African Journal of Mathematics and Statistics Studies

ISSN: 2689-5323

Volume 5, Issue 1, 2022 (pp. 67-81)

www.abjournals.org

HYBRID FALKNER - TYPE BLOCK METHODS FOR SOLUTION OF SECOND ORDER BOUNDARY VALUE PROBLEMS

Yakusak N.S.[1] and Adeniyi R.B.[2]

[1]Department of Mathematics Federal College of Education Okene Kogi State

[2]Department of Mathematics University of Ilorin, Ilorin

Cite this article:

Yakusak N.S., Adeniyi R.B. (2022), Hybrid Falkner - Type Block Methods for Solution of Second Order Boundary Value Problems. African Journal of Mathematics and Statistics Studies 5(1), 67-81. DOI: 10.52589/AJMSS-I01PYJA7

Manuscript History

Received: 7 March 2022

Accepted: 31 March 2022

Published: 10 April 2022

ABSTRACT: *Hybrid Falkner-type block methods for the general solution of second-order boundary value problem of second-order ordinary differential equations is developed base on collocation and interpolation approach and implemented as block. The resulting scheme is zero-stable, consistent and convergent with a good region of absolute stability. The tabular and graphical presentations of the numerical results to the problems considered demonstrate the effectiveness and good accuracy of the scheme in comparison with other methods.*

KEYWORDS: Second Order Ordinary Differential Equation, Boundary Value problem, Block Methods, Hybrids Methods.

INTRODUCTION

The goal of this paper, is to consider two-point Boundary Value Problems (BVPs) constituted by the Ordinary Differential Equation (ODEs)

$$y''(x) = f\big(x, y(x), y'(x)\big),$$
$$x \in [a, b] \qquad (1)$$

Where the prime denotes differentiation with respect to$x$, with the condition

$$y(a) = \alpha, y(b) = \beta \qquad (2)$$

It is assumed that; the function f in (1) satisfies the condition to assure the existence and uniqueness theorems. The differential equation in (1) arises in modelling real-life situations such as: Chemical reaction, deflection and deformation of beam, heat transmission, whether prediction, stuck trend among others. The development of numerical methods to obtain approximate solutions becomes necessary, because most of the equations obtained, often do not have analytic solutions.

A number of researchers which includes [1, 2, 3, 9, 13, 15, 19, 20] have worked on the solution of second-order ordinary differential equations via collocation and interpolation techniques, among others. [14] developed a procedure to obtain the k-step Falkner-type method in both explicit and implicit forms in their variable step-size version, providing recurrent formula to compute the coefficients efficiently. [5] derived a new method called the Extended Cubic B-spline method (ECBIM) for solving the two-point Boundary Value Problem (BVP). [6] also derived a numerical base interpolation method by Quartic spline function to solve second order BVPs based on Neumann condition. [10], proposed a three-point block one-step method for solving second order BVPs directly without reducing to the system of first-order. [11] proposed a Bernoulli polynomial over the interval [0,1] for the solution of second-order, linear and nonlinear BVPs with Dirichlet, Neumann and Robin's conditions. [4] Proposed the adapted Falkner-type method for a system of oscillatory second-order ODEs. They showed that the error bound for the global error of the solution is independent. [21] derived a direct three-step block one-step method for solving second order linear Dirichlet and Neumann boundary conditions. [16] propose the Falkner-type block method for $k = 2$.

$$\left.\begin{aligned}
y_{n+2} &= y_{n+1} + hy'_{n+1} + \frac{1}{24}h^2(3f_{n+2} + 10f_{n+1} - f_n) \\
y'_{n+2} &= y'_{n+1} + \frac{1}{12}h(35f_{n+2} + 8f_{n+1} - f_n) \\
y_n &= y_{n+1} + \frac{1}{24}h^2(3f_n + 10f_{n+1} - f_{n+2}) \\
y'_n &= \frac{1}{12}h(5f_n + 8f_{n+1} - f_{n+2})
\end{aligned}\right\}$$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

For the solution of the second-order ODE, [17] propose a third derivative two-step block Falkner-type method for solving general second-order boundary value systems.

$$\left.\begin{aligned} y_n &= y_{n+1} - hy'_{n+1} + h^2\left(\frac{187}{1680}f_n + \frac{11}{30}f_{n+1} + \frac{37}{1680}f_{n+2}\right) + h^3\left(\frac{2}{105}g_n - \frac{19}{210}g_{n+1} - \frac{1}{168}g_{n+2}\right) \\ y'_n &= y'_{n+1} - h\left(\frac{100}{240}f_n + \frac{8}{15}f_{n+1} + \frac{11}{240}f_{n+2}\right) - h^2\left(\frac{13}{240}g_n - \frac{1}{6}g_{n+1} - \frac{1}{80}g_{n+2}\right) \\ y_0 &= y_1 - hy'_1 + h^2\left(\frac{187}{1680}f_0 + \frac{11}{30}f_1 + \frac{37}{1680}f_2\right) + h^3\left(\frac{2}{105}g_0 - \frac{19}{210}g_1 - \frac{1}{168}g_2\right) \\ y_0 &= y'_1 - h\left(\frac{101}{240}f_n + \frac{8}{15}f_{n+1} - \frac{11}{240}f_{n+2}\right) - h^2\left(\frac{13}{40}g_0 - \frac{1}{6}g_1 - \frac{1}{80}g_2\right) \end{aligned}\right\} \quad (4)$$

In this manuscript, we derive a Falkner-type method by introducing and off-step which was not included in [16] and [17] for the solution of second-order BVPs.

