-5}	8.717 X 10^{4}	39.96	41.87	66.61	70.07
16	165	28	1.699	2	3	1	62.28	8.022 X10^{-6}	1.247 X 10^{5}	30.35	44.96	65.78	92.25
17	166	28	1.721	2	3	2	60.82	9.431 X 10^{-6}	1.060 X 10^{5}	30.23	46.04	66.14	92.63
18	167	28	1.742	2	3	3	59.47	1.093 X 10^{-5}	9.151 X 10^{4}	30.14	47.09	66.50	92.90
19	205	28	1.788	3	1	1	139.8	3.022 X 10^{-7}	3.309 X 10^{6}	101.1	20.03	58.50	27.69
20	206	28	1.854	3	1	2	124.3	6.425 X 10^{-7}	1.557 X 10^{6}	98.44	22.53	60.10	28.44
21	207	28	1.918	3	1	3	111.9	1.291 X 10^{-6}	7.746 X 10^{5}	96.68	25.04	61.62	28.96
22	217	28	1.741	3	2	1	91.19	2.600 X 10^{-6}	3.845 X 10^{5}	56.16	30.71	63.16	49.86
23	218	28	1.774	3	2	2	86.53	3.661 X 10^{-6}	2.731 X 10^{5}	55.43	32.36	63.95	50.52
24	219	28	1.807	3	2	3	82.38	5.101 X 10^{-6}	1.961 X 10^{5}	54.86	33.99	64.71	51.04
25	229	28	1.717	3	3	1	73.90	5.842 X 10^{-6}	1.712 X10^{5}	40.77	37.89	65.02	68.68
26	230	28	1.739	3	3	2	71.69	7.144 X 10^{-6}	1.400 X 10^{5}	40.48	39.06	65.49	69.18
27	231	28	1.761	3	3	3	69.65	8.696 X 10^{-6}	1.150 X10^{5}	40.24	40.20	65.95	69.59

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Table 10 Optimality properties for FCCD with Full Factorial Replicates in k = 4 variables

Design type: FCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$ max$\lvert\left(\frac{X'X}{N}\right)\rvert$	$Dopt$ $min\lvert\left(\frac{X'X}{N}\right)^{-1}$	$Gopt$	$Aeff$	$Deff$	$Geff$
1	25	15	1	1	1	1	58.86	5.356×10^{-6}	1.867×10^{5}	16.48	25.49	44.52	91.00
2	26	15	1	1	1	2	60.22	3.528×10^{-6}	2.834×10^{5}	17.14	24.91	43.30	87.53
3	27	15	1	1	1	3	61.79	2.318×10^{-6}	4.314×10^{5}	17.79	24.28	42.11	84.31
4	33	15	1	1	2	1	48.90	1.846×10^{-6}	5.418×10^{5}	21.03	30.68	41.47	71.35
5	34	15	1	1	2	2	49.92	1.304×10^{-6}	7.668×10^{5}	21.65	30.05	40.52	69.28
6	35	15	1	1	2	3	51.00	9.249×10^{-7}	1.081×10^{6}	22.28	29.41	39.60	67.33
7	41	15	1	1	3	1	48.20	5.099×10^{-7}	1.961×10^{6}	25.38	31.12	32.10	59.11
8	42	15	1	1	3	2	49.07	3.818×10^{-7}	2.619×10^{6}	25.98	30.57	37.34	57.74
9	43	15	1	1	3	3	49.97	2.870×10^{-7}	3.485×10^{6}	26.59	30.02	36.63	56.42
10	41	15	1	2	1	1	49.97	5.211×10^{-6}	3.485×10^{6}	20.78	17.87	44.44	72.20
11	42	15	1	2	1	2	84.46	4.299×10^{-6}	2.326×10^{5}	20.63	17.76	43.88	72.72
12	43	15	1	2	1	3	85.32	$3.490 \times 10^{=6}$	2.865×10^{5}	20.63	17.48	43.27	72.72
13	49	15	1	2	2	1	48.29	6.575×10^{-6}	1.521×10^{5}	16.16	25.73	45.14	92.83
14	50	15	1	2	2	2	58.86	5.356×10^{-6}	1.867×10^{5}	16.48	25.49	44.52	91.00
15	51	15	1	2	2	3	59.51	4.350×10^{-6}	2.299×10^{5}	16.81	25.21	43.91	89.23
16	57	15	1	2	3	1	51.03	4.134×10^{-6}	2.419×10^{5}	18.46	29.39	43.76	81.24
17	58	15	1	2	3	2	51.56	3.414×10^{-6}	2.930×10^{5}	18.78	29.09	43.21	79.86
18	59	15	1	2	3	3	52.11	2.819×10^{-6}	3.547×10^{5}	19.10	28.78	42.66	78.53
19	57	15	1	3	1	1	110.7	2.970×10^{-6}	3.367×10^{5}	28.35	13.55	42.81	52.92
20	58	15	1	3	1	2	110.6	2.708×10^{-6}	3.693×10^{5}	27.94	13.57	42.54	53.69
21	59	15	1	3	1	3	110.9	2.420×10^{-6}	4.132×10^{5}	27.74	13.52	42.23	54.07
22	65	15	1	3	2	1	70.69	7.049×10^{-6}	1.419×10^{5}	17.10	21.22	45.35	87.75
23	66	15	1	3	2	2	70.98	6.178×10^{-6}	1.619×10^{5}	17.02	21.13	44.95	88.12
24	67	15	1	3	2	3	71.39	5.386×10^{-6}	1.857×10^{5}	17.00	21.01	44.54	88.25
25	73	15	1	3	3	1	58.12	7.035×10^{-6}	1.422×10^{5}	16.05	25.81	45.34	93.46
26	74	15	1	3	3	2	58.47	6.143×10^{-6}	1.628×10^{5}	16.27	25.66	44.93	92.21
27	75	15	1	3	3	3	58.86	5.356×10^{-6}	1.867×10^{5}	16.48	25.49	44.52	91.00

