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A SURVEY OF ADVANCES IN MAGNIFYING ELEMENTS IN SEMIGROUPS

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ABSTRACT: *An element s of a semigroup S is said to be a right (resp. left) magnifying (magnifier) element if there exists a proper subset M of S such that $Ms = S$ [resp. $sM = S$]. If M is a proper subsemigroup of S, then s in S is called strong right (resp. strong left) magnifying element. In this paper, we present a panorama of various research articles in magnifying elements in semigroups from its inception in 1963. We present this via the various existing notions and related results.*

KEYWORDS: Semigroup; subsemigroup; left (right) magnifying element.

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INTRODUCTION

A semigroup is simply a set S which is closed under an associative binary operation. A clear and obvious example of semigroups is the set of natural numbers N under addition. Thus, the notion of a semigroup is a natural one since this example has been with mathematicians even before the inception of semigroups. The collection of all self-maps of a set X into itself under composition of maps is called a full transformation semigroup.

An element s of a semigroup S is said to be a right (resp. left) magnifying (magnifier) element if there exists a proper subset M of S such that $Ms = S$ [resp. $sM = S$]. The magnifying elements appear to be involved in a natural way in different situations and relate in a strong way to the algebraic structure of the semigroup. It is not immediately obvious from the definition what sort of semigroups will have magnifying elements or not, but regardless, the notion is beautiful in its way and has proven to have interesting properties in any structure it appears. In this paper, we present an overview of the advances that the notion of magnifying elements underwent from its inception. This is with a view to arouse interest and focus attention of researchers to the area. We shall present this by defining related concepts and results associated with each article that will be considered. It would be rather ambitious to try to trace the advances in magnifying elements in semigroups up to the present day. We will therefore only take the survey as far as 2019.

From here on, we note that the usage of S and $T(X)$ will always mean a semigroup and the semigroup of full transformation on X respectively (should they be omitted anywhere in this survey). Again, all definitions and results (theorems) that will be presented hereafter are to the credit of the work of the researcher under discussion, except if otherwise stated. Proofs to the theorems are omitted. They can be found handy in the various researches they represent. We shall present the survey on article base and make the author's particulars the heading. We begin from inception in the work of Lyapin (1963).

Lyapin (1963)

The idea and concept of magnifying elements started in the book by Lyapin in 1963. An element s of a semigroup S is said to be a right (resp. left) magnifying element if there exists a proper subset M of S such that $Ms = S$ [resp. $sM = S$]. He gave a holistic approach to the idea and touched it from various standpoints. He began with a fascinating result by clarifying that no single element in a semigroup can be both a left magnifying and a right magnifying element at the same time, thus making the elements unique in this way. Lyapin presented this fact in the result that follows:

Theorem 1: No element of a semigroup is simultaneously a left and a right magnifier.

It is obvious at this opening stage that the definition of magnifying elements appears to be without restriction as to the type of environment where it will exist. We cannot go on without pointing out that not all semigroups contain magnifying elements. We present in the points below semigroups without magnifying elements while still providing corresponding justification. This is due to Lyapin (1963):

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i. Finite semigroups have no magnifying elements. This follows from the fact that given any semigroup S of finite order n i. e., $|S|=n$ with $s \in S, M \subset S,$ then, $|Ms|[|sM|]<n$ which ultimately contradicts the meaning of magnifying elements. This is no doubt a strong statement.

ii. If a semigroup S is commutative, then $\forall s \in S,\ sM = Ms$ and any proper subset $M \subset S$. This contradicts Theorem 1. Hence, magnifying elements do not exist in a commutative semigroup.

iii. Let S be a semigroup with two-sided cancellation such that $s \in S$ and $M \subset S$. We then have that $Ms = S$. Suppose $s' \in S$ such that $s' \notin M,$ then there exists $s_\circ \in M$ such that $s_\circ s = s's$. This is not true in a semigroup with right cancellation unless $s_\circ = s'$, which contradicts our supposition. Hence, semigroups with two-sided cancellation have no magnifying elements.

iv. Since every group is a semigroup and we know from elementary group theory that groups have the property of two-sided cancellation, it follows therefore from (iii) above that groups do not contain magnifying elements.

