



OPTIMAL TRADING USING BLACK-SCHOLES EQUATION WITH TRANSACTION COSTS

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ABSTRACT: *This work deals with optimal trading using Black-Scholes equation with transaction costs. The partial differential equation for option pricing with transaction costs on the domain $(P, T) \in (0, \infty) \times (0, T)$ with terminal condition $C(P, T) = \text{Max}(P - E, 0)$, $P \in (0, \infty)$ for European call options with strike price, E , and a suitable terminal condition for European puts was obtained and then solved to obtain the optimal value function.*

KEYWORDS: Black-Scholes equation with transaction costs, Optimal trading, Option pricing, European call and put options, Portfolio of options.



INTRODUCTION

Money can be invested and thus produce more money. An investor is always faced with the problems of where and when to invest, ability to regularly and dynamically build his portfolio of investment, ability to find investment strategies that give profits with zero initial expenditures, option pricing and many more. Hence, there is need to understand the random fluctuations of assets prices. This work considers a market model in which trading the asset requires paying transaction fees which are proportional to the quantity and the value of the asset traded. This is a problem with a long history in mathematical finance. In a complete frictionless (i.e. without transaction costs) financial market, the Black-Scholes model (1973) [1] provides a hedging strategy for any European type contingent claim. Hence, in the Black-Scholes hedging portfolio, the total turnover of stock in the time interval $[0, T]$ is infinite since it requires trading at all times instants. Accordingly, when transactions cost that is directly proportional to trading is incorporated in the Black-Scholes model the resulting hedging portfolio is prohibitively expensive. It is, therefore, acceptable that there is no portfolio that can replicate the European call option with finite transaction costs in a continuous time model with transaction costs. The condition under which hedging can take place has to be relaxed such that the portfolio only dominates rather than replicates the value of the European call option at maturity. With this relaxation, there is always the trivial dominating hedging strategy of buying and holding one share of the stock on which the call is written. The price of an option should not be greater than the smallest initial capital that can support a dominating portfolio, that is, from arbitrage pricing theory. Interesting results have evolved from this line of approach to pricing option without transaction costs, however, in the presence of constraints (see Cvitanic and Karatzas [2]), and Broadie *et al* [3]), in the presence of transaction costs, Soner *et al* [4], proved that the minimal hedging portfolio that dominates a European call option is the trivial one. Leland [5] uses a relaxation with the effect that his model allows transactions only at discrete times. Leland's derivation makes an assumption of the convexity of the resulting option price. Avellanda and Paras [6] have made an extension to this approach to general prices. In another approach by Boyle and Vorst [7], pricing option is looked at in discrete time with a binomial tree model for the stock value. By applying the theorem of central limit, they showed that as the time step and the transaction cost tend to zero, the price of the discrete option converges to a Black-Scholes price with adjusted volatility. In a related paper [8], Bensaid *et al* investigated the discrete time, dominating policies. In [9], Hodges and Neuberger maximized the utility of the difference between final wealth when there is no option liability and when there is such a liability. They then postulate that the price of the option should be equal to the unique cash increment which offsets this differences. It is, therefore, surprising that in the absence of market frictions, the price obtained from maximizing utility is the same as the Black-Scholes option price. Thus, the Black-Scholes option pricing theory can be extended by use of utility maximization approach. Davis [10] has further work on this in the presence of transaction costs. This is to say that the price depends on the initial wealth of the investor, the chosen portfolio, the type of utility function, and the mean return rate of the stock. This work deals with optimal trading using the Black-Scholes equation with transaction costs. The partial differential equation for option pricing with transaction costs on the domain $(P, T) \in (0, \infty) \times (0, T)$ with terminal condition $C(P, T) = \text{Max}(P - E, 0), P \in (0, \infty)$ for European call options with strike price, E , and a suitable terminal condition for European puts was obtained and then solved to obtain the optimal value function.