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Table 11 Optimality properties for FCCD with Full Factorial Replicates in k = 5 variables

Design type: FCCD	N	P	α	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								max$\lvert\left(\frac{X'X}{N}\right)\rvert$	$min\lvert\left(\frac{X'X}{N}\right)^{-1}$				
1	43	21	1	1	1	1	113.2	4.834 X 10^{-8}	2.069 X 10^{7}	22.19	18.55	44.84	94.66
2	44	21	1	1	1	2	115.0	3.390 X 10^{-8}	2.950 X 10^{7}	22.25	18.27	44.09	94.38
3	45	21	1	1	1	3	116.9	2.369 X 10^{-8}	4.222 X 10^{7}	22.39	17.97	43.34	93.78
4	53	21	1	1	2	1	82.12	2.379 X 10^{-8}	4.205 X 10^{7}	25.58	25.57	43.35	82.11
5	54	21	1	1	2	2	83.33	1.725 X 10^{-8}	5.797 X 10^{7}	26.06	25.20	42.69	80.59
6	55	21	1	1	2	3	84.57	1.255 X 10^{-8}	7.971 X 10^{7}	26.54	24.83	40.05	79.14
7	63	21	1	1	3	1	74.43	6.126 X 10^{-9}	1.633 X 10^{8}	29.94	28.21	40.64	70.14
8	64	21	1	1	3	2	75.41	4.626 X 10^{-9}	2.162 X 10^{8}	30.41	27.85	40.10	69.05
9	65	21	1	1	3	3	76.40	3.504 X 10^{-9}	2.854 X 10^{8}	30.89	27.49	39.57	67.99
10	75	21	1	2	1	1	180.0	2.302 X 10^{-8}	4.344 X 10^{7}	37.62	11.67	43.28	55.82
11	76	21	1	2	1	2	180.9	1.980 X 10^{-8}	5.051 X 10^{7}	37.35	11.61	42.97	56.23
12	77	21	1	2	1	3	182.0	1.685 X 10^{-8}	5.936 X 10^{7}	37.21	11.54	42.64	56.43
13	85	21	1	2	2	1	112.4	5.757 X 10^{-8}	1.737 X 10^{7}	22.19	18.68	45.21	94.63
14	86	21	1	2	2	2	113.2	4.834 X 10^{-8}	2.069 X 10^{7}	22.19	18.55	44.84	94.66
15	87	21	1	2	2	3	114.1	4.051 X 10^{-8}	2.468 X 10^{7}	22.21	18.41	44.46	94.57
16	95	21	1	2	3	1	91.14	4.771 X 10^{-8}	2.096 X 10^{-7}	23.11	23.04	44.81	90.86
17	96	21	1	2	3	2	91.81	4.022 X 10^{-8}	2.486 X 10^{7}	23.36	22.87	44.45	89.92
18	97	21	1	2	3	3	92.51	3.391 X 10^{-8}	2.949 X 10^{7}	23.60	22.70	44.09	89.00
19	107	21	1	3	1	1	248.3	8.253 X $10^{=9}$	1.212 X 10^{8}	53.15	8.456	41.22	39.51
20	108	21	1	3	1	2	248.6	7.709 X 10^{-9}	1.297 X 10^{8}	52.53	8.449	41.08	39.98
21	109	21	1	3	1	3	249.2	7.111 X 10^{-9}	1.406 X 10^{8}	52.14	8.428	40.93	40.28
22	117	21	1	3	2	1	145.7	4.173 X 10^{-8}	2.396 X 10^{7}	30.00	14.42	44.52	70.01
23	118	21	1	3	2	2	146.2	3.745 X 10^{-8}	2.670 X 10^{7}	29.88	14.37	44.30	70.28
24	119	21	1	3	2	3	146.8	3.350 X 10^{-8}	2.985 X 10^{7}	29.81	14.31	44.06	70.44
25	127	21	1	3	3	1	112.2	6.100 X 10^{-8}	1.640 X10^{7}	22.20	18.73	45.34	94.59
26	128	21	1	3	3	2	112.7	5.433 X 10^{-8}	1.841 X 10^{7}	22.19	18.64	45.09	94.65
27	129	21	1	3	3	3	113.2	4.834 X 10^{-8}	2.069 X 10^{7}	22.19	18.55	44.84	94.66

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Table 12 Optimality properties for FCCD with Full Factorial Replicates in k = 6 variables

