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ON INTRACTABILITY IN G-CLASS OF LIFETIME PROBABILITY DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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ABSTRACT: *This research seeks to project the relevance of incorporating intractability in G-Class probability model development. Juchez distribution represents intractability among the conventional tractable distributions, as comparatively studied in their different G-Class dimensions. Some properties were derived; although nonintegrable properties really were a confirmation note to the intractable nature of the distribution. The L-shape hazard function as a special feature across the G-Class derived distributions suggests that the distribution is fit to model outcomes that do not wear out as time or cycle elapses. Finally, the performance comparison reveals that intractability could be embraced in the development of probability models, as the derived distributions showed to be a better fit than the existing G-class baseline-tractable distributions.*

KEYWORDS: Intractability, G-Class distributions, Flexibility, AIC, L-Hazard shape, Complex.

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INTRODUCTION

Oguntunde (2016) stated in his work the consideration or preference for the choice of probability distribution development for modeling real life phenomena: it could be based on either the tractability or flexibility of distribution. Tractable distribution as implied in this study refers to distributions whose algebraic properties are incomplex or lesser degree polynomial; where intractable distribution, on the one hand, is such whose algebraic properties are complex or characterized by high degree polynomials. Tractable distributions may be theory friendly because of its ease of applicability and algebraic property computations; however, professionals attach greater importance to the flexibility of a distribution (Oguntunde, 2016). Exponential and Inverse Exponential distribution, Weibull distribution, Frechet distribution, Lindley distribution, Uniform distribution, Lomax distribution, Pareto distribution, Laplace, among others, exemplify our intended meaning of tractability. In addition, we have Cauchy distribution, Logistics distribution, Wald distribution, Meixner Hypergeometric distribution, Skewed t distribution, Hyperbolic distribution, F-distribution; and in the class of mixture models, Juchez distribution, Devya distribution, Shambhu distribution, Odoma distribution, among others, are present.

The G-Class of distributions describes models that allow for the extension of other distributions, with the objective to incorporate extra parameter(s) to existing distributions. The inclusion of these parameters modifies a distribution into a robust model which varies the tail weight or adjusts the skewness of the resulting compound distribution. These modifications improve the flexibility of a distribution as fit for modeling, especially wild data observations. The G-class of distributions cover these developments: Kumaraswamy-G family of distributions, Gamma-G (type 1) family of distributions, McDonald-G family of distributions, Log-Gamma-G family of distributions, Exponentiated-G (EG) family of distributions, Logistic-G family of distributions, Weibull-G family of distributions, Marshall-Olkin family of distributions, Jones-G family of distributions, Juchez-G family of distributions, among others.

In recent studies, Cordeiro (2013), Bourguignon (2014), Owoloko (2015), Cordeiro (2016), Merovci (2015), Tahir and Mansoor (2014), Torabi and Montazari (2014), Gomez-Deniz (2010), Cordeiro and de Castro (2011) Cordeiro and Cintra (2012), Abd-Elfattah and Assar (2016), and Abouammoh and Alshingiti (2009) have all advanced the G-Glass models with the common goal of developing generalized distributions that would be robust and more flexible than the existing standard theoretical distributions. Nadarajah and Eljabri (2013) developed the kumaraswamy generalized pareto distribution, Oguntunde and Odetunmibi (2015) also developed the exponentiated generalized weibull distribution; pareto and weibull distribution as employed distributions are tractable and hence reduced the complexities that come with the host distribution. Jones (2009) worked on Kumaraswamy distribution: a beta-type distribution with some tractability advantages; Bourguignon and Cordeiro (2014) developed the weibull-g family of probability distributions; Alizadeh and Cordeiro (2015), the beta Marshall-Olkin family of distributions. The Frechet distribution as used in Marshall-Olkin frechet distribution is another example of tractable distribution in G-Class (Krishna & Jose, 2013). More on tractability in G-Class development is Marshall-Olkin extended uniform distribution (Jose, 2011).

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Universally, tractability, mildness of algebraic properties and explorative rarity are usually the factors considered while developing probability distributions, alongside their applications and properties. While those might be advantageous, researchers or professionals might greatly lose out subjecting themselves following those factors; where greater flexibility could be achieved in making substitutions in these G-classes of distributions, employing intractable distributions. As studied, hardly would you observe any G-Class of distribution families incorporating a complex distribution; where the obvious reason is that the host compound distribution is already complex by default.

The major aim in this study is to explore the incorporation of an intractable or complex probability model, Juchez distribution to be precise, into a G-Class development of some family of distributions. Some properties like Shape of the distributions, Hazard function, Maximum likelihood estimator, Inverse cumulative function and simulation studies are considered; where the model fit comparison across the categorical G-Class development follows suit in a wrap.

The next sections cover the model development, using the combination of the intractable and G-Class distributions, and its properties. The binomial representations of the derived models are not left out as well and the performance comparison among the vari ous counterpart G-class distributions.

MODEL DEVELOPMENT

In this section, we develop four generalized distribution models, adopting Juchez distribution proposed by Echebiri and Mbegbu (2022), as the intractable baseline distribution; where ω: $\omega > 0$ represents the vector of parameters as used through the study.