The manuscript is outlined as follows, section (1) introduce the paper. In section (2), we present the procedure for the developed method. Section (3) presents the analysis of the methods. Section (4) presents the numerical implementation of the methods. Section (5) presents some numerical examples followed by a conclusion.

Derivation of the methods

In this section, hybrid Falkner-type methods for solving problem (1) is derived. We seek an approximation of the from

$$Y(x) = \sum_{j=0}^{2k-n} a_j \emptyset_j(x) + \sum_{j=n+1}^{k+3} a_j P_j \quad (5)$$

Where $\emptyset_j(x)$ is orthogonal polynomials with respect to the weight function $w(x) = x^2 + 5$ over an interval $[-1, 1]$ and $P_j(x) = x^j$ is power series function, k is the step number, $n = k$ are unknown coefficients to be determined. Taking $k = 4$ as a step function, $v = k - r$ as the off-step points and $j = 0,1,2,3, v, 4$ the continuous approximation is obtain as

$$\left.\begin{aligned} y_{n+k-r} &= Y(x_{n+k-r}) \\ y'_{n+k-r} &= Y(x_{n+k-r}) \\ y''_{n+j} &= Y''^{(x_{n+j})} = f(x_{n+j}) \end{aligned}\right\} \quad (6)$$

Which lead to the system of $k + 3$ equation and $k + 3$ unknown written in the form $Ax = B$, where $r = \frac{1}{2}$ is the selected off-step point. Solving equation (6) by Gussian elimination

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

method, the coefficients a_j can be obtained. Substituting the coefficient a_j into (5) yields the continued scheme

$$Y(x) = \alpha_j(x)y'_{n+j} + \alpha'(x)hy'_{n+1} + h^2\sum_{j=0}^{k}\beta_j(x)f_{n+k} + h^2\beta_v(x)f_{n+v} \quad (7)$$

Where

$$\left.\begin{array}{c}
\alpha_{\frac{7}{2}}(z) = 1 \\
\alpha'_{\frac{7}{2}}(z) = \frac{7}{2} + z \\
\beta_0(z) = -\frac{7147}{2340}z - \frac{1}{3528}z^7 + \frac{3}{560}z^6 - \frac{1}{24}z^5 + \frac{115}{772}z^4 - \frac{199}{504}z^3 + \frac{1}{2}z^2 + \frac{637}{9216} \\
\beta_1(z) = -\frac{323}{240}z + \frac{1}{630}z^7 - \frac{1}{36}z^6 + \frac{23}{120}z^5 - \frac{24}{36}z^4 - \frac{14}{15}z^3 + \frac{2401}{2304} \\
\beta_2(z) = -\frac{5831}{11520}z + \frac{35619}{23040} - \frac{1}{252}z^7 + \frac{23}{360}z^6 - \frac{47}{120}z^5 + \frac{157}{144}z^4 - \frac{7}{6}z^3 \\
\beta_3(z) = -\frac{4459}{2880}z + \frac{1}{126}z^7 - \frac{7}{60}z^6 + \frac{77}{120}z^5 - \frac{19}{12}z^4 + \frac{14}{9}z^3 + \frac{31213}{11520} \\
\beta_{\frac{7}{2}}(z) = \frac{49}{180} + \frac{77}{180}z - \frac{128}{105}z^3 - \frac{8}{15}z^5 + \frac{80}{63}z^4 - \frac{16}{2205}z^7 + \frac{32}{315}z^6 \\
\beta_4(z) = -\frac{343}{2560}z + \frac{1}{504}z^7 - \frac{19}{720}z^6 + \frac{2}{15}z^5 - \frac{89}{288}z^4 + \frac{7}{24}z^3 + \frac{2401}{46080} \\
\vdots
\end{array}\right\}$$