Design type: FCCD	**N**	**P**	**α**	r_f	r_α	n_c	$Aopt$	$Dopt$		$Gopt$	$Aeff$	$Deff$	$Geff$
								max$\lvert\left(\frac{X'X}{N}\right)\rvert$	$min\lvert\left(\frac{X'X}{N}\right)^{-1}$				
1	77	28	1.0000	1	1	1	227.5	$1.770 \text{ X } 10^{-10}$	$5.650 \text{ X } 10^{9}$	38.98	12.31	44.85	71.83
2	78	28	1.0000	1	1	2	229.5	$1.366 \text{ X } 10^{-10}$	$7.320 \text{ X } 10^{9}$	39.93	12.20	44.43	71.93
3	79	28	1.0000	1	1	3	231.6	$1.049 \text{ X } 10^{-10}$	$9.531 \text{ X } 10^{9}$	38.96	12.09	44.02	71.88
4	89	28	1.0000	1	2	1	146.2	$2.223 \text{ X } 10^{-10}$	$4.499 \text{ X } 10^{9}$	30.10	19.15	45.21	93.02
5	90	28	1.0000	1	2	2	147.5	$1.719 \text{ X } 10^{-10}$	$5.819 \text{ X } 10^{9}$	30.44	18.98	44.80	91.99
6	91	28	1.0000	1	2	3	148.9	$1.329 \text{ X } 10^{-10}$	$7.522 \text{ X } 10^{9}$	30.78	18.81	44.39	90.98
7	101	28	1.0000	1	3	1	121.5	$8.563 \text{ X } 10^{-11}$	$1.168 \text{ X } 10^{10}$	33.91	23.05	44.70	82.58
8	102	28	1.0000	1	3	2	122.5	$6.752 \text{ X } 10^{-11}$	$1.481 \text{ X } 10^{10}$	34.24	22.86	43.33	81.77
9	103	28	1.0000	1	3	3	123.5	$5.331 \text{ X } 10^{-11}$	$1.876 \text{ X } 10^{10}$	34.58	22.67	42.96	80.98
10	141	28	1.0000	2	1	1	393.5	$2.981 \text{ X } 10^{-11}$	$3.355 \text{ X } 10^{10}$	70.33	7.12	42.08	39.82
11	142	28	1.0000	2	1	2	394.5	$2.708 \text{ X } 10^{-11}$	$3.692 \text{ X } 10^{10}$	69.80	7.10	41.94	40.11
12	143	28	1.0000	2	1	3	395.9	$2.441 \text{ X } 10^{-11}$	$4.097 \text{ X } 10^{10}$	69.45	7.07	41.78	40.32
13	153	28	1.0000	2	2	1	226.6	$2.010 \text{ X } 10^{-10}$	$4.975 \text{ X } 10^{9}$	39.05	12.36	45.05	71.71
14	154	28	1.0000	2	2	2	227.5	$1.770 \text{ X } 10^{-10}$	$5.650 \text{ X } 10^{9}$	38.98	12.31	44.85	71.83
15	155	28	1.0000	2	2	3	228.5	$1.556 \text{ X } 10^{-10}$	$6.426 \text{ X } 10^{9}$	38.94	12.26	44.64	71.91
16	165	28	1.0000	2	3	1	172.0	$2.968 \text{ X } 10^{-10}$	$3.369 \text{ X } 10^{9}$	28.56	16.28	45.68	98.06
17	166	28	1.0000	2	3	2	172.7	$2.603 \text{ X } 10^{-10}$	$3.842 \text{ X } 10^{9}$	28.56	16.21	45.47	98.03
18	167	28	1.0000	2	3	3	173.5	$2.282 \text{ X } 10^{-10}$	$4.382 \text{ X } 10^{9}$	28.58	16.14	45.25	97.96
19	205	28	1.0000	3	1	1	560.8	$6.081 \text{ X } 10^{-12}$	$1.645 \text{ X } 10^{11}$	101.73	4.99	39.76	27.53
20	206	28	1.0000	3	1	2	561.0	$5.875 \text{ X } 10^{-12}$	$1.702 \text{ X } 10^{10}$	100.74	4.99	3971	27.79
21	207	28	1.0000	3	1	3	561.6	$5.628 \text{ X } 10^{-12}$	$1.778 \text{ X } 10^{11}$	100.00	4.99	39.65	28.00
22	217	28	1.0000	3	2	1	309.5	$7.965 \text{ X } 10^{-11}$	$1.256 \text{ X } 10^{10}$	54.85	9.05	43.58	51.05
23	218	28	1.0000	3	2	2	310.1	$7.400 \text{ X } 10^{-11}$	$1.351 \text{ X } 10^{10}$	54.64	9.03	43.47	51.25
24	219	28	1.0000	3	2	3	310.9	$6.860 \text{ X } 10^{-11}$	$1.458 \text{ X } 10^{10}$	54.48	9.01	43.35	51.40
25	229	28	1.0000	3	3	1	226.3	$2.096 \text{ X } 10^{-10}$	$4.771 \text{ X } 10^{9}$	39.08	12.38	45.12	71.65
26	230	28	1.0000	3	3	2	226.9	$1.927 \text{ X } 10^{-10}$	$5.190 \text{ X } 10^{9}$	39.02	12.34	44.98	71.76
27	231	28	1.0000	3	3	3	227.5	$1.770 \text{ X } 10^{-10}$	$5.650 \text{ X } 10^{9}$	38.98	12.31	44.85	71.83

The best Optimality and the best efficiency values for SCCD, RCCD, OCCD and FCCD have been summarized in Tables 13-15 with best optimal combinations.