BASIC TOOLS AND PRELIMINARIES

The random walk is governed in discrete time by

$$\delta P = \sigma P \varphi \sqrt{\delta t} + \mu P \delta t \quad (1)$$

where φ is drawn from a standardized normal distribution. The expected return on the hedged portfolio is equivalent to the return on a bank deposit. The portfolio is redefined or revised after a finite fixed time-step δt . Re-hedging the portfolio could be done weekly. Transaction costs in buying and selling the asset are proportional to the monetary value of the transaction. At a price of P and a constant k depending on an individual's aversion to risk, the transaction costs are $P|N|k$, where N is the number of shares bought ($N > 0$) or sold ($N < 0$). On an infinitesimal level, rebalancing the portfolio between stocks and bonds does not change the value of the hedging portfolio. Hausdorff, [11] introduced the first notion of dimension, providing a measure for filling space which allows for the possibility of non-integral dimensions. Then D -dimensional Hausdorff measure on a set A is given by

$$M_D(A) = \sum_k r_k^D ; D > 0; r_k < \varepsilon \quad (2)$$

Define the optimal covering of this set using spheres of variable radius, r_k . The Hausdorff dimension D_H is the value of D at which M_D jumps from ∞ to 0, while the dimension D can be calculated as $= \frac{\log \log N_1}{\log \log S_1}$, where N_1 is the number of small pieces that go into the larger one and S_1 is the scale to which the smaller pieces compare to the larger one.

THE MODEL

We first create a hedging portfolio that has ω stocks and τ bonds in X . Hence, the initial value of the hedging portfolio that is at time t_0 is

$$\omega_{t_0} P_{t_0} + \tau_{t_0} B_{t_0} = \omega(t_0) P(t_0) + \tau(t_0) B(t_0). \quad (3)$$

At the final time T , we intend to find

$$\omega(T) P(T) + \tau(T) B(T) = \rho(P(T)), \quad (4)$$

where ρ is called the contract function. Equation (4) is the value of the portfolio. If we denote the value of the option at time t by $W(t, P(t))$, we have

$$W(t, P(t)) = \omega(t) P(t) + \tau(t) B(t). \quad (5)$$

Since both sides should have the same dynamics, we have

$$d(W(t, P(t))) = d(\omega(t) P(t) + \tau(t) B(t)), \quad (6)$$

where d is an infinitesimal change. If the portfolio is self-financing, we have

$$d(\omega(t) P(t) + \tau(t) B(t)) = \omega(t) dP(t) + \tau(t) dB(t). \quad (7)$$



But

$$dB = rBdt \quad (7i)$$

Equation (7) becomes

$$\omega(\mu P\delta t + \sigma\varphi P\sqrt{\delta t}) + \tau rB\delta t = (\omega\mu P + \tau rB)\delta t + \omega\sigma\varphi P\sqrt{\delta t}, \quad (8)$$

since $\delta P = \mu P\delta t + \sigma\varphi P\sqrt{\delta t}$. Thus

$$\delta(W(t, P(t))) = (\omega\mu P + \tau rB)\delta t + \omega\sigma\varphi P\sqrt{\delta t}. \quad (9)$$

By Ito's lemma and replacing $W(t, P(t))$ by W , we have

$$\delta W = \frac{\partial W}{\partial t} \delta t + \frac{\partial W}{\partial P} \delta P + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} (\delta P)^2, \quad (10)$$

where $\delta P = \sigma P\varphi\sqrt{\delta t} + \mu P\delta t$. Hence, $(\delta P)^2 = (\sigma P\varphi\sqrt{\delta t} + \mu P\delta t)^2 = \sigma^2\varphi^2 P^2 \delta t$, since $(\delta t)^2 \approx 0$ and $\delta t\sqrt{\delta t} = 0$. Equation (10) becomes

$$\delta W = \frac{\partial W}{\partial t} \delta t + \frac{\partial W}{\partial P} (\mu P\delta t + \sigma P\varphi\sqrt{\delta t}) + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2\varphi^2 P^2 \delta t. \quad (11)$$

Rearranging, we have

$$\delta W = \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2} \right) \delta t + \sigma\varphi P \frac{\partial W}{\partial P} \sqrt{\delta t}. \quad (12)$$

By the assumption about the existence of transaction costs, we have

$$\delta W = \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2} \right) \delta t + \sigma\varphi P \frac{\partial W}{\partial P} \sqrt{\delta t} - k|N|P. \quad (13)$$

From equations (9) and (13), we have

$$\begin{aligned} \delta W &= \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2} \right) \delta t + \sigma\varphi P \frac{\partial W}{\partial P} \sqrt{\delta t} - k|N|P \\ &= (\omega\mu P + \tau rB)\delta t + \omega\sigma\varphi P\sqrt{\delta t}. \end{aligned} \quad (14)$$

Let $\omega = \frac{\partial W}{\partial P}$ where ω is the number of assets. Given that ω is evaluated at the asset value P and time t , we have $\omega = \frac{\partial W}{\partial P}(P, t)$. Re-hedging after a finite time δt gives $\frac{\partial W}{\partial P}(P + \delta P, t + \delta t)$. To find the number N of assets bought or sold at the new time and asset price, we have

$$N = \frac{\partial W}{\partial P}(P + \delta P, t + \delta t) - \frac{\partial W}{\partial P}(P, t). \quad (15)$$