The derivative of (7) is given as

$$Y'(x) = \alpha'_j(x)y'_{n+j} + h\sum_{j=0}^{k}\beta'_j(x)f_{n+k} + h\beta'_v(x)f_{n+v} \quad (8)$$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

where

$$\begin{aligned}\alpha'_{\frac{7}{2}}(z) = 1\ \beta'_0(z) &= z - \frac{1}{504}z^6 + \frac{9}{280}z^5 - \frac{5}{24}z^4 + \frac{155}{168}z^3 - \frac{199}{168}z^2 - \frac{7147}{23040}\ \ \beta'_1(z)\\ &= \frac{1}{90}z^6 - \frac{1}{6}z^5 + \frac{23}{24}z^4 - \frac{23}{9}z^3 + \frac{14}{5}z^2 - \frac{343}{240}\ \beta'_2(z)\\ &= -\frac{5831}{11520} - \frac{1}{36}z^6 + \frac{23}{60}z^5 - \frac{47}{24}z^4 + \frac{157}{36}z^3 - \frac{7}{2}z^2\ \beta'_3(z)\\ &= \frac{1}{18}z^6 - \frac{7}{10}z^5 + \frac{77}{24}z^4 - \frac{19}{3}z^3 - \frac{4459}{2880} + \frac{14}{3}z^2\ \beta'_{\frac{7}{2}}(z)\\ &= \frac{77}{180} - \frac{128}{35}z^2 - \frac{8}{3}z^4 + \frac{320}{63}z^3 - \frac{16}{315}z^6 + \frac{64}{105}z^5\ \beta'_4\\ &= \frac{1}{72}z^6 - \frac{19}{120}z^4 + \frac{2}{3}z^4 - \frac{89}{72}z^3 + \frac{7}{8}z^2 - \frac{343}{2560}\}\end{aligned}$$

Evaluating equation (7) and (8) at $z = 4$ we obtained the main equations.

$$\left.\begin{aligned}y_{n+4} &= y_{n+\frac{7}{2}} + \frac{1}{2}hy'_{n+\frac{1}{2}} + h^2\left(\frac{457}{2257}f_n - \frac{25}{16128}f_{n+1} + \frac{989}{161280}f_{n+2} - \frac{2389}{80640}f_{n+3} + \frac{1109}{8820}f_{n+\frac{7}{2}} + \frac{155}{64412}f_{n+4} +\right)\\ y'_{n+4} &= y_{\prime n+\frac{7}{2}} + h\left(\frac{7}{7680}f_n - \frac{1}{144}f_{n+1} + \frac{313}{11520}f_{n+2} - \frac{121}{960}f_{n+3} + \frac{77}{180}f_{n+\frac{7}{2}} + \frac{4081}{23040}f_{n+4}\right)\end{aligned}\right\} \quad (9)$$

Similarly, evaluation (7) and (8) at $z = 0,1,2,3,\frac{7}{2}$ yield the additional equations.

$$\left.\begin{aligned}y_n &= y_{n+\frac{7}{2}} + \frac{7}{2}hy'_{n+\frac{7}{2}} + h^2\left(\frac{637}{9216}f_n + \frac{2401}{2304}f_{n+1} + \frac{45619}{23040}f_{n+2} + \frac{31213}{11520}f_{n+3} + \frac{49}{180}f_{n+\frac{7}{2}} + \frac{2401}{46080}f_{n+4}\right)\\ y'_n &= y'_{n+\frac{7}{2}} + h\left(\frac{-7147}{23040}f_n - \frac{343}{240}f_{n+1} - \frac{5831}{11520}f_{n+2} - \frac{4459}{2880}f_{n+3} + \frac{77}{180}f_{n+\frac{7}{2}} - \frac{343}{3560}f_{n+4}\right)\\ y_{n+1} &= y_{n+\frac{7}{2}} - \frac{5}{2}hy'_{n+\frac{7}{2}} + h^2\left(\frac{-625}{451584}f_n + \frac{1175}{16128}f_{n+1} + \frac{34375}{32256}f_{n+2} + \frac{26875}{16128}f_{n+3} + \frac{275}{882}f_{n+\frac{7}{2}} + \frac{625}{64512}f_{n+4}\right)\\ y'_{n+1} &= y'_{n+\frac{7}{2}} + h\left(\frac{125}{10752}f_n - \frac{55}{144}f_{n+1} - \frac{2875}{2304}f_{n+2} - \frac{125}{192}f_{n+3} - \frac{65}{252}f_{n+\frac{7}{2}} + \frac{125}{4608}f_{n+4}\right)\\ y_{n+2} &= y_{n+\frac{7}{2}} - \frac{3}{2}hy'_{n+\frac{7}{2}} + h^2\left(\frac{333}{250880}f_n - \frac{117}{8960}f_{n+1} + \frac{453}{3584}f_{n+2} + \frac{722}{8960}f_{n+3} + \frac{39}{196}f_{n+\frac{7}{2}} + \frac{171}{35840}f_{n+4}\right)\\ y'_{n+2} &= y'_{n+\frac{7}{2}} + h\left(\frac{-69}{17920}f_n + \frac{3}{80}f_{n+1} - \frac{591}{1280}f_{n+2} - \frac{339}{320}f_{n+3} + \frac{3}{140}f_{n+\frac{7}{2}} - \frac{87}{2560}f_{n+4}\right)\\ y_{n+3} &= y_{n+\frac{7}{2}} - \frac{1}{2}hy'_{n+\frac{7}{2}} + h^2\left(\frac{-289}{2257920}f_n + \frac{83}{80640}f_{n+1} - \frac{737}{161280}f_{n+2} + \frac{755}{16128}f_{n+3} + \frac{38}{441}f_{n+\frac{7}{2}} - \frac{1391}{322560}f_{n+4}\right)\\ y'_{n+3} &= y'_{n+\frac{7}{2}} + h\left(\frac{83}{161280}f_n - \frac{1}{240}f_{n+1} + \frac{217}{11520}f_{n+2} - \frac{787}{2880}f_{n+3} - \frac{65}{252}f_{n+\frac{7}{2}} + \frac{41}{2560}f_{n+4}\right)\end{aligned}\right\} \quad (10)$$