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Table 13: Optimality Values and Efficiency Values (%) k = 4

Design Size	Design Variable K	Parameter P	Optimal Combination with corresponding Design Size and Axial Distance for A-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for D-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for G-Optimality Criterion				
SCCD	4	15	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	2.000	1	1	3	51	2.000	2	2	3	26	2.000	1	1	2
			Aopt value = 28.70(min) Aeff value = 52.29(max)					Dopt Value: max|M|= 0.0210 min|M^{-1}|= 47.72 Deff value = 77.28(max)					Gopt value = 15.17(min) Geff value = 98.90(max)				
Best Optimal Combination for SCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	2.000	1	1	3	27	2.000	1	1	3	27	2.000	1	1	3
			$Aopt$ = 28.70					D_{opt}^{max} = 0.0178 D_{opt}^{min} = 56.241					$Gopt$ = 15.75				
			$Aeff$ = 52.29					$Deff$ =76.44					$Geff$ = 95.24				
RCCD	4	15	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			59	2.632	3	1	3	57	2.632	3	1	1	26	2.000	1	1	2
			Aopt value = 25.17(min) Aeff value = 59.60(max)					Dopt Value: max|M|= 0.8071 Min|M^{-1}|= 1.2390 Deff value = 98.59(max)					Gopt value = 15.17(min) Geff value = 98.90(max)				
Best Optimal Combination for RCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	2.000	1	1	3	27	2.000	1	1	3	27	2.000	1	1	3
			$Aopt$ = 28.70					D_{opt}^{max} = 0.0178 D_{opt}^{min} = 56.241					$Gopt$ = 15.75				
			$Aeff$ = 52.29					$Deff$ =76.44					$Geff$ = 95.24				
OCCD	4	15	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	1.547	1	1	3	43	1.596	2	1	3	73	1.366	3	3	1
			Aopt value = 31.35(min) Aeff value = 47.85(max)					Dopt Value: max|M|= 7.803 X 10^{-4} Min|M^{-1}|= 1.282 X 10^{3} Deff value = 62.06(max)					Gopt value = 15.56(min) Geff value = 96.41(max)				
Best Optimal Combination for OCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	1.547	1	1	3	27	1.547	1	1	3	27	1.547	1	1	3
			$Aopt$ = 28.70					D_{opt}^{max} = 0.0178 D_{opt}^{min} = 56.241					$Gopt$ = 15.75				
			$Aeff$ = 52.29					$Deff$ =76.44					$Geff$ = 95.24				
FCCD	4	15	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			41	1.000	1	3	1	65	1.000	3	2	1	73	1.000	3	3	1
			Aopt value = 48.20(min) Aeff value = 31.12(max)					Dopt Value: max|M|= 7.049 X 10^{-64} Min|M^{-1}|= 1.419 X 10^{5} Deff value = 45.35(max)					Gopt value = 16.05(min) Geff value = 93.46(max)				
Best Optimal Combination for FCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			25	1.000	1	1	1	25	1.000	1	1	1	25	1.000	1	1	1
			$Aopt$ = 58.86					D_{opt}^{max} = 5.356 X 10^{-6} D_{opt}^{min} = 1.857 X 10^{5}					$Gopt$ = 16.48				
			$Aeff$ = 25.49					$Deff$ = 44.52					$Geff$ = 91.00				

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Table 14: Optimality Values and Efficiency Values (%) k = 5

Design Size	Design Variable K	Parameter P	Optimal Combination with corresponding Design Size and Axial Distance for A-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for D-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for G-Optimality Criterion				
SCCD	5	21	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			45	2.236	1	1	3	44	2.236	1	1	2	55	2.236	1	2	3
			Aopt value = 41.22(min) Aeff value = 50.93(max)					Dopt Value: $\max\lvert M\rvert = 0.0119$ $\mathrm{Min}\lvert M^{-1}\rvert = 84.28$ Deff value = 80.96(max)					Gopt value = 23.53(min) Geff value = 89.23(max)				
Best Optimal Combination for SCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			45	2.236	1	1	3	45	2.236	1	1	3	45	2.236	1	1	3
			$Aopt = 41.22$					$D_{opt}^{max} = 0.0111$ $D_{opt}^{min} = 90.06$					$Gopt = 24.43$				
			$Aeff = 50.93$					$Deff = 80.71$					$Geff = 85.97$				
RCCD	5	21	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			59	2.632	3	1	3	57	2.632	3	1	1	26	2.000	1	1	2
			Aopt value = 31.57(min) Aeff value 66.53(max)					Dopt Value: $\max\lvert M\rvert = 2.2902$ $\mathrm{Min}\lvert M^{-1}\rvert = 0.4366$ Deff value = 104.03(max)					Gopt value = 15.17(min) Geff value = 98.90(max)				
Best Optimal Combination for RCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			45	2.378	1	1	3	45	2.378	1	1	3	45	2.378	1	1	3
			$Aopt = 39.37$					$D_{opt}^{max} = 0.0386$ $D_{opt}^{min} = 25.9008$					$Gopt = 25.90$				
			$Aeff = 53.34$					$Deff = 76.44$					$Geff = 83.04$				
OCCD	5	21	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			55	1.577	1	2	3	45	1.724	1	1	3	95	1.526	2	3	1
			Aopt value = 42.31(min) Aeff value = 49.64(max)					Dopt Value: $\max\lvert M\rvert = 1.317 \times 10^{-4}$ $\mathrm{Min}\lvert M^{-1}\rvert = 7.595 \times 10^{3}$ Deff value = 65.36(max)					Gopt value = 22.40(min) Geff value = 93.73(max)				
Best Optimal Combination for OCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			45	1.724	1	1	3	45	1.724	1	1	3	45	1.724	1	1	3
			$Aopt = 44.77$					$D_{opt}^{max} = 1.317 \times 10^{-4}$ $D_{opt}^{min} = 7.595 \times 10^{3}$					$Gopt = 23.31$				
			$Aeff = 49.11$					$Deff = 65.36$					$Geff = 90.09$				
FCCD	5	21	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			63	1.000	1	3	1	127	1.000	3	3	1	86	1.000	2	2	2
			Aopt value = 74.43(min) Aeff value = 28.21(max)					Dopt Value: $\max\lvert M\rvert = 1.640 \times 10^{7}$ $\mathrm{Min}\lvert M^{-1}\rvert = 1.640 \times 10^{7}$ Deff value = 45.34(max					Gopt value = 22.19(min) Geff value = 94.66(max)				
Best Optimal Combination for FCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			53	1.000	1	2	1	53	1.000	1	2	1	53	1.000	1	2	1
			$Aopt = 82.12$					$D_{opt}^{max} = 2.379 \times 10^{-8}$ $D_{opt}^{min} = 4.205 \times 10^{7}$					$Gopt = 25.58$				
			$Aeff = 25.57$					$Deff = 43.35$					$Geff = 82.11$				