We further summarize the following result from Lyapin (1963):

Theorem 2: Suppose that some elements of a semigroup S are right magnifiers and some are left magnifiers. The collection of all right (resp. left) magnifiers in S is a subsemigroup of S and so is their compliment in S.

If U and V are the collections of right (resp. left) magnifiers of S and $W,$ the collection of elements of S that satisfy neither, then as a consequence of theorem 2, $U \cap V = \emptyset$ and $U \cup W$ and $V \cup W$ are both subsemigroups of S except if they are empty.

Definition: A semigroup generated by a single element is called a monogenic semigroup (Howie, 1995).

Definition: An element a of a semigrup S is called regular if there exists an elemnts x in S such that $axa = a.$ If all the elements of S are regular, S is called a regular semigroup (Howie, 1995).

Furthermore, he considered a semigroup with units as one semigroup which contains magnifying elements. He furnished us with the following results:

Theorem 3: In a semigroup with unit, every magnifying element is regular and it generates an infinite monogenic semigroup.

Theorem 4: If a semigroup with unit contains right magnifiers, it also contains left magnifiers and vice versa.

Definition: An element $s \in S$ is said to be right (resp. left) invertible if it is a left (resp. right) divisor of every element of S. An element of S that is both left and right invertible is said to be a two-sidedly invertible element.

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Lyapin showed that the property of magnifying elements is closely related with the idea of invertibility of elements in the following results.

Theorem 5: Every right (resp. left) magnifying element of a semigroup is left (resp. right) invertible but not right (resp. left) invertible.

Further connections between invertibility and magnifying elements can be found in Lyapin (1963).

Tolo (1969)

He undertook work on factorizable semigroups. He found the idea of magnifying elements instrumental in finding the sufficient condition for a semigroup S to be factorizable. To complement on the work of Lyapin (1963) and to achieve the aim of his work, Tolo introduced the concepts of strong magnifying elements. This however led him to call Lyapin's magnifying element weak magnifying element as against the strong magnifying element he initiated. He put forward the following definitions:

Definition: A multiplicative semigroup S is said to be factorizable if it can be written as the set product AB of proper subsemigroups of A and B of S. If this is possible, AB is called a factorization of S, with factors A and B. These factors are not required to be unique.

Definition: An element r of S is said to be a strong right (resp. left) magnifying element if there exists a proper subsemigroup M of S such that Mr (resp. rM) $= S$.

Tolo presented the sufficient condition for a semigroup to be factorizable in the result that follows:

Theorem 6: If a semigroup S contains either a strong right magnifying element or a strong left magnifying element, then S is factorizable.

Theorem 7: Let S be a semigroup containing a weak right magnifying element and assume that S satisfies any one of the following conditions:

i. S has a left identity element

ii. S contains a weak left magnifying element

iii. S is regular

Then S is factorizable.

Migliorini (1971)

Migliorini raised the question of the existence of a minimal subset M of S for which $sM[Ms] = S$ for $s \in S$. He succeeded in answering this question in the affirmative in the course of his work. He was able to find out that if $s \in S$ and $M \subset S$ such that $sM = S$, we can find a minimal subset M' of S such that $sM' = S$ but for which if s' is any of its element, $s(M' - s') \neq S$. In a more general sense he established that:

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Theorem 8: Let $S = sM$ such that M is a minimal subset with respect to $s(M \neq S)$. Then for any $n \in N$, $M^{(n+1)} \subset M^n(M^{(n+1)} \neq M^n)$, $S = a^{n+1}M^{n+1}$ and M^{n+1} is minimal with respect to a^{n+1}.

The above translates to saying that there is in M an infinite chain of subsets.

$$M = M^{(1)} \supset M^{(2)} \supset M^{(3)} \supset \cdots \supset M^{(n)} \supset M^{(n+1)} \supset \cdots$$

This is a chain in which the nth element $M^{(n)}$ is such that $S = s^n M^n$ and is minimal with respect to s^n. Thus, if the n as represented in s^n is increased, we can make small the corresponding subset M^n for which $S = s^n M^n$.