Intractability in G-Class

Let $X \sim Juchez(x,\ \theta)$, then the PDF and CDF is given as:

$$g_{juc}(x) = \ e^{-\theta x}(\ 1 + x + \ x^3\ )\theta^4[\theta^3 + \theta^2 + 6]^{-1}\ ,\ x > 0,\ \theta > \ 0 \tag{1}$$

$$G_{juc}(x) = \ 1 - e^{-\theta x}(1 + \ [\ \theta^3\, x + \ \theta^2\, x^2 \ + \ 3\, \theta^3\, x^3 \ \ + \ 6\, \theta x][\theta^3 + \theta^2 \ + \ 6]^{-1}) \tag{2}$$

where $x > 0\ ,\ \theta > 0.$

Let $X \sim Kumaraswamy\ G\ (x,\ \alpha,\ \beta)$, then the PDF and CDF of the distribution is given as:

$$F(x) = 1 - (1 - [G(x)]^{\alpha})^{\beta} \tag{3}$$

$$f(x) = \alpha\beta g(x)[G(x)]^{\alpha-1}\{1 - \ [G(x)]^{\alpha}\}^{\beta-1} \tag{4}$$

Where $x > 0\ ,\ \alpha > 0,\ \beta > 0.$

Let $X \sim MarshalOlkinG\ (x, \alpha)$, then the PDF and CDF of the distribution is given as:

$$f(x) = \frac{\alpha g(x)}{\{1-(1-\alpha)[1-G(x)]\}^2}\ ; \qquad x > 0, \alpha > 0 \tag{5}$$

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$$F(x) = 1 - \frac{\alpha[1-G(x)]}{1-(1-\alpha)\,[1-G(x)]} \tag{6}$$

where $\alpha > 0$ is the shape parameter, (Marshall & Olkin, 1997).

Let $X \sim EGJD\ (x, \alpha)$, then the PDF and CDF of the distribution is given as

$$f(x) = \alpha\beta g(x)[1 - G(x)]^{\alpha-1}\ \{1 - [1 - G(x)]^{\alpha}\}^{\beta-1} \tag{7}$$

$$F(x, \alpha, \beta) = \{1 - [1 - G(x)]^{\alpha}\}^{\beta} \tag{8}$$

where $x > 0,\ \alpha > 0,\ \beta > 0$, Cordeiro and Ortega (2013)

Equations (1) & (2) clearly show the algebraic structure of the PDF and CDF of the Juchez distribution. Now, subjecting them to mathematical or statistical derivations or integral operations would obviously lead to a huge bottleneck task owing to its polynomial nature; how much more when their substitution is made into these complex G-Class of distributions in equations (3 - 8). This is a confirmatory note that the baseline distribution is intractable. Hence, it suffices to note that these terminologies: "tractability and intractability" are common observations in both classroom theory and software computations.

Let X denotes a non-negative continuous random variable such that; $X \sim Juchez(x, \theta)$, then the PDF and CDF of Kumaraswamy-G Juchez distribution are respectively given as:

$$f(x, \theta, \alpha, \beta) = \alpha\beta\ e^{-\theta x}(\ 1 + x + \ x^{3}\ )\ \theta^{4}[\theta^{3} + \theta^{2} + 6]^{-1}\ [1 - q]^{\alpha-1}\ \ \{1 - [1 - q]^{\alpha}\}^{\beta-1} \tag{9}$$

$$F(x, \theta, \alpha, \beta) = 1 - \{1 - [1 - q]^{\alpha}\}^{\beta} \tag{10}$$

$$q = (1 + w)e^{-\theta x} \text{ and } w = \frac{\theta^{3}x + \theta^{3}x^{3} + 3\theta^{2}x^{2} + 6\theta x}{\theta^{3} + \theta^{2} + 6}$$

This is obtained by inserting equations (1) and (2) into equations (3) and (4); where the alternative binomial representation of the PDF and CDF are given as

$$f(x, \theta, \alpha, \beta) = \alpha\beta g_{juc}(x)\left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\alpha-1}{i}\binom{\beta}{j}\binom{j}{k}(-1)^{i+j+k}q^{i}\,[1 - (1 - q)^{\alpha}]^{k}\right] \tag{11}$$

$$F(x, \theta, \alpha, \beta) = 1 - \left\{\sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\beta}{i}\binom{i}{j}[-1]^{i+j}[1 - (1 - q)^{\alpha}]^{j}\right\} \tag{12}$$

Let X denote a non-negative continuous random variable such that $X \sim Juchez(x, \theta)$, then the PDF and CDF of Marshall Olkin-G Juchez distribution are respectively given as:

$$f(x, \theta, \alpha) = \frac{\alpha g_{juc}(x)}{\{1-(1-\alpha)[q]\}^{2}} \tag{13}$$

$$F(x, \theta, \alpha) = 1 - \frac{\alpha q}{1-(1-\alpha)[q]} \tag{14}$$

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Let X denote a non-negative continuous random variable such that; $X \sim Juchez(x, \theta)$, then the cdf and pdf of Exponentiated-G Juchez distribution are respectively given as:

$$f(x, \theta, \alpha, \beta) = \alpha\beta g_{juc}(x)(q)^{\alpha-1} \{1 - (q)^{\alpha}\}^{\beta-1} = \alpha\beta g_{juc}(x)(q)^{\alpha-1} \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i q^{\alpha i} \quad (15)$$

$$F(x, \theta, \alpha, \beta) = \{1 - [(1 + w)e^{-\theta x}]^{\alpha}\}^{\beta} = \sum_{i=0}^{\infty} \binom{\beta}{i} (-1)^i q^{\alpha i} \quad (16)$$

Properties of the G-Class distributions

To verify the validity or properness of the selected G-Class distributions, it suffices that:

$$lim_{x\to\infty} F(x, \omega) = 1 \quad (17)$$

where ω is a vector of parameters.