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Table 15: Optimality Values and Efficiency Values (%) K = 6

Design Size	Design Variable K	Parameter P	Optimal Combination with corresponding Design Size and Axial Distance for A-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for D-Optimality Criterion					Optimal Combination with corresponding Design Size and Axial Distance for G-Optimality Criterion				
			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
SCCD	6	28	79	2.450	1	1	3	91	2.236	1	2	3	91	2.450	1	2	3
			Aopt value = 61.13(min) Aeff value = 45.80(max)					Dopt Value: max|M|= 0.0079 Min|M^{-1}|= 126.39 Deff value = 84.13 (max)					Gopt value = 30.33(min) Geff value = 92.31(max)				
Best Optimal Combination for SCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			91	2.450	1	2	3	91	2.450	1	2	3	91	2.450	1	2	3
			$Aopt$ = 66.25					D_{opt}^{max} = 0.0078 D_{opt}^{min} = 126.93					$Gopt$ = 24.43				
			$Aeff$ = 42.26					$Deff$ = 84.13					$Geff$ = 85.97				
RCCD	6	28	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			207	3.722	3	1	3	205	3.722	3	1	1	91	2.378	1	2	3
			Aopt value = 37.27(min) Aeff value = 75.14(max)					Dopt Value: max|M|= 7.8319 Min|M^{-1}|= 0.1277 Deff value =107.64(max)					Gopt value = 29.75(min) Geff value = 94.12(max)				
Best Optimal Combination for RCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			91	2.378	1	2	3	91	2.378	1	2	3	91	2.378	1	2	3
			$Aopt$ = 66.22					D_{opt}^{max} = 0.0039 D_{opt}^{min} = 254.3159					$Gopt$ = 29.75				
			$Aeff$ = 42.28					$Deff$ = 82.06					$Geff$ = 94.12				
OCCD	6	28	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			27	1.547	1	1	3	43	1.596	2	1	3	73	1.366	3	3	1
			Aopt value = 53.09(min) Aeff value = 52.74(max)					Dopt Value: max|M|= 2.408 X 10^{-5} Min|M^{-1}|= 4.152 X 10^{4} Deff value = 68.40 (max)					Gopt value = 29.31(min) Geff value = 95.53(max)				
Best Optimal Combination for OCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			91	1.755	1	1	3	91	1.755	1	1	3	91	1.755	1	1	3
			$Aopt$ = 54.01					D_{opt}^{max} = 9.308 X 10^{-6} D_{opt}^{min} = 1.074 X 10^{5}					$Gopt$ = 29.86				
			$Aeff$ = 51.84					$Deff$ = 66.11					$Geff$ = 93.78				
FCCD	6	28	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			41	1.000	1	3	1	65	1.000	3	2	1	73	1.000	3	3	1
			Aopt value = 121.5(min) Aeff value = 23.05(max)					Dopt Value: max|M|= 2.968 X 10^{-10} Min|M^{-1}|= 3.369 X 10^{9} Deff value = 45.68(max					Gopt value = 28.56(min) Geff value = 98.06(max)				
Best Optimal Combination for FCCD			N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C	N	α	r_f	r_α	n_C
			89	1.000	1	2	1	89	1.000	1	2	1	89	1.000	1	2	1
			$Aopt$ = 146.20					D_{opt}^{max} = 2.223 X 10^{-10} D_{opt}^{min} = 4.499 X 10^{9}					$Gopt$ = 30.10				
			$Aeff$ = 19.15					$Deff$ = 45.21					$Geff$ = 93.02				