Lyapin established that the Croissot-Tiessier semigroup is an example of a semigroup which has all its elements to be right magnifying. Migliorini on the other hand established the property of such semigroup. We recall the following definitions for the purpose of clarity in the results that will follow them which characterizes semigroups whose elements are right magnifying.

Definition: An element s of a semigroup S is said to be an idempotent if $s^2 = s$.

Definition: A nonempty subset M of S is called a left (resp. right) ideal of S if SM (resp. MS) $\subseteq M$. If M is both a right and left ideal, it is called a two-sided ideal.

Definition: A semigroup is called simple if it contains no two-sided ideals.

Definition: A minimal right (resp. left, two-sided) ideal of S is a right (resp. left, two-sided) ideal which contains no proper subset that is a right (resp. left, two-sided) ideal of S.

Theorem 9: If all the element in S are right magnifying, then S is a semigroup with left invertibility, without idempotent elements and no element has a right identity.

Theorem 10: If S has all its elements to be right magnifying, then S is left simple and contains no minimal right ideal.

Migliorini (1974)

This is an extension to his work of 1971. Here, he established the fact that among the minimal subsets relative to the same magnifying elements s, one of these subsets can be found to be a subsemigroup of S. This is an extension to Theorem 8. He further studies the structure of such semigroup.

Catino and Migliorini (1985)

They studied the concept of quasi increasing (q-increasing) elements in semigroups as a generalization of the concept of magnifying elements. They defined q-increasing elements as follows:

Definition: Let S be a semigroup and T be a subset of S. An element a in T such that $aT = T[Ta = T]$ is said to be a left (resp. right) q-increasing element relative to T if there exists a subset $T'$of T with $aT' = T[T'a = T]$.

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Since the concept of q-increasing elements is a generalization of magnifying elements, they commented that a left (resp. right) magnifying element is left (resp. right) q-increasing. Interestingly, the immediate properties of q-increasing elements is the same as those of magnifying elements since periodic (finite) semigroups, commutative semigroups, two-sided cancellative semigroups and groups do not contain q-increasing elements. The following results were observed:

Theorem 11: No element of S can be both left (resp. right) q-increasing.

Theorem 12: A completely regular element of S is not a left (resp. right) q-increasing element of S. Hence, a completely regular semigroup contains no left (resp. right) q-increasing element.

For more on completely regular semigroups, Howie (1995) will be resourceful.

Theorem 13: Let S^r (S^l) be the sets of all right (resp. left) q-increasing elements of S. Then S^r (S^l) has no left (resp. right) q-increasing elements.

The rest of the work is dedicated to proving the sufficient condition for a semigroup to contain q-increasing elements. It is that the semigroup has to be regular with two distinct D-related idempotents. (D-Green's D relation, Howie, 1995). He further defined minimal subsets relative to q-increasing elements showing their existence in a semigroup.

Catino and Migliorini (1992)

Here, they improved the work of Tolo (1969) by making it possible to determine the existence of a subsemigroup M of S such that $sM[Ms] = S$ for $s \in S$. They summarized this finding in the following results:

Theorem 14: If S is a semigroup with magnifying elements and with a left identity, then S contains a strong left magnifying element.

The above result in summary is that if S contains a left identity, then we can be sure that in S there will be a strong left magnifying element.

Theorem 15: Let S be a semigroup which contains right magnifying elements. Then,

i. If S contains left magnifying elements, then S contains some strong left (resp. right) magnifying elements.

ii. If S is regular, all the right magnifying elements are strong.

They remarked that no example was yet discovered about a semigroup which contains both strong and weak right (resp. left) magnifying elements.

Definition: A semigroup without zero is called simple if it has no proper ideals.

Definition: A semigroup is called right simple if $R = S \times S$ and left simple if $L = S \times S$. The R and L Green's relations and their meanings can be found in Howie (1995).