For small δP and δt and by Taylor's series expansion of the first term, we have

$\delta P = \sigma P \varphi \sqrt{\delta t} + \mu P \delta t = \sigma P \varphi \sqrt{\delta t} + O(\delta t)$. Therefore, the first term expands as

$\frac{\partial W}{\partial P}(P + \delta P, t + \delta t) = \frac{\partial W}{\partial P}(P, t) + \frac{\partial^2 W}{\partial P^2} \delta P + \frac{1}{2} \frac{\partial^3 W}{\partial P^3} (\delta P)^2 + \dots$. The third P partial derivative of the option price is removed since it does not play any role. Hence,

$$\frac{\partial W}{\partial P}(P + \delta P, t + \delta t) = \frac{\partial W}{\partial P}(P, t) + \frac{\partial^2 W}{\partial P^2} \delta P. \quad (15a)$$

Substituting for δP gives

$$N = \frac{\partial^2 W}{\partial P^2} \delta P \approx \frac{\partial^2 W}{\partial P^2} \sigma P \varphi \sqrt{\delta t}. \quad (16)$$

Hence, the expected transaction cost in a finite time-step, $E[k|N|P]$ is given by

$$E[k|N|P] = kPE[|N|] = kPE\left[\left|\frac{\partial^2 W}{\partial P^2} \sigma P \varphi \sqrt{\delta t}\right|\right] = k\sigma P^2 \left|\frac{\partial^2 W}{\partial P^2}\right| E[|\varphi|] \sqrt{\delta t}. \quad (17)$$

Now,

$$E[|\varphi|] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi e^{-0.5\varphi^2} d\varphi = 2 \sqrt{\frac{1}{2\pi}} \int_0^{\infty} \varphi e^{-0.5\varphi^2} d\varphi = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \varphi e^{-0.5\varphi^2} d\varphi. \quad (18)$$

Solving equation (18) by substitution: let $\varphi^2 = V$, then $\varphi d\varphi = \frac{dV}{2}$. Equation (18) becomes

$$\frac{1}{2} \int_0^{\infty} e^{-\frac{V}{2}} dV = -[e^{\infty} - e^0] = 1. \text{ Hence,}$$

$$E|\varphi| = \sqrt{\frac{2}{\pi}}. \quad (19)$$

From equations (14) and (17) we have

$\left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left|\frac{\partial^2 W}{\partial P^2}\right|\right) \delta t + \sigma P \frac{\partial W}{\partial P} \sqrt{\delta t}$. Substituting into equation (13) gives

$$\begin{aligned} \delta W &= \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2}\right) \delta t + \sigma P \frac{\partial W}{\partial P} \sqrt{\delta t} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left|\frac{\partial^2 W}{\partial P^2}\right| \delta t \\ &= \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 \varphi^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left|\frac{\partial^2 W}{\partial P^2}\right|\right) \delta t + \sigma P \frac{\partial W}{\partial P} \sqrt{\delta t}. \text{ It follows that} \\ E[\delta W] &= \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left|\frac{\partial^2 W}{\partial P^2}\right|\right), \end{aligned} \quad (20)$$



since $E[\phi^2] = 1$. From equation (9), we have $\omega = \frac{\partial W}{\partial P}$ and

$$\omega \mu P + \tau r B = \left(\frac{\partial W}{\partial t} + \mu P \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left| \frac{\partial^2 W}{\partial P^2} \right| \right). \quad (21)$$

From equation (5), we have $\tau B = W - \omega P = W - P \frac{\partial W}{\partial P}$. Since $\omega = \frac{\partial W}{\partial P}$, equation (21) becomes

$$\begin{aligned} \tau r B &= \left(\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left| \frac{\partial^2 W}{\partial P^2} \right| \right) \\ W - P \frac{\partial W}{\partial P} &= \left(\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left| \frac{\partial^2 W}{\partial P^2} \right| \right) \left(\frac{1}{r} \right). \end{aligned} \quad (22)$$

Equation (22) becomes

$$\begin{aligned} rW - rP \frac{\partial W}{\partial P} &= \left(\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left| \frac{\partial^2 W}{\partial P^2} \right| \right). \text{ Hence,} \\ \left(\frac{\partial W}{\partial t} + rP \frac{\partial W}{\partial P} + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 W}{\partial P^2} - \sqrt{\frac{2}{\pi}} k \sigma P^2 \sqrt{\delta t} \left| \frac{\partial^2 W}{\partial P^2} \right| \right) &= rW. \end{aligned} \quad (23)$$

Equation (23) is one form of Black-Scholes equation that incorporates transaction cost.