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DISCUSSION OF RESULTS

With reference to Table1, for SCCD full factorial replicates, the optimal combination of [1:1:3] yielded the best A_{opt} value of 28.70 with A-efficiency value of 52.29%. The associated design is of size N = 27 and contains one complete 2^4 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:1:2] yielded the best G_{opt} value of 15.17 with the best G-efficiency value of 98.90%. The associated design is of size N = 26 and contains one complete 2^4 factorial points, one complete 2k axial points and two center points. It is interesting to note that the design associated with the [1:1:2] combination resulted in the second best A-Optimal value of 33.05 as well as second best A-efficiency value of45.40% Optimal combination of [2:2:3] produced the best D-efficiency value of 77.28%. The associated design is of size N = 51. It is necessary to note that the design associated with [1:1:2] resulted in the second best D-efficiency value and exhibits a smaller design size of N = 26 and has a run size efficiency advantage over the combination of [2:2:3] with N = 51. The optimal combination of [1:1:3] yielded the most efficient design with design size of N = 27, and also has a run size efficiency advantage over other combinations. Similarly, the combination of [1:1:3] producing best A_{opt} value of 28.70 and best A-efficiency value of 52.29% was also very good in terms of G_{opt} and G-efficiency value. Precisely, the combination of [1:1:3] produced the second best G_{opt} value of 15.75 and G-efficiency value of 95.24%. The optimal combinations of [1:1:2] and [2:2:3] produced best D_{opt}^{max} values of 0.0209 and 0.0210, respectively. It is important to observe that the slight difference in these values is due to round-up error. The corresponding G-efficiency values are 77.27% and 77.28%, respectively. Of the two D-Optimal combinations, the combination [1:1:2], exhibits run size efficiency having a smaller design size of 26 against the combination [2:2:3] with N =51. [1:1:3] yielded the best optimal combination and can be considered the most efficient design

Similarly, from Table 2, for SCCD full factorial replicates, the optimal combination of [1:1:3] yielded the best A_{opt} value of 41.22 with A-efficiency value of 50.93%. The associated design is of size N = 45 and contains one complete 2^5 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:2:3] yielded the best G_{opt} value of 23.53 with the best G-efficiency value of 89.23%. The associated design is of size N = 55 and contains one complete 2^5 factorial points, two complete 2k axial points and three center points. Optimal combination of [1:1:2] yielded the best D_{opt}^{max}value of 0.0119 with the best D-efficiency value of 80.96%. The associated design is of size N = 44 and contains one complete 2^5 factorial points, one complete 2k axial points and two center points. Optimal combination of [1:1:2] yielded an A_{opt} value of 49.10 and A-efficiency value of 42.75%, best D_{opt}^{max} value of 0.0119 and best D-efficiency value of 80.96, and G_{opt} value of 23.89 with a corresponding G-efficiency value of 87.92%. The associated design size is N = 44. Optimal combination of [1:1:3] yielded best A_{opt} value of 41.22 and best A-efficiency value of 50.93%, a D_{opt}^{max} value of 0.0111 and D-efficiency value of 80.71 and G_{opt} value of 23.89 with a corresponding G-efficiency value of 87.92%. The associated design size is N = 45. Optimal combination of [1:2:3] yielded the second best A_{opt}

From Table 3, for SCCD full factorial replicates, the optimal combination of [1:1:3] yielded the best A_{opt} value of 61.13 with A-efficiency value of 45.80%. The associated design is of size N = 79 and contains one complete 2^6 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:2:3] yielded the best G_{opt} value of 30.33 with the best G-efficiency value of 92.31%, and also the best D_{opt}^{max} value of 126.39 with corresponding best

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D-efficiency value of 84.13%. The associated design is of size N = 91 and contains one complete 2^6 factorial points, two complete 2k axial points and three center points. Notice that the design associated with [1:1:3] recorded the best A-efficiency value of 45.80% with corresponding D and G-efficiency values of 82.54% and 69.66%, respectively. The design associated with [1:2:3] yielded the second best A-efficiency value of 42.26% with corresponding best D and G-efficiency values of 84.13% and 92.31%, respectively. Though, with larger design size of N = 91 and G-efficiency value of 92.31%, the [1:2:3] combination exhibits a comparative advantage over the [1:1:3] combination with smaller design size of N = 79 and G-efficiency value of 69.66%.

From Table 4, for RCCD full factorial replicates, the optimal combination of [3:1:3] yielded the best A_{opt} value of 25.17 with A-efficiency value of 59.60%. The associated design is of size N = 59 and contains three complete 2^4 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:1:2] yielded the best G_{opt} value of 15.17 with the best G-efficiency value of 98.90%. The associated design is of size N = 26 and contains one complete 2^4 factorial points, one complete 2k axial points and two center points. Optimal combination of [3:1:1] yielded the best D_{opt}^{max} value of 0.0807 with the best D-efficiency value of 98.59%. The associated design is of size N = 57 and contains three complete 2^4 factorial points, one complete 2k axial points and one center point. Observe that the designs associated with best A and D criteria produced G-efficiency values below 50% of the maximum G-efficiency value. Optimal combination of [1:1:3] recorded best A, D and G-efficiency values with run-size efficiency advantage over other combinations.