Equation (9) and (10) form the required scheme called Hybrid Falkner-type Block method (HFBM)

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Analysis Of The Methods

Local truncation error and order

The linear difference operator associated with the general k −step if of the form

$$\mathcal{L}(x) = \left[\sum_{j=0}^{k} \alpha_j y(x_n + jh) - \beta_j h y'(x_n + jh) - h^2 \beta_j y''(x_n + jh)\right] \quad (10)$$

Where $y(x)$ is an arbitrary test function

Where $y(x)$ is an arbitrary test function continuously differential on $[a,\ b]$. Expanding$y$

$(x_n + jh), y'(x_n + jh)\ and\ y''(x_n + jh)$ in Taylor series, if we assume that $y(x)$ has many higher derivatives and collecting the terms we have

$$\mathcal{L}(y(x)) = \sum_{j=0}^{r+1} C_p h^p y^p + 0(h^{r+2})$$

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_q(-1)^q = \left[\frac{1}{q}\sum_{j=1}^{k} j\alpha_j + \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j\right]$$

$$q = 1,2,3 \ldots \ldots$$

Definition 1. We say that the methods (8) and (9) is of order $p \geq 1$, if $C_0 = C = \cdots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$. In this case, expanding the proposed schemes (8) and (9) in Taylor series yields the order of the method is obtained as $pp = (6,6,6,6,6)^T$ (See [8])

$$C_{p+2} = \left(\frac{-389}{442380}, \frac{305}{6193152}, \frac{-393}{1146880}\frac{-1375}{6193152}\frac{-2401}{884736}\right)$$

Definition 2. If the hybrid method (8) and (9) has order $p \geq 1$, its said to be consistence. In this case the proposed scheme is consistence since its of order $P = 6$.

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Zero-stability and Convergence

This is the concept concerning the behaviour of a numerical method with stability of the first characteristic polynomial as $h \to 0$, can be written in matrix form as

$$A^0\tilde{Y}_\mu - A'\tilde{Y}_{\mu-1} = 0$$

Where

$$\bar{Y}_\mu = (y_{n+1}, y_{n+2}, \dots, y_{n+k})^T$$

$$\bar{Y}_{\mu-1} = (y_n, y_{n+1}, y_{n+k-r})^T$$

A^0 is the identity matrix. Following the procedure in [15] and [17] the proposed methods can be shown that,

$$A^0 = \begin{bmatrix} 1000000000 \\ 0100000000 \\ 0010000000 \\ 0001000000 \\ 0000100000 \\ 0000010000 \\ 0000001000 \\ 0000000100 \\ 0000000010 \\ 0000000001 \end{bmatrix} \qquad A' = \begin{bmatrix} 0000100000 \\ 0000100000 \\ 0000100000 \\ 0000100000 \\ 0000100000 \\ 0000000000 \\ 0000000000 \\ 0000000000 \\ 0000000000 \\ 0000000000 \end{bmatrix}$$

Definition 3. The block method (8) and (9) is zero stable provided the roots $R_j, j = 1, \dots, 2k$, of its first characteristic polynomial satisfy $|R_j| = 1$, the multiplicity does not exceed 2.

Theorem 3.1 A Linear Multistep Methods (LMMs) is said to be convergent if it is consistent and zero-stable [18].

Remarks: The roots of the proposed schemed is obtained as $\rho(r) = r^9(r-1), r = 0000000001$. This implies that the proposed method is zero stable. The proposed method is convergent.

Region of Absolute Stability

As we mentioned before, zero-stability is a concept concerning the behavior of a numerical method for $h \to 0$. In order to know if a numerical method will give reasonable result for a given $h > 0$, we need a concept of stability different from zero-stability. Considering the stability function inform

$$MM(z)z = h(A - Cz - Dz^2)$$

where $zz = \lambda\lambda h$ and A, B, C, D are obtained from interpolating and collocating points of the method. Computing the stability function and its first derivative gives the polynomial which can be plotted via Matlab environment.