They provided a necessary and sufficient condition for which any bisimple semigroup will contain a magnifying element. The result to this is presented below:

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Theorem 16: If S is a bisimple semigroup, then

i. S will contain a left magnifier if S contains a left identity and is not a right group.

ii. S will contain a left magnifier if S contains no idempotent and is right simple.

Magill (1994)

A further studies saw Magill extend the work of Catino and Migliorini (1992) which was done in the abstract semigroup to the transformation semigroup. Infact, this is the first time the magnifying element was discussed in terms of transformation. Magill mentioned that if s is a left magnifier in a semigroup S with identity, then there exists a proper ideal R such that $R = S$ and since the transformations of S has an identity, there can never be a distinction between weak and strong magnifying elements in this environment. Magill improved on the result of Lyapin on the characterization of magnifying elements, and equally introduced the R and L Green's relations to some of his findings. This can be seen in the following results:

Theorem 17: Let S be a semigroup with identity e. An element $a \in S$ is a left magnifying element if and only if there exists an element $b \in S$ such that $ab = e$ but $ba \neq e$. Similarly, an element $b \in S$ is a right magnifying element if and only if there is an element $a \in S$ such that $ab = e$ but $ba \neq e$.

The above results translates to saying that if S has an identity, then S has a right magnifying element if and only if it has a left magnifying element.

Theorem 18: Let S be a semigroup with identity e. Then $a \in S$ is a left magnifying element if and only if it is not a unit and aRe and it is a right magnifying element if and only if it is not a unit and aLe.

Definition: Let $A, B \in X$. A map $\alpha \in T(X)$ maps AT-isomorphically onto B if $\alpha(A) \subseteq B$ and there exists a mapping $\beta \in T(X)$ such that $\beta(B) \subseteq A$, $\alpha \circ \beta|B = \delta|B$ and $\beta \circ \alpha|A = \delta|A$. When this happens, we say A and B are T-isomorphic. $\alpha \circ \beta|B$ and $\beta \circ \alpha|A$ are restriction maps.

Definition: A subset of X is a T-retract of X if it is the range of an idempotent mapping in $T(X)$.

He characterizes magnifying elements in $T(X)$ below:

Theorem 19: A map $\alpha \in T(X)$ is a left magnifying element if and only if it maps some proper T-retract of XT-isomorphically onto X and it is a right magnifying element if and only if it maps XT-isomorphically onto some proper T-retract of X.

He rounded off his work by applying the result considered above to the semigroups of all continuous self-maps of a topological space and to the semigroup of all linear transformations of a vector space.

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Gutan (1996)

Gutan's short paper was an answer to what Magill (1992) and Catino and Migliorini (1992) mentioned. They remarked as at the time that no example was yet found for a semigroup which contains both strong and weak magnifying elements. Gutan therefore supplied the answer to this concern in the affirmative. He considered $\underline{LM(S)}$ and $LM(S)$ to be the collections of all strong left magnifying elements and weak left magnifying elements respectively. In a semigroup with left identity, $\underline{LM(S)}$ and $LM(S)$ coincide. He mentioned that the bicyclic semigroup is an example of this. Regarding the issue of finding a semigroup which contains both strong and weak magnifying elements, he noted that such a semigroup must be one without left identity. He described such a semigroup as one that is a direct product $S \times T$ of a semigroup S with no left identity and T with left identity. He showed that $S \times T$ will be a semigroup without left identity and will be one containing both strong and weak magnifying elements. He illustrated this using the bicyclic and the Baer-Levi semigroups.

Gutan (1997)

Here Gutan provides an extension to the work of Tolo (1969) and thus solved the problem of finding the necessary and sufficient condition for a semigroup with magnifying element to be factorizable. Gutan established primarily and in strong terms that every semigroup that contains left and right magnifying elements is factorizable. This furnishes us with a condition stronger than what was presented in Tolo (1969) which only suggests sufficiently that semigroups with strong right magnifying elements and weak right magnifying elements are factorizable.

Remark: We remark at this point that Gutan (2000) introduced a new concept called good and very good magnifiers and again, Gutan and Kisielewicz (2003) introduced good and bad magnifiers in semigroups. We omit this concept in this review as our focus is only on the advances in magnifying elements alone.