From Table 5, for RCCD full factorial replication, the optimal combination of [3:1:3] yielded the best A_{opt} value of 31.57 with A-efficiency value of 66.53%. The associated design is of size N = 109 and contains three complete 2^5 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:2:2] yielded the best G_{opt} value of 23.81 with the best G-efficiency value of 88.19%. The associated design is of size N = 54 and contains one complete 2^5 factorial points, two complete 2k axial points and two center points. Optimal combination of [3:1:1] yielded the best D_{opt}^{max} value of 2.2902 with super D-efficiency value of 104.03%. The associated design is of size N = 107 and contains three complete 2^5 factorial points, one complete 2k axial points and one center point. Optimal combinations of [3:1:1], [3:1:2] and [3:1:3], yielded best D_{opt}^{max} values of 2.2902, 2.1325, and 1.9621, respectively and corresponding super D-efficiency values of 104.03%, 103.68% and 103.27% respectively and large A-efficiency values of 61.10%, 64.05 and 66.53%, respectively with design sizes of N = 107, 108 and 109, respectively. The corresponding G_{opt} values of 61.56, 61.36 and 61.30, respectively with G-efficiency values of 34.11%, 34.23% and 34.26%, respectively recorded less than 50% of the maximum G-efficiency value of 88.19%. Comparatively, the design associated with [1:1:3] seems to yield the best optimal combination with a smaller design size of N = 45.

From Table 6, for RCCD full factorial replicates, the optimal combination of [3:1:3] yielded the best A_{opt} value of 37.27 with A-efficiency value of 75.14%. The associated design is of size N = 207 and contains three complete 2^6 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:2:3] yielded the best G_{opt} value of 29.75 with the best G-efficiency value of 94.12%. The associated design is of size N = 91 and contains one complete 2^6 factorial points, two complete 2k axial points and three center points. Optimal combination of [3:1:3] resulted in best D_{opt}^{max} value of 7.8319 with super D-efficiency value of

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107.64%. It is worthy to note that the designs associated with [2:1:1], [2:1:2], [2:1:3], [3:1:1], [3:1:2], [3:1:3], [3:2:1], [3:2:2], and [3:2:3], recorded D_{opt}^{max} values of 2.8628, 2.6400, 2.4083, 7.8319, 7.2505, 6.6946, 1.0821, 1.0673 and 1.0409, respectively with corresponding super D-efficiency values of 103.82%, 103.52%, 103.18%, 107.64%, 107.30%, 107.0%, 100.30%, 100.20% and 100.20% respectively and corresponding large A-efficiency values of 62.50%, 65.24%, 67.55%, 74.06%, 75.14%, 50.39%, 50.39% and 55.84% respectively. Their corresponding G-efficiency values of 35.00%, 35.13%, 35.19%, 24.42%, 24.45%, 24.47%, 44.71%, 44.97% and 45.15%, respectively are all less than 50% of the best G-efficiency value of 94.12% and as such cannot be considered a good optimal combination. The design associated with [1:2:3] tends to be comparatively better than other combinations.

Similarly, from Table 7 for OCCD full factorial replicates, the optimal combination of [1:1:3] yielded the best A_{opt} value of 31.35 with A-efficiency value of 47.85%. The associated design is of size N = 27 and contains one complete 2^4 factorial points, one complete 2k axial points and three center points. Optimal combination of [3:3:1] yielded the best G_{opt} value of 15.56 with the best G-efficiency value of 96.41%. The associated design is of size N = 73 and contains three complete 2^4 factorial points, three complete 2k axial points and one center point. Optimal combination of [2:1:3] yielded the best D_{opt}^{max} value of 7.803 X 10^{-4} with the best D-efficiency value of 62.06%. The associated design is of size N = 43 and contains two complete 2^4 factorial points, one complete 2k axial points and three center points. Optimal combination of [1:1:3] yielded the best A_{opt} value of 31.35 with A-efficiency value of 47.85%. The associated design size is N = 27 with a corresponding D_{opt}^{max} value of 4.924 X 10^{-4} and D-efficiency value of 60.18%, exhibits a run-size efficiency advantage over the maximum D-efficiency value of 62.06% with associated design size of N = 43. The corresponding G_{opt} value of 16.82 and G-efficiency value of 90.07% also exhibits a run-size efficiency advantage over the maximum G-efficiency value of 96.41% with a design size of N = 73. Therefore, the optimal combination of [1:1:3] tends to yield better efficient design compared to other combinations

From Table 8, for OCCD full factorial replication, the optimal combination of [1:2:3] yielded the best A_{opt} value of 42.31 with A-efficiency value of 49.64%. The associated design is of size N = 55 and contains one complete 2^5 factorial points, two complete 2k axial points and three center points. Optimal combination of [2:3:1] yielded the best G_{opt} value of 22.40 with the best G-efficiency value of 93.73%. The associated design is of size N = 95 and contains two complete 2^5 factorial points, three complete 2k axial points and one center point. Optimal combination of [1:1:3] yielded the best D_{opt}^{max}value of 1.317 X 10^{-4} with best D-efficiency value of 65.36%. The associated design is of size N = 45 and contains one complete 2^5 factorial points, one complete 2k axial points and three center points. Comparatively, the optimal combination of [1:1:3] yielded the best optimal combination.