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

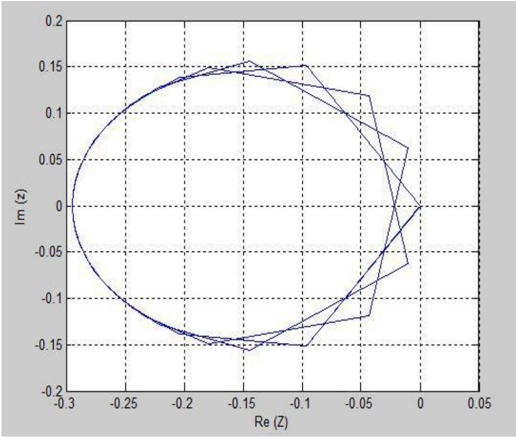

Figure 1. Stability region of the proposed scheme

The required absolute stability region of the proposed methods are plotted in Figure 1 with an interval of absolute stability $(-0.29, 0)$.

Numerical Experiments.

In this section, the proposed methods will be applied to second-order linear boundary value problems with Dirichlet via the shooting method. We add a sufficient number of conditions at one endpoint and adjust these conditions until the required conditions are satisfied at the other end. The related IVP is solved using our proposed Falkner-type method. With the notations below, we present numerical and graphical results obtained for some test problems.

Table:1. Numerical notations used

x	Point of evaluation
k	Step number
CPU	Central processing unit time in seconds
Maxero	Maximum Error
T.S	Total step taken
h	Step length
HFBM	Hybrid Fulker-type Blook method

Problem 1 (Source, [23])

$$y''(x) - y(x)\, cos\, cos\, (x) = 0, \qquad y(x) = 0, y(1) = 1, h = 0.125$$

Exact Solution

$$y(x) = \frac{-\cosh 1 + \sinh 1 + \cos 1 + 2}{4\sinh 1} e^{x} + \frac{\cosh 1 + \sinh 1 - \cos 1 - 2}{4\sinh 1} e^{-x} - \frac{1}{2}\cos(x)$$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Table:1a. Numerical and Error Results for Problem 1

x	Exact	HFBM	Error	[23]
0.125	0.0609853491005539	0.0609853489587224103	$1.400000E-10$	$8.267309E-10$
0.250	0.138427934741475654	0.138427934476724589	$2.000000E-10$	$7.993385E-10$
0.375	0.233175541509714373	0.233175541113614575	$4.000000E-10$	$7.801650E-10$
0.500	0.346110454006479368	0.346110453478098929	$5.000000E-10$	$1.666651E-09$
0.625	0.478172624479587739	0.478172624022991395	$5.000000E-10$	$1.794876E-09$
0.750	0.630387283060996859	0.630387282766213004	$3.000000E-10$	$7.693057E-10$
0.875	0.803897221213436799	0.803897221056497294	$1.000000E-10$	$2.635849E-10$
1.000	1.000000000000000000	1.000000000000000000	$0.000000E+00$	$0.000000E+00$
CPU		0.12600 s	–	–

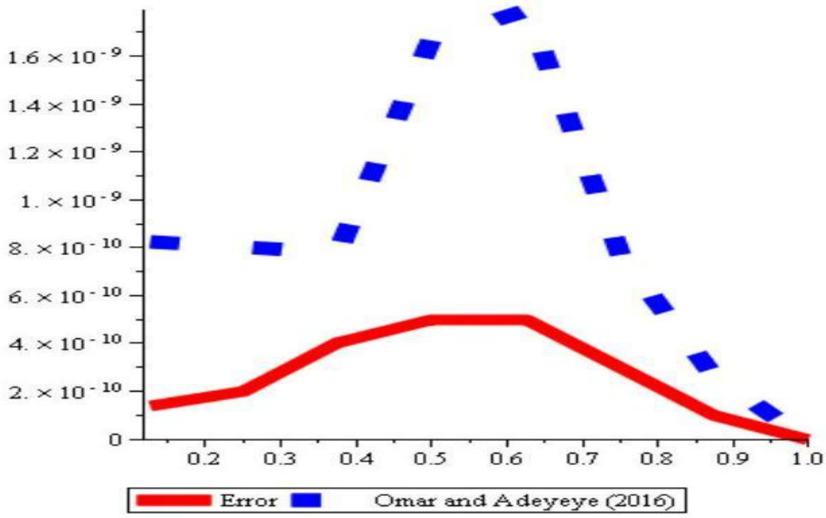

Figure 2: Graphical Presentation of Error Results for Problem 1

Problem 2. (Source, [22])

$$y''(x) + 3y'(x) + 2y(x) - 4x^2 = 0, \quad y(1) = 1, y(2) = 6, h = 0.1$$

Exact Solution

$$y(\mathrm{x}) = \mathrm{C}_1\mathrm{e}^{-\mathrm{x}} - \mathrm{C}_2\mathrm{e}^{-2\mathrm{x}} + 7 - 6\mathrm{x} + 2\mathrm{x}^2$$