$X \sim KGJD$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = 1 - \{1 - [1 - (1 + w)e^{-\theta x}]^{\alpha}\}^{\beta} \quad (18)$$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = 1 - \{1 - [1 - (1 + \infty)0]^{\alpha}\}^{\beta} \quad (19)$$

Since $e^{-\theta x} = \frac{1}{e^{\theta x}}$; $\frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = 1$$

$X \sim MOGJD$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) =._{x\to\infty} \left[1 - \frac{\alpha\left(1 + \frac{[\theta^3 x + \theta^3 x^3 + 3\theta^2 x^2 + 6\theta x]}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x}}{1 - (1-\alpha)\left[\left(1 + \frac{[\theta^3 x + \theta^3 x^3 + 3\theta^2 x^2 + 6\theta x]}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x}\right]}\right] \quad (20)$$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = 1 - \frac{\alpha(1+\infty)0}{1-(1-\alpha)(1+\infty)0} = 1 \quad (21)$$

$X \sim EGJD$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = \{1 - [(1 + w)e^{-\theta x}]^{\alpha}\}^{\beta} \quad (22)$$

$$lim_{x\to\infty} F(x, \theta, \alpha, \beta) = \{1 - [(1 + \infty)0]^{\alpha}\}^{\beta} = 1 \quad (23)$$

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Shapes of the G-Class Distribution

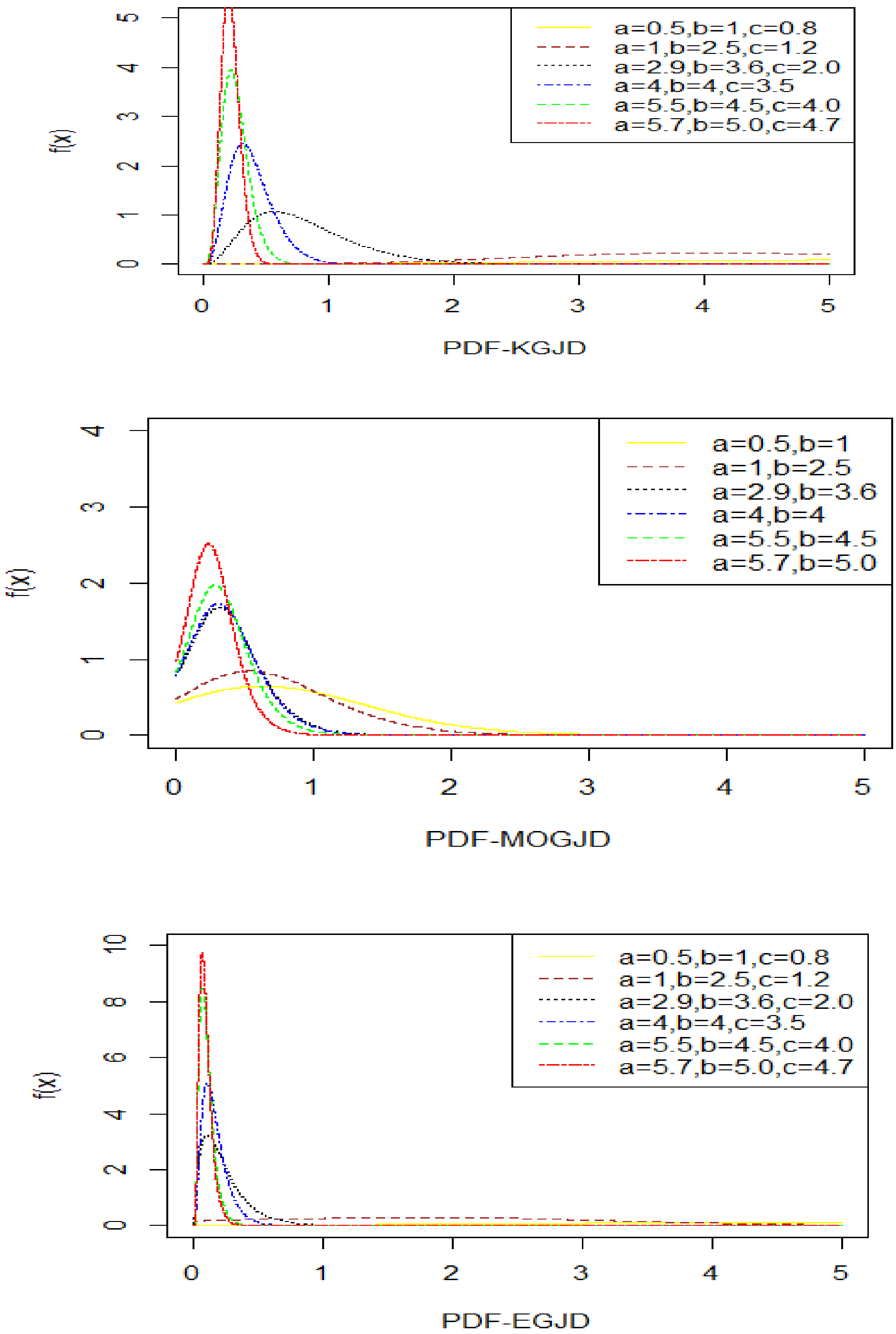

Figure 1 (a)(b)(c)(d). The shape of the PDFs of G-Class distributions

The shape of the G-Class distributions is not far-fetched from the fact that they all fit to model heavily skewed and long-tailed datasets.

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Reliability, Cumulative Hazard and Failure Rate Functions.