Let us denote an optimal trading strategy π_t^* for which we define

$$H_{\pi_t^*}(t, r, P) = E_{\pi_t^*}[U(W(T))]. \quad (24)$$

Our objective here is to find the optimal value function such that

$$H(t, r, P) = H_{\pi_t^*}(t, r, P). \quad (25)$$

From [12], for $D_1 \neq 0$ together with the optimal π_t^* , equation (23) becomes

$$\frac{\partial W}{\partial t} + (rP - D_1) \frac{\partial W}{\partial P} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 \pi_t^* P^2 - \sqrt{\frac{2}{\pi}} \frac{k \sigma P^2}{\sqrt{\delta t}} \pi_t^* \left| \frac{\partial^2 W}{\partial P^2} \right| = rW. \quad (26)$$

Put

$$z = \frac{\alpha}{P}; \quad W(P) = z^\beta H(Z). \quad (27)$$

Thus

$$\frac{\partial W}{\partial P} = -\frac{1}{\alpha} \left(\beta z^{\beta+1} H + z^{\beta+2} \frac{\partial H}{\partial z} \right) \quad (28)$$



and

$$\frac{\partial^2 W}{\partial P^2} = -\frac{1}{\alpha} z^2 \left(\beta(\beta + 1) z^\beta H + \beta z^{\beta+1} \frac{\partial H}{\partial z} + (\beta + 2) z^{\beta+1} \frac{\partial H}{\partial z} + z^{\beta+2} \frac{\partial^2 H}{\partial z^2} \right) \quad (29)$$

Since W is not dependent on r , substituting into equation (26) gives

$$\begin{aligned} r z^\beta H = & -\frac{\sigma^2}{2} \left(\beta(\beta + 1) z^\beta H + \beta z^{\beta+1} \frac{\partial H}{\partial z} + (\beta + 2) z^{\beta+1} \frac{\partial H}{\partial z} + z^{\beta+2} \pi_t^* \frac{\partial^2 H}{\partial z^2} \right) - r \beta z^\beta H - \\ & r z^{\beta+1} \frac{\partial H}{\partial z} + \frac{D_1 \beta z^{\beta+1} H}{\alpha} + \frac{D_1 z^{\beta+2} \frac{\partial H}{\partial z}}{\alpha} - \sqrt{\frac{2}{\pi}} \frac{k \sigma}{\sqrt{\delta t}} \left(\beta(\beta + 1) z^\beta H + \beta z^{\beta+1} \frac{\partial H}{\partial z} + (\beta + \right. \\ & \left. 2) z^{\beta+1} \frac{\partial H}{\partial z} + z^{\beta+2} \frac{\partial^2 H}{\partial z^2} \right) \end{aligned} \quad (30)$$

Cancelling by z^β and collecting like terms give

$$\begin{aligned} 0 = & \left(\frac{-\sigma^2 \pi_t^* z^2}{2} - \sqrt{\frac{2}{\pi}} \frac{k \sigma \pi_t^* z^2}{\sqrt{\delta t}} \right) \frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z} z \left(-\sigma^2 \beta - \sigma^2 - r + \frac{D_1}{\alpha} z - 2 \sqrt{\frac{2}{\pi}} \frac{k \sigma \beta}{\sqrt{\delta t}} - 2 \sqrt{\frac{2}{\pi}} \frac{k \sigma}{\sqrt{\delta t}} \right) + \\ & H \left(-\frac{\sigma^2}{2} \beta(\beta + 1) - r(\beta + 1) + \beta \frac{D_1}{\alpha} z - \sqrt{\frac{2}{\pi}} \frac{k \sigma \beta}{\sqrt{\delta t}} (\beta + 1) \right) \end{aligned} \quad (31)$$