$$\mathrm{C}_1 = \frac{\mathrm{e}^{-4}(3\mathrm{e}^2 + 2)}{-\mathrm{e}^{-5} + \mathrm{e}^{-4}}, \mathrm{C}_2 = \frac{\mathrm{e}^{-2}(3\mathrm{e}^1 + 2)}{-\mathrm{e}^{-5} + \mathrm{e}^{-4}}$$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Table:2. Numerical and Error Results for Problem 2

x	Exact	HFBM	Error	[22]
1.1	2.39360540	2.39360558	$1.77E-07$	$7.61E-05$
1.2	3.42671629	3.42671654	$2.50E-07$	$1.21E-04$
1.3	4.18289731	4.18289763	$3.20E-07$	$1.31E-04$
1.4	4.72952175	4.72952209	$3.40E-07$	$1.29E-04$
1.5	5.12080584	5.12080611	$2.70E-07$	$1.12E-04$
1.6	5.40028355	5.40028376	$2.10E-07$	$9.23E-05$
1.7	5.60282433	5.60282450	$1.70E-07$	$6.70E-05$
1.8	5.75627727	5.75627740	$1.30E-07$	$4.41E-05$
1.9	5.88281025	5.88281032	$7.00E-08$	$2.01E-05$
2.0	6.00000000	6.00000000	$0.00+00$	$0.00+00$
CPU		$0.327\ s$	Maxero at $x=1.4$	Maxero at $x=1.3$

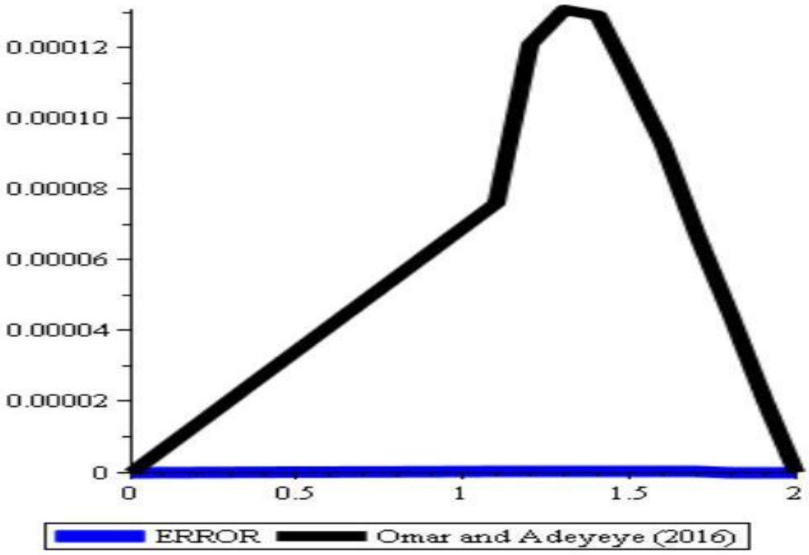

Figure 3: Graphical Presentation of Error Results for Problem 2

Problem 3. (Source, [11])

$$y'' = \frac{2}{x^2} - \frac{1}{x},\ 2 < x < 3,\ y(2) = 0, y(3) = 0, h = 0.1$$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Exact Solution

$$y(x) = \frac{1}{38}\left(-5x^2 + 19x - \frac{36}{x}\right)$$

Table:3. Numerical and Error Results for Problem 3

x	EXACT	HFBM	Error	[11]
2.1	0.0186090225	0.01860902372	$1.220000E-9$	$8.710000E-09$
2.2	0.0325358852	0.03253588663	$1.430000E-9$	$4.662072E-09$
2.3	0.0420480549	0.04204805687	$1.970000E-9$	$1.138989E-09$
2.4	0.0473684210	0.04736842332	$2.320000E-9$	$5.421880E-09$
2.5	0.0486842105	0.04868421266	$2.160000E-9$	$9.479982E-10$
2.6	0.0461538462	0.04615384797	$1.770000E-9$	$4.318545E-09$
2.7	0.0399122807	0.03991228224	$1.540000E-9$	$2.126246E-09$
2.8	0.0300751884	0.0300751892	$8.000000E-10$	$2.000938E-09$
2.9	0.0167422871	0.0167422876	$5.000000E-10$	$2.571030E-09$
3.0	$3.000000E-10$	$3.000000E-10$	$0.000000+00$	$0.000000+00$
CPU		$0.985\ s$	Maxero at $x = 2.4$	Maxero at $x = 2.4$