Reliability model for any lifetime distribution can be expressed as

$$\underline{F}\,(x) = exp\{-H(x)\} = \int_x^\infty f(t)dt \tag{24}$$

Where $H(x)$ is the cumulative hazard function and given as: $H(x) = \int_0^x h(t)dt = -ln\underline{F}\,(x)$

$$h(x) = -\frac{d}{dx} ln\underline{F}\,(x) = \frac{f(x)}{\underline{F}(x)} \tag{25}$$

Let $X \sim KGJD$, then

$$\underline{F}_{KGJD}(x) = \left\{\sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\beta}{i}\binom{i}{j}[-1]^{i+j}[1-(1-q)^{\alpha}]^{j}\right\} \tag{26}$$

$$h_{KGJD}(x) = \frac{\alpha\beta g_{juc}(x)\left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\alpha-1}{i}\binom{\beta}{j}\binom{j}{k}(-1)^{i+j+k}q^{i}\,[1-(1-q)^{\alpha}]^{k}\right]}{\left\{\sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\beta}{i}\binom{i}{j}[-1]^{i+j}[1-(1-q)^{\alpha}]^{j}\right\}} \tag{27}$$

$X \sim MOGJD$

$$\underline{F}_{MOGJD}(x) = \frac{\alpha q}{1-(1-\alpha)[q]} \tag{28}$$

$$h_{MOGJD}(x) = \frac{\dfrac{\alpha\left[\frac{\theta^4}{\theta^3+\theta^2+6}\left(1+x+x^3\right)e^{-\theta x}\right]}{\{1-(1-\alpha)[q]\}^2}}{\dfrac{\alpha q}{1-(1-\alpha)[q]}} \tag{29}$$

$$= \frac{\alpha g_{juc}(x)}{[1-(1-\alpha)[q]][\alpha q]} \tag{30}$$

$X \sim EGJD$

$$\underline{F}_{EGJD}(x) = 1 - [1-q]^{\alpha} = 1 - \sum_{i}^{\infty} \binom{\alpha}{i}(-1)^{i}q^{i} \tag{31}$$

$$h_{EGJD} = \frac{\alpha\beta\, g_{juc}(x)(q)^{\alpha-1} \sum_{i}^{\infty} \binom{\beta-1}{i}(-1)^{i}\, q^{i\alpha}}{1-\sum_{i}^{\infty} \binom{\alpha}{i}(-1)^{i}\, q^{i}} \tag{32}$$

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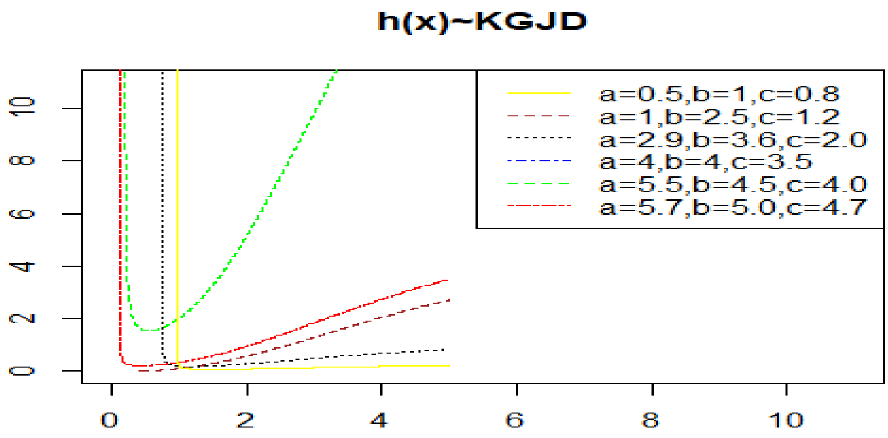

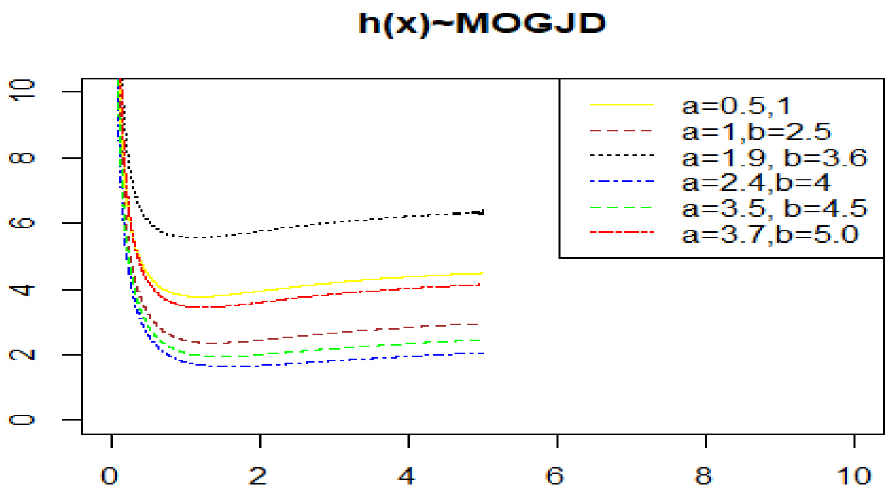

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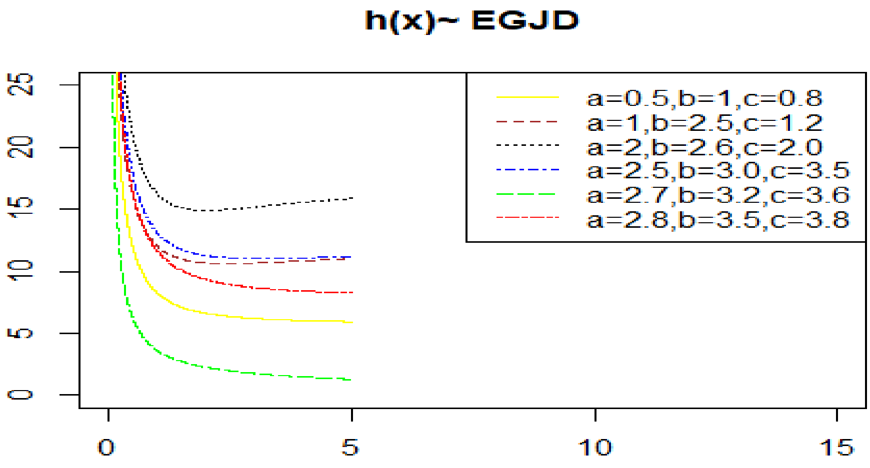

Figure 2 (a),(b),(c)&(d). The hazard function of the G-Class distributions

The exploration of the intractable ''Juchez'' distribution in G-Class distributions as seen in figure 2 reveals a trend, an L-shape hazard model, which projects the capacity of the distributions to model outcomes that exhibit short-lived decreasing rate followed by a constant rate. In other words, the derived G-Class distributions are fit for modeling outcomes that do not wear out as time or cycle elapses.