Set
$$r = \frac{\sqrt{\pi \delta t} \sigma^2 \beta + 2^{\frac{3}{2}} k \sigma \beta}{2 \sqrt{\pi \delta t}} \quad (32)$$

$$\frac{\frac{D_1}{\alpha}}{\frac{\sqrt{\pi \delta t} \sigma^2 + 2^{\frac{3}{2}} k \sigma}{2 \sqrt{\pi \delta t}}} = -1 \quad (33)$$

and
$$\gamma = \frac{\sigma \sqrt{\pi \delta t}}{\sqrt{\pi \delta t} \sigma + 2^{\frac{3}{2}} k} \quad (34)$$

to have
$$\beta = \frac{2r \sqrt{\pi \delta t}}{\sqrt{\pi \delta t} \sigma + 2^{\frac{3}{2}} k \sigma} \quad (35)$$

$$\alpha = \frac{-2D_1 \sqrt{\pi \delta t}}{\sqrt{\pi \delta t} \sigma + 2^{\frac{3}{2}} k \sigma} \quad (36)$$

and
$$\frac{\sigma \sqrt{\pi \delta t}}{\gamma} = \sqrt{\pi \delta t} \sigma + 2^{\frac{3}{2}} k \quad (37)$$



Substituting into equation (31) to obtain

$$0 = \pi_t^* z^2 \frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z} z \left(2\gamma\beta + 2\gamma + \beta + z + 2 \cdot 2^{\frac{3}{2}} \frac{k\beta\gamma}{\sigma\sqrt{\pi\delta t}} + 2 \cdot 2^{\frac{3}{2}} \frac{k\gamma}{\sigma\sqrt{\pi\delta t}} \right) + H \left(\beta\gamma + \gamma + \beta + 1 + z + 2\sqrt{2} \frac{k\beta\gamma}{\sigma\sqrt{\pi\delta t}} + 2\sqrt{2} \frac{k\gamma}{\sigma\sqrt{\pi\delta t}} \right) \quad (38)$$

Let $\varepsilon = \frac{k\gamma}{\sigma\sqrt{\pi\delta t}}$ (39)

Equation (38) becomes

$$0 = \pi_t^* z^2 \frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z} z \left(2\gamma\beta + 2\gamma + \beta + z + 2 \cdot 2^{\frac{3}{2}} \beta\varepsilon + 2 \cdot 2^{\frac{3}{2}} \varepsilon \right) + H \left(\beta\gamma + \gamma + \beta + 1 + z + 2\sqrt{2} \beta\varepsilon + 2\sqrt{2} \varepsilon \right) \quad (40)$$

Solving for π_t^* gives

$$\pi_t^* = \frac{-zH_z \left(2\gamma\beta + 2\gamma + \beta + z + 2 \cdot 2^{\frac{3}{2}} \beta\varepsilon + 2 \cdot 2^{\frac{3}{2}} \varepsilon \right)}{z^2 H_{zz}} - \frac{H(\beta\gamma + \gamma + \beta + 1 + z + 2\sqrt{2} \beta\varepsilon + 2\sqrt{2} \varepsilon)}{z^2 H_{zz}} \quad (41)$$

where γ , β and ε are as defined in equations (34), (35) and (39).

CONCLUSION

The model has some successes in explaining excess volatility of stock returns compared to fundamentals and negative skewness of equity returns. The random component does not feature in the Black-Scholes equation and this is precisely because of the manner in which we chose the number of stocks, ω . The second derivative of the option price in the transaction cost term apparently suggests that it is closely linked to the extent of re-hedging in the following time interval. The equation is nonlinear, hence, the value of a portfolio of options is different from the sum of the individual components. If we have a position in two call options which have the same exercise price and the same expiry date and on the same underlying asset. If one is held short and the other long, our net position is zero. There is a cancellation effect of the two opposite positions that goes unnoticed if our basket of options is large. Due to this effect, we decide to hedge each of them separately. Re-hedging at each time step on both options implies that we drain the option price. At maturity, the costs are still present and we have a negative net balance, since the two payoffs offset each other. Analyzing the result from the buyer's and seller's prices perspectives we see that given $\frac{\partial^2 W}{\partial P^2} > 0$ for a single call or put held long, if the modulus sign is dropped and σ^2 is replaced with $\hat{\sigma}^2 = \sigma^2 - 2\sqrt{\frac{2}{\pi}} \frac{k\sigma}{\sqrt{\delta t}}$, the resulting option price can be viewed as the buyer's price. Similarly, replacing σ^2 with $\hat{\sigma}^2 = \sigma^2 + 2\sqrt{\frac{2}{\pi}} \frac{k\sigma}{\sqrt{\delta t}}$, we view the resulting option price as the seller's price. This holds because the seller is like a market maker who must sell the option for more than its worth to make a profit and edge away all the risk. Unlike the seller, the buyer of an option by convention cannot hedge his position.



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