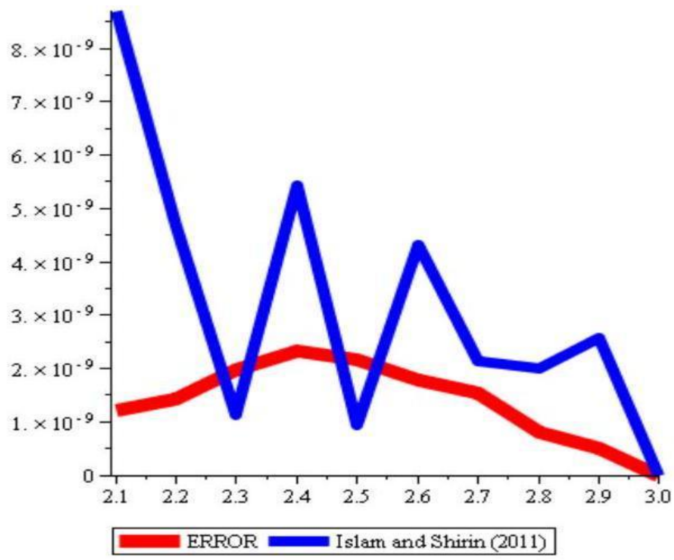

Figure 4: Graphical Presentation of Error Results for Problem 3

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

Problem 4. (Source, [23])

$$y''(x) + \frac{2}{x}y' - \frac{2}{x^2} - \frac{sin(ln\, x)}{x^2}, \quad 1 < xx < 2$$
$$yy(1) = 1, yy(2) = 2$$

Exact Solution

$$y(x) = C_1 x + \frac{C_2}{x^2} - \frac{3}{10} sin\, sin\, (ln) - \frac{1}{10} sin\, sin\, (ln)$$

$$y\ \ C_1 = \frac{11}{10} - C_2, C_2 = \frac{1}{70}[8 - 12sin\ (ln\ 2) - 4sin\ (ln\ 2)]$$

Table:4. Numerical and Error Results for Problem 4

x	Exact	HFBM	Error	[23]
1.1	1.0926292985	1.092629365	$6.70E-08$	$1.60E-06$
1.2	1.1870848405	1.187084903	$6.30E-08$	$2.95E-06$
1.3	1.2833823641	1.283382435	$7.10E-08$	$2.71E-06$
1.4	1.3814459517	1.381446020	$6.80E-08$	$2.54E-06$
1.5	1.4811594170	1.481159473	$5.60E-08$	$2.03E-06$
1.6	1.5823924608	1.582392503	$4.20E-08$	$1.62E-06$
1.7	1.6850139617	1.685013991	$2.90E-08$	$1.15E-06$
1.8	1.7888985346	1.788898554	$1.90E-08$	$7.40E-07$
1.9	1.8939295092	1.893929519	$1.00E-08$	$3.48E-07$
2.0	2.000000000	2.000000000	$0.00+00$	$0.00+00$
CPU		$0.298\ s$	Maxero at $x = 1.3$	Maxero at $x = 1.2$

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

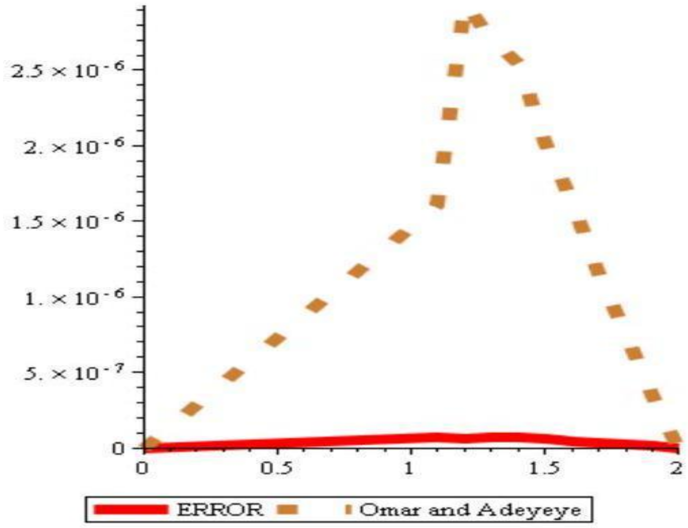

Figure 5: Graphical Presentation of Error Results for Problem 4

Remark: From all the tables above, we see the better effectiveness of the proposed method in terms of accuracy when compared with other existing methods. The graph shows the effectiveness of the proposed method.

DISCUSSION OF RESULTS

The analysis of the proposed scheme shows that the method HFBM is of order$p$ $p = (6)$, zero stable, consistent and convergent with a good region of absolute stability. From Table 3, it is observed that the proposed method compared with the numerical results obtained using Bernoulli polynomial by [11] are closer to the exact solution and Tables 1, 2 and 4 show that the proposed methods compared favourably. The proposed method has the least error compared to the methods observed from the figures.

CONCLUSION

In this study, the Hybrid Falkner-type Block Method (HFBM) for the solution of second-order ODEs without reducing to the system of first-order ODEs has been proposed. The derived method effectively solves second-order linear boundary value problems with Dirichlet. The tabular and graphical presentations of the numerical results and computing time of the problems considered demonstrate its effectiveness and good accuracy when compared with other methods.