Mean Time to Failure (MTTF) and Mean Residual Life (MRL).

Using Laplace transform given as $"l^*(t) = \int_0^\infty e^{-tx}u(x)dt"$, we derive the MTTF of the distributions, where the Laplace transform of the survivor function is

$$\underline{F}^*(t) = \int_0^\infty e^{-tx}\,\underline{F}(x)\,dx \qquad (33)$$

When $t = 0$, we obtain

$$\underline{F}^*(0) = \int_0^\infty \underline{F}(x)\,dx = MTTF = \mu \qquad (34)$$

The mean residual (remaining) life of an item at age t is

$$MRL(t) = \frac{1}{\underline{F}(x)}\int_t^\infty \underline{F}(x)\,dx = \mu(t) \qquad (35)$$

For intractability reasons, these are left without application to the derived G-Class of distributions for further exploration.

Parameter Estimation

Let $X_i, i = 1,2,3,\dots,n,$ be a random variable from KGJD, the log-likelihood function $lnLf(x,\theta,\alpha,\beta)$, is obtained as:

$$lnLf(x,\theta,\alpha,\beta) = 4nln\theta + nln\alpha + nln\beta - nln(\theta^3 + \theta^2 + 6) + \sum_{i=1}^{n} ln(1 + x_i + x_i^3)$$

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$$-\theta \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} lnA_i(\theta) + (\beta - 1) \sum_{i=1}^{n} lnA_i(\theta,\alpha) \quad (36)$$

$$A_i(\theta) = 1 - \left(1 + \frac{\theta^3 x_i + \theta^3 x_i^3 + 3\theta^2 x_i^2 + 6\theta x_i}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x_i}$$

$$A_i(\theta,\alpha) = 1 - \left(1 - [1 + \frac{\theta^3 x_i + \theta^3 x_i^3 + 3\theta^2 x_i^2 + 6\theta x_i}{\theta^3 + \theta^2 + 6}] e^{-\theta x_i}\right)^{\alpha}$$

The MLE's are maximized at $\frac{\partial\, lnL}{\partial\theta} = \frac{\partial\, lnL}{\partial\alpha} = \frac{\partial\, lnL}{\partial\beta} = 0$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2 + 2\theta)}{\theta^3 + \theta^2 + 6} - \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \frac{A_i'(\theta)}{A_i(\theta)} + (\beta - 1) \sum_{i=1}^{n} \frac{A_{i\theta}'(\theta,\alpha)}{A_i(\theta,\alpha)} \quad (37)$$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} lnA_i(\theta) + (\beta - 1) \sum_{i=1}^{n} \frac{A_{i\alpha}'(\theta,\alpha)}{A_i(\theta,\alpha)} \quad (38)$$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} lnA_i(\theta,\alpha) \quad (39)$$

$$A_{i\theta}'(\theta,\alpha) = \frac{\partial A_i(\theta,\alpha)}{\partial\theta} \text{ and } A_{i\alpha}'(\theta,\alpha) = \frac{\partial A_i(\theta,\alpha)}{\partial\alpha}$$

Let $X_i, i = 1,2,3,\ldots,n,$ be a random variable from MOGJD, the log-likelihood function $lnLf(x,\theta,\alpha)$, is obtained as:

$$lnLf(x,\theta,\alpha) = 4nln\theta + nln\alpha - nln(\theta^3 + \theta^2 + 6) + \sum_{i=1}^{n} ln(1 + x_i + x_i^3) - \theta \sum_{i=1}^{n} x_i$$

$$-2 \sum_{i=1}^{n} lnB_i(\theta,\alpha) \quad (40)$$

$$B_i(\theta,\alpha) = 1 - (1 - \alpha)\left(1 + \frac{\theta^3 x_i + \theta^3 x_i^3 + 3\theta^2 x_i^2 + 6\theta x_i}{\theta^3 + \theta^2 + 6}\right) e^{-\theta x_i}$$

The MLE's are maximized at $\frac{\partial\, lnLf}{\partial\theta} = \frac{\partial\, lnLf}{\partial\alpha} = 0$

$$\frac{\partial\, lnLf(x,\theta,\alpha)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2 + 2\theta)}{\theta^3 + \theta^2 + 6} - \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \frac{B_{i\theta}'(\theta,\alpha)}{B_i(\theta,\alpha)} \quad (41)$$

$$\frac{\partial\, lnLf(x,\theta,\alpha)}{\partial\alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{B_{i\alpha}'(\theta,\alpha)}{B_i(\theta,\alpha)} \quad (42)$$

$$B_{i\theta}'(\theta,\alpha) = \frac{\partial B_i(\theta,\alpha)}{\partial\theta} \text{ and } B_{i\alpha}'(\theta,\alpha) = \frac{\partial B_i(\theta,\alpha)}{\partial\alpha}$$

Let $X_i, i = 1,2,3,\ldots,n,$ be a random variable from EGJD, the log-likelihood function $lnLf(x,\theta,\alpha,\beta)$, is obtained as:

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$$lnLf(x,\theta,\alpha,\beta) = 4nln\theta + nln\alpha + nln\beta - nln(\theta^3 + \theta^2 + 6) + \sum_{i=1}^{n} ln\left(1 + x_i + x_i{}^3\right)$$

$$-\theta \sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n} lnE_i(\theta) + (\beta - 1)\sum_{i=1}^{n} lnE_i(\theta,\alpha) \quad (43)$$

Where $E_i(\theta) = \left(1 + \frac{\theta^3 x_i + \theta^3 x_i{}^3 + 3\theta^2 x_i{}^2 + 6\theta x_i}{\theta^3 + \theta^2 + 6}\right)e^{-\theta x_i}$

$$E_i(\theta,\alpha) = \left[1 - \left((1 + \frac{\theta^3 x_i + \theta^3 x_i{}^3 + 3\theta^2 x_i{}^2 + 6\theta x_i}{\theta^3 + \theta^2 + 6})e^{-\theta x_i}\right)^{\alpha}\right]$$

The MLE's are maximized at $\frac{\partial\, lnL}{\partial\theta} = \frac{\partial\, lnL}{\partial\alpha} = \frac{\partial\, lnL}{\partial\beta} = 0,$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2 + 2\theta)}{\theta^3 + \theta^2 + 6} - \sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n} \frac{E_i'(\theta)}{E_i(\theta)} + (\beta - 1)\sum_{i=1}^{n} \frac{E_{i\theta}'(\theta,\alpha)}{E_i(\theta,\alpha)} \quad (44)$$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} lnE_i(\theta) + (\beta - 1)\sum_{i=1}^{n} \frac{E_{i\alpha}'(\theta,\alpha)}{E(\theta,\alpha)} \quad (45)$$

$$\frac{\partial\, lnLf(x,\theta,\alpha,\beta)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} lnE_i(\theta,\alpha) \quad (46)$$

$$E_{i\theta}'(\theta,\alpha) = \frac{\partial E_i(\theta,\alpha)}{\partial\theta},\ E_{i\alpha}'(\theta,\alpha) = \frac{\partial E_i(\theta,\alpha)}{\partial\alpha}$$

The ambiguity in the parameter estimation models can be efficiently and or numerically handled by the use of statistical software.

Statistical Orderings

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution. Let $X_1 < X_2 <, \dots, < X_n$ denote the corresponding order statistics. The PDF and the CDF of the kth order statistics, say $Y = X_k$ is given by:

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} f(y)\, F^{k-1}(y)\, \{1 - F(y)\}^{n-k} \quad (47)$$

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} f(y) \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l\, F^{k+l-1}(y) \quad (48)$$

$$F_Y(y) = \sum_{k=m}^{n} \binom{n}{k} F^k(y)\, \{1 - F(y)\}^{n-k} \quad (49)$$

$$F_Y(y) = \sum_{k=m}^{n} \sum_{l=0}^{n-k} \binom{n}{k}\binom{n-k}{l}(-1)^l\, F^{k+l}(y) \quad (50$$

Thus the pdf and the cdf of kth order statistics of KGJD distribution are given by

$$f_{Y-kgjd}(y) = \frac{n!\,\alpha\beta\, g_{juc}(y)}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{t=0}^{p} \binom{n-k}{l}\binom{k+l-1}{s}\binom{\alpha-1}{m}\binom{\beta-1}{p}\binom{p}{t}(-1)^{l+s+m+p+t}$$

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$$q^m(1-[1-q]^\alpha)^t \left\{\sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\beta}{i}\binom{i}{j}(-1)^{i+j}(1-[1-q]^\alpha)^j\right\}^i \quad (51)$$

$$F_{Y-kgjd}(y) = \sum_{k=m}^{n} \sum_{l=0}^{n-k} \sum_{j=0}^{\infty} \binom{n}{k}\binom{n-k}{l}\binom{i+l}{j}(-1)^{j+l}\left[\sum_{b=0}^{\infty} \binom{\beta}{b}(-1)^b \left(\sum_{m=0}^{\infty} \binom{\alpha}{m}(-1)^m q^m\right)^b\right]^j \quad (52)$$

The pdf and the cdf of kth order statistics of MOGJD distribution are given by

$$f_Y(y) = \frac{n!\,\alpha g_{juc}(y)}{(k-1)!(n-k)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{n-k} \binom{-2}{i}\binom{k+l-1}{j}\binom{n-k}{l}(-1)^{l+i+j}[(1-q)]^i \left\{\frac{\alpha q}{1-(1-\alpha)q}\right\}^j \quad (53)$$

$$F_Y(y) = \sum_{i=j}^{n} \sum_{l=0}^{n-i} \binom{n-1}{l}(-1)^l \sum_{k=0}^{\infty} \binom{i+l}{k}(-1)^k \left[\frac{\alpha q}{1-(1-\alpha)q}\right]^k \quad (54)$$

More so, the pdf and the cdf of kth order statistics of EGJD distribution are given by

$$f_Y(y) = \frac{n!\,\alpha\beta\, g_{juc}(y)}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \sum_{i=0}^{\infty} \binom{\beta-1}{i}\binom{n-k}{l}(-1)^{i+l} q^{\alpha i}\left\{\sum_{j=0}^{\infty} \binom{\beta}{j}(-1)^j q^{\alpha j}\right\}^{k+j-1} \quad (55)$$

$$F_Y(y) = \sum_{i=j}^{n} \sum_{l=0}^{n-i} \binom{n}{i}\binom{n-i}{l}(-1)^l \left\{\sum_{k=0}^{\infty} \binom{\beta}{k}(-1)^k q^{\alpha k}\right\}^{i+l} \quad (56)$$

Consequently, the pdf of minimum order statistics $f_{1:n}$ is obtained by substituting $k = 1$ in equations (51), (53) and (55); the corresponding pdf of maximum order statistics $f_{n:n}$ is obtained at $k = n$ in a like manner.