From Table 9, for OCCD full factorial replicates, the optimal combination of [1:3:3] yielded the best A_{opt} value of 53.09 with A-efficiency value of 52.74%. The associated design is of size N = 103 and contains one complete 2^6 factorial points, three complete 2k axial points and three center points. Optimal combination of [1:2:1] yielded the best G_{opt} value of 29.31 with the best G-efficiency value of 95.53%. The associated design is of size N = 89 and contains one complete 2^6 factorial points, two complete 2k axial points and one center point. Optimal combination of [1:1:3] yielded the best D_{opt}^{max} value of 2.408 X 10^{-5} with D-efficiency value of 68.40%. The associated design is of size N = 79 and contains one complete 2^6 factorial points,

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one complete 2k axial point and three center points. Comparatively, the optimal combination of [1:1:3] also yielded the best efficient design.

Similarly, from Table 10, for FCCD full factorial replicates, the optimal combination of [1:3:1] yielded the best A_{opt} value of 48.20 with A-efficiency value of 31.12%. The associated design is of size N = 41 and contains one complete 2^4 factorial points, three complete 2k axial points and one center point. Optimal combination of [3:3:1] yielded the best G_{opt} value of 16.05 with the best G-efficiency value of 93.46%. The associated design is of size N = 73 and contains three complete 2^4 factorial points, three complete 2k axial points and one center point. Optimal combination of [3:2:1] yielded the best D_{opt}^{max} value of 7.049 X 10^{-6} with the best D-efficiency value of 45.35%. The associated design is of size N = 65 and contains three complete 2^4 factorial points, two complete 2k axial points and one center point. Observe that optimal combinations of [1:1:1], [2:2:2] and [3:3:3] yielded the same A_{opt}, D_{opt}, and G_{opt} values. Their corresponding A, D and G-efficiencies, also yielded the same values, but different design sizes of N = 25, 50 and 75, respectively. Though the optimality and efficiency values are the same, the [1:1:1] combination is preferred over the [2:2:2] and [3:3:3] because it has the smallest design size efficiency value, and therefore can be considered the best optimal and efficient combination.

From Table 11, for FCCD full factorial replication, the optimal combination of [1:3:1] yielded the best A_{opt} value of 74.43 with A-efficiency value of 28.21%. The associated design is of size N = 63 and contains one complete 2^5 factorial points, three complete 2k axial points and one center point. Optimal combination of [3:3:1] yielded the best D_{opt}^{max} value of 6.100 X 10^{-8} with best D-efficiency value of 45.34%. Observe that optimal combinations of [1:1:1], [2:2:2] and [3:3:3] yielded the same best G_{opt} values of 22.19 and G-efficiency values of 94.66%, respectively. The design associated with [1:1:1] combination is preferred over the [2:2:2] and [3:3:3] because it has the smallest design size efficiency value of N = 43 against the combination of [2:2:2] and [3:3:3] with design sizes of N = 86 and 128, respectively. Comparatively, the design associated with [1:2:1], yielded the best optimal combination.

Finally, from Table 12, for FCCD full factorial replicates, the optimal combination of [1:3:1] yielded the best A_{opt} value of 121.50 with A-efficiency value of 23.05%. The associated design is of size N = 101 and contains one complete 2^6 factorial points, three complete 2k axial points and one center point. Optimal combination of [2:3:1] yielded the best G_{opt} value of 28.56 with the best G-efficiency value of 98.06% as well as best D_{opt}^{max} value of 2.968 X 10^{-10}, with best D-efficiency value of 45.68% The associated design is of size N = 165 and contains two complete 2^6 factorial points, three complete 2k axial points and one center point. Comparatively, the design associated with [1:2:1] tends to yield the best optimal combination.

It is observed that, A-Optimal design satisfies the criterion A= mintrace$[(X^TX)^{-1}]$ and D-Optimal design satisfies the criterion D = min$|(X^TX)^{-1}|$ or equivalently D = max$|(X^TX)|$. A design with minimum prediction variance $min[N\hat{\sigma}_{max}^2]$ is G-Optimal. Conversely, an efficient design is a design with high efficiency values. . When one has various designs at one's disposal, the most efficient design to choose is the one with larger efficiency values with smaller design size. The overall results for the twenty seven number combinations of different replications of the various portions of all the CCDs under study show that the efficiency of design is dependent on the optimality values; optimality values tend to influence the efficiency of design either negatively or positively. Larger optimality values depreciate the efficiency of the design while

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smaller optimality values improve efficiency of the design. The results, however, show that all the points where A, D and G are optimal with maximum efficiency values were all partially replicated except for FCCD with k = 5 where the various portions were held fixed. Replicating a complete SCCD, RCCD, OCCD and FCCD with one $r_{f,}$ r_{α} and n_c r-times, yielded the same A, D and G-efficiency values. Generally, the best G-efficiencies for the various CCDs recorded overall superior performance of 85% and above efficiency values. The best D-efficiency recorded above 70% for SCCD and RCCD and below 70% for OCCD and FCCD. The RCCD recorded the best A-efficiency value of 75.14%. Super D-efficiency values of 104.03%, 103.68% and 103.27% were also recorded by RCCD for factor k = 5 and 103.82%, 103.52%, 103.18%, 107.64%, 107.30%, 107.0%, 100.30%, 100.20% and 100.20% for factor k = 6. To obtain optimal values and to have a good and efficient design, there is a need to replicate the cube, the axial and the center points. Partial replication of the various portions of the CCD tends to yield better results than equal replications. The best designs in terms of optimal and efficiency values seem to put more emphasis on replication of center point for SCCD, RCCD and OCCD but places more emphasis on replication of the axial point for FCCD.

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