Gutan (1999)

Gutan characterized semigroups with magnifying elements that admit subsemigroups as minimal subsets, and presented a general method for obtaining this subsemigroup. Gutan proved that every such semigroup can be obtained using this method or construction. Specifically, he was able to show that every semigroup S with left magnifiers admitting subsemigroups as minimal subsets is an extension of a semigroup M which contains left magnifiers and left identities by an endomorphism of M. He further presents examples to these results.

Kaewnoi et al. (2018)

They studied $P(X,E) = \{\alpha \in P(X): (x,y) \in E \Longrightarrow (\alpha(x), \alpha(y)) \in E\}$, a subsemigroup of $P(X)$, the partial transformation of the set X. Here, E is an equivalence relation on the non-empty set X. With respect to the semigroup $P(X,E)$, they studied left (resp. right) magnifying element and provided necessary and sufficient conditions for elements of the semigroup to be right (left) magnifying. They characterized right (left) magnifying elements in $P(X,E)$ in the following results:

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Theorem 20: If α is a right (resp. left) magnifying element in $P(X, E)$, then α is onto (resp. one-one).

By the result above, it means that in $P(X, E)$, a right (resp. left) magnifying element is an onto (resp. one-one) map.

They summarized the necessary and sufficient condition in the result below:

Theorem 21: α in $P(X, E)$ is right magnifying if and only if α is onto, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = \alpha(a), y = \alpha(b)$ and either $dom\alpha \neq X$ or $dom\alpha = X$ and α is not one-one.

Theorem 22: α in $P(X, E)$ is left magnifying if and only if α is one-one but not onto, $dom\alpha = X$ and for any $x, y \in X$, $(\alpha(x), \alpha(y)) \in E$ implies $(x, y) \in E$.

Chinram et al. (2018)

They studied $T(X, P) = \{\alpha \in T(X): \alpha(X_i) \subseteq X_i, \forall\, i \in N\}$ where $P = \{X_i: i \in N\}$ is a partition of the set X. They established that $T(X, P)$ is a subsemigroup of the full transformation semigroup $T(X)$. Just as in the work of Kaewnoi et al. (2018), they equally characterized elements of $T(X, P)$ that are right (resp. left) magnifying and presented necessary and sufficient conditions under which an element of $T(X, P)$ will be right (resp. left) magnifying. These characterizations are presented in the following results:

Theorem 23: $\alpha \in T(X, P)$ is right (resp. left) magnifying if it is onto (resp. one-one).

The necessary and sufficient conditions for right (resp. left) magnifying elements was characterized in the following results:

Theorem 24:

i. $T(X, P)$ has a right (resp. left) magnifying element if and only if X_i is infinite for some $i \in N$

ii. $\alpha \in T(X, P)$ is right (resp. left) magnifying if and only if it is onto (resp. one-one) but not one-one (resp. onto).

The remainder of the work was dedicated to applying the above conditions to give necessary and sufficient conditions for some elements in some generalized linear transformation groups.

Chinram and Baupradist (2019)

They considered the semigroup of transformation with invariant set. Given $T(X)$ is a semigroup of full transformation on the set X, and Y be a fixed non-empty subset of X, they considered the semigroup $S(X, Y) = \{\alpha \in T(X): Y\alpha \subseteq Y\}$. They found the necessary and sufficient condition for the elements in $S(X, Y)$ to be left or right magnifying. The following three results summarized this finding:

Theorem 25: Let $\alpha \in S(X, Y)$. Then α is right magnifying in $S(X, Y)$ if and only if α is onto but not one-one and such that $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$.

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Theorem 26: Let $\alpha \in S(X, Y)$. Then α is left magnifying in $S(X, Y)$ if and only if α is one-one but not onto and such that $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap im\alpha$. In fact,

Theorem 27: If $Y = X$, then α is left magnifying in $S(X, Y)$ if and only if α is one-one but not onto.

CONCLUSION

We have presented a critical literature survey on the concept of magnifying elements as various researches unfold from its inception. Various results which form the very essence of these researches have been presented and addendums to these results are presented where necessary.

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