Inverse cumulative function

The mathematical expression for the G-Class quantile function is derived thus:

$$F(x) = u \quad \rightarrow \quad x = F^{-1}(u) \quad ; 0 \le u \le 1. \quad (57)$$

To obtain the quantile function for KGJD we recall equation (10).

$$u = 1 - \left\{1 - \left[1 - (1+w)e^{-\theta x}\right]^\alpha\right\}^\beta \quad (58)$$

$$(1-u)^{\frac{1}{\beta}} = 1 - \left[1 - (1+w)e^{-\theta x}\right]^\alpha \quad (59)$$

$$[1 - (1-u)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}} = 1 - (1+w)e^{-\theta x}$$

$$1 - [1 - (1-u)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}} = (1+w)e^{-\theta x}$$

$$ln\left\{1 - [1 - (1-u)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}}\right\} = ln(1+w) - \theta x$$

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KGJD quantile is given as

$$ln(1+w) - ln\left\{1 - [1-(1-u)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}}\right\} - \theta x = 0 \quad (60)$$

However, following the same procedure, the quantile functions for MOGJD and EGJD are respectively obtained as:

- $$ln(1+w) - ln\ ln\ (1-u)\ - \theta x(2-\alpha) = 0 \quad (61)$$
- $$ln(1+\ w) - ln\left(1-u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} - \theta x = 0 \quad (62)$$

where $u \sim uniform(0,1)$, and at $u = 0.5$, the median for the respective distributions are obtained.

C-Moments of G-Class distribution

The skewness and kurtosis as introduced by Kenney and Keeping (1962), and Moor (1986) respectively are obtained by appropriately making substitutions into $xQ(u,\omega)$, which represents the function for any of the derived inverse cumulative distributions.

$$X_{sk} = \frac{Q\left[\frac{9}{12},\omega\right]+Q\left[\frac{5}{20},\omega\right]-2Q\left[\frac{9}{18},\omega\right]}{Q\left[\frac{15}{20},\omega\right]-Q\left[\frac{6}{24},\omega\right]} \quad (63)$$

$$X_{k} = \frac{Q\left[\frac{14}{16},\omega\right]-Q\left[\frac{15}{24},\omega\right]-Q\left[\frac{12}{32},\omega\right]+Q\left[\frac{3}{24},\omega\right]}{Q\left[\frac{3}{4},\omega\right]-Q\left[\frac{5}{40},\omega\right]} \quad (64)$$

L-Moment of G-Class distribution

Hosking (1990) introduced this statistic to suffice for the limitation of the C-moments, which is - its high susceptibility to the presence of outliers. In other word, L-moments are not greatly influenced by wild observations as observed for the conventional moments.

Let X be a random variable distributed with the CDF $F(x)$ and quantile function $xQ(.)$; and let $X_{1:n} \le X_{2:n} \le L \le X_{n:n}$ be order statistics of a random sample of the sample size n which comes from the distribution of the random variable X. Then the statistics is defined as

$$ℷ_r = \frac{1}{r}\sum_{j=0}^{r=1} (-1)^j \binom{r-1}{j} E(X_{r-j:r}),\ r = 1,2,L$$

(65)

Where the expected value of the r-th order statistics of the random sample of size n is

$$E(X_{r:n}) = \frac{n!}{(r-1)!j!}\int_0^1 xQ(.)[F(x)]^{r-1}[1-F(x)]^{n-r}dF(x) \quad (66)$$

$$\rightarrow \quad ℷ_r = \frac{1}{r}\sum_{j=0}^{r=1} (-1)^j \binom{r-1}{j} \frac{r!}{(r-j-1)!j!}\int_0^1 xQ(.)[F(x)]^{r-1}[1-F(x)]^{n-r}dF(x) \quad (67)$$

$$\rightarrow \quad ℷ_{r=1} = \int_0^1 xQ(.)dF(x) \quad (68)$$

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$$\beth_{r=2} = \int_0^1 \; xQ(.)[2F(x) - 1]\, dF(x) \tag{69}$$

$$\beth_{r=3} = \int_0^1 \; xQ(.)\{6[F(x)]^2 - 6F(x) + 1\}\, dF(x) \tag{70}$$

$$\beth_{r=4} = \int_0^1 \; xQ(.)\{20[F(x)]^3 - 30[F(x)]^2 + 12F(x) - 1\}\, dF(x) \tag{71}$$

Demonstrating with KGJD, the L- moments are derived thus

$$\beth_{r=1} = \int_0^1 \; xQ_{KGJD}(.)dF_{KGJD}(x) = xQ_{KGJD}(.)F_{KGJD}(x)]_0^1 \tag{72}$$

$$\beth_{r=2} = \int_0^1 \; xQ_{KGJD}(.)[2F_{KGJD}(x) - 1]\, dF(x) = \left\{xQ_{KGJD}(.)\{\,[F_{KGJD}(x)]^2 - F_{KGJD}(x)\}\right\}]_0^1$$
(73)

$$\beth_{r=3} = \int_0^1 \; xQ_{KGJD}(.)\left\{6[F_{KGJD}(x)]^2 - 6F(x) + 1\right\}\, dF_{KGJD}(x)$$

$$= \left\{xQ_{KGJD}(.)\{\,2[F_{KGJD}(x)]^3 - [F_{KGJD}(x)]^2 + F_{KGJD}(x)\}\right\}]_0^1 \tag{74}$$

$$\beth_{r=4} = \int_0^1 \; xQ_{KGJD}(.)\left\{20[F_{KGJD}(x)]^3 - 30[F_{KGJD}(x)]^2 + 12F_{KGJD}(x) - 1\right\}\, dF_{KGJD}(x)$$

$$= \left\{xQ_{KGJD}(.)\{\,5[F_{KGJD}(x)]^4 - 10[F_{KGJD}(x)]^3 + 6[F_{KGJD}(x)]^2 - F_{KGJD}(x)\}\right\}]_0^1 \tag{75}$$

In like manner, other L - moments can be obtained across the derived G-Class of distributions.