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

REFERENCES

[1] Bongsoo, J. Two-point boundary value problems by the extended Adomian decomposition method. *Journal of Computational and Applied Mathematics,* 219(1):253-262.

[2] Caglar H., Caglar N. and Elfaituri K. (2006). B-spline interpolation compared with a finite difference, finite element and finite volume methods applied to two-point boundary value problems. *Applied mathematics and computation*, 175:72-79.

[3] Fang Q, Tsuchiya T. and Yamamoto T. (2002). Finite difference, finite element and finite volume methods applied to two-point boundary value problems. *Journal of Computational and Applied Mathematics*, 139:9-19.

[4] Jiyong *Li* and Xinyouan Wu (2013). Adapted Falkner-type methods for solving oscillatory second order differential equations. *Journal of Numerical Algorithm* 62(3): 355 - 381

[5] Hamid, N. N. A., Majid, A. A. and Ismail, A. I. M. (2011). Extended cubic B-spline method for linear two-point boundary value problems. *Sains Malaysiana*, 40:1285-1290

[6] Liu, L.B., Liu, H.-W. and Chen, Y. (2011). Polynomial spline approach for solving second-order boundary-value problems with Neumann conditions. *Applied Mathematics and Computation*, 217: 6872-6882.

[8] Lambert, J. D. (1973). *Computational methods in ordinary differential equations.* John Wiley and Son, New York.

[9] Mughti, M. H., Zanariah, A. M., Norazak, S., Sokkalingam, R., Hanita, D and Muhamad, A. D. (2006). Direct Fifth Order Block Method for Solving Second Order Neumann and Singular Perturbed Boundary Value Problem. Citation: AIP Conference Proceedings 1787-020006 doi: 10.1063/1.4968055

[10] Mukhtar, N. Z., Majid, Z. A. and Ismail, F (2011). Solutions of General Second-order ODEs using direct block method of a Runge-Kutta type. *Journal of Quality Measurement and Analysis (JQMA)* 7(2): 145-154.

[11] Shafiqul, M. I. and Shirin, A. (2011). Numerical Solutions of a Class of Second-Order Boundary Value Problems on Using Bernoulli Polynomials. *Applied Mathematics* 1059(2)-1067.

[12] Pablo, M., David, J. L. and Amelia, G (2000). Implementation of Falkner method for the problem of second-order ordinary differential equations. *Journal of Applied Mathematics and Computation*. 2-3(109):183-187

[13] Raft Abdelrahim and Omar Z. (2016). Directly solving second order linear boundary value problems of ordinary differential equations. *International Journal of Pure and Applied Mathematics*, 106(3):841-849.

[14] Ramos H. and Vigo-Aguiar J., (2006). Variable step-size implementation of multistep methods for second-order ordinary differential equations. *Journal of Computational and Applied Mathematics* 192:114-131

[15] Ramos H. and Lorenzo C. (2010). Review of explicit Falkner methods and their modification on special second-order initial value problems. *Comput. Phys. Commun*. 181: 1833-1841

[16] Ramos H., Mehta S. and Vigo-Aguiter J. (2016). A unified approach for k-step block Falkner methods for solving general second-order problems in ODEs. *Journal of Computation and Applied Mathematics.* htt:dx.doi.org/10.1016.

 Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7

[17] Ramose, H. and Rufai, A (2019). Third derivative two-step block Falkner-type method for solving general second-order boundary value system. *Mathematics and Computer in Simulation(2019)htt:/doi.org/10.1016/J.matcom 03.003*

[18] Henrici, P (1962). *Discrete variable methods in ordinary differential equation*'', John Wiley and son new York USA

[19] Yakusak, N. S. and Adeniyi, R. B. (2017). Three off-step hybrid Falkner-type Block Method for Solution of Second-order Initial Value Problems. *ABACUS, Mathematical Series* 44(1):78-86.

[20] Yakusak, N. S. and Owolanke, A. O. (2018),. A Class of Linear multistep Method for Direct Solution of Second-order Initial Value Problems in Ordinary Differential Equation. *Journal of Advances in Mathematics and Computer Sciences*, 26(1):1-11.

[21] Zanariah, A. M., Mohd, M. H. and Norazak, S. (2013) . Solving Second Order Linear Dirichlet and Neumann Boundary Value Problems by Block Method. *International Journal of Applied Mathematics,* 43:2.

[22] Omar, Z. and Oluwaseun, A. (2016) . Order Five Block Method for the Solution of Second-order Boundary Value Problem. *Indian Journal of Science and Technology* 9(31): 0974-6846.

[23] Omar, Z. and Oluwaseun, A. (2016). Solving Two-Point Second-order Boundary Value Problems Using Two-Step Block Method with Starting and Non-Starting Values. *International Journal of Applied Engineering Research* 11(4):2407-2410.

Article DOI: 10.52589/AJMSS-I01PYJA7
DOI URL: https://doi.org/10.52589/AJMSS-I01PYJA7