MODEL FIT COMPARISON

In this section, we carry out a model fit comparative study among tractable distributions and the intractable distribution as expressed in equation (1), all in the derived G-Class dimensions. However, we list out the PDFs and CDFs of the baseline tractable distributions as would be used in this comparison; and in their G-Class dimensions.

Table 1: Tractable Distributions

Model	PDF	CDF
L – Lindley	$\theta^2 e^{-\theta x}(1+x)(\theta+1)^{-1}$	$1 - e^{-\theta x}(1+\theta+\theta x)(1+\theta)^{-1}$
Gu – Gumbel	$e^{-\left(x+e^{-(\beta^{-1}(x-\mu))}\right)}$	$e^{-e^{-(\beta^{-1}(x-\mu))}}$
F – Fretchet	$\beta\alpha^{\beta}x^{-(\beta+1)}e^{-\left[\frac{\alpha}{x}\right]^{\beta}}$	$e^{-\left[\frac{\alpha}{x}\right]^{\beta}}$
E – Exponential	$\theta e^{-\theta x}$	$1 - e^{-\theta x}$
P – Pareto	$\alpha x[x^{\alpha+1}]^{-1}$	$1 - [x^{\alpha}]^{-1}$
W – Weibul	$\alpha\gamma^{-1}[x\gamma^{-1}]^{\alpha-1}$	$1 - e^{-(x\gamma^{-1})^{\alpha}}$
Lo – Lomax	$\alpha\gamma^{-1}[1+x\gamma^{-1}]^{-(\alpha+1)}$	$1 - [1+x\gamma^{-1}]^{-\alpha}$

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Data Description and Application

Data Set 1: Lifetime data on three months (daily) average one liners in a live streaming program. The firm targets an average number of 10000 viewers as a threshold for outreach-success. Echebiri and Mbegbu (2022)

11.2, 10.9, 13.2, 12.0, 11.5, 11.1, 10.8, 10.3, 13.8, 12.5, 12.3, 12.0, 11.1, 13.7, 14.3, 15.3, 13.1, 12.0, 11.8, 10.9, 13.5, 12.6, 13.4, 14.2, 11.6,13.7, 12.6, 11.6, 14.0, 11.0, 13.6, 12.0, 11.5, 11.9, 10.7, 12.6, 12.5, 13.7, 13.5, 12.4, 13.0, 13.2, 12.0, 14.3, 14.3, 12.5, 11.0, 12.0, 13.2, 12.0, 13.5, 13.2, 12.5, 11.6, 14.0, 12.9, 10.5, 13.4, 14.0, 10.5, 12.6, 13.4, 14, 12.6, 13, 12.8, 13.7, 12.7, 13.6, 14.5, 13.4, 12.9, 11.0, 15.1, 13.6, 12.4, 12.9, 11.2, 10.7, 12.3, 13.5, 12.6, 13.5, 12.3, 13.5, 12.4, 12.3, 11.2, 13.5, 10.3, 11.3 (in thousands).

Data Set II: Lifetime data primarily obtained from 80 days calibration of oil/winding temperature (measured in Celsius). These measurements indicate system (transformer) normalcy, and once the temperature reads above 65˚C, the system has gone wrong or is said to have failed (Maintenance department of First Independent Power Limited Company, Afam, River State, Nigeria).

60.5, 40.3, 50.2, 65, 48.6, 59.6, 60.4, 49.3, 43.2, 46.9, 53.1, 58.6, 47.6, 63.2, 49.0, 45.7, 54.1, 62.4, 60.8, 40.9, 48.6, 49.7, 58.7, 60.3, 47.9, 45.2, 46.3, 50.7, 52.3, 49.6, 54.3, 63.7, 60.2, 63.1, 60.4, 65.0, 49.5, 49.5, 46.9, 50.8, 43.8, 47.3, 47.5, 48.0, 64.8, 49.0, 50.2, 57.6, 62.1, 48.7, 60.7, 56.7, 59.7, 53.6, 61.0, 62.1, 54.9, 54.0, 64.7, 53.9, 48.3, 56.9, 49.7, 63.8, 64.1, 63.8, 60.0, 45.3, 65.0, 63.2, 64.2, 46.2, 48.3, 46.3, 48.3, 57.9, 59.6, 53.9, 56.3, 64.0.

Data Set III: Recuperation time (days) of 152 covid-19 patients who were tested and received treatment using sotrovimab at Federal Medical Center, Asaba, Nigeria.

16, 14, 17, 16, 30, 44, 25, 25, 26, 38, 42, 40, 37, 41, 36, 27, 39, 41, 37,18,30,27,19,25,18,37,42,44,14, 21,29,18,19,40,43,23,18,38,39,34,44,27,14,16,41,18,19,20,35,41,36,41,17,40,39,41,43,15,39, 39,43,17,29,40,38,42,16,25,24,28,19,14,40,37,14, 40, 37, 22, 43, 17, 19, 38, 35, 41, 20, 16, 19, 33, 22, 17, 18, 44, 32, 18, 29, 31, 19,28, 21, 40, 31, 28, 18, 14,30, 44, 31, 17, 28,25, 40, 15, 18, 20, 39, 25, 26, 38, 42, 40, 37, 41, 36, 27, 39, 41, 37,18,30,27,19,25,18,37,42,44,14, 21,29,18,19,40,43,23,18,38,39,34,44,27,14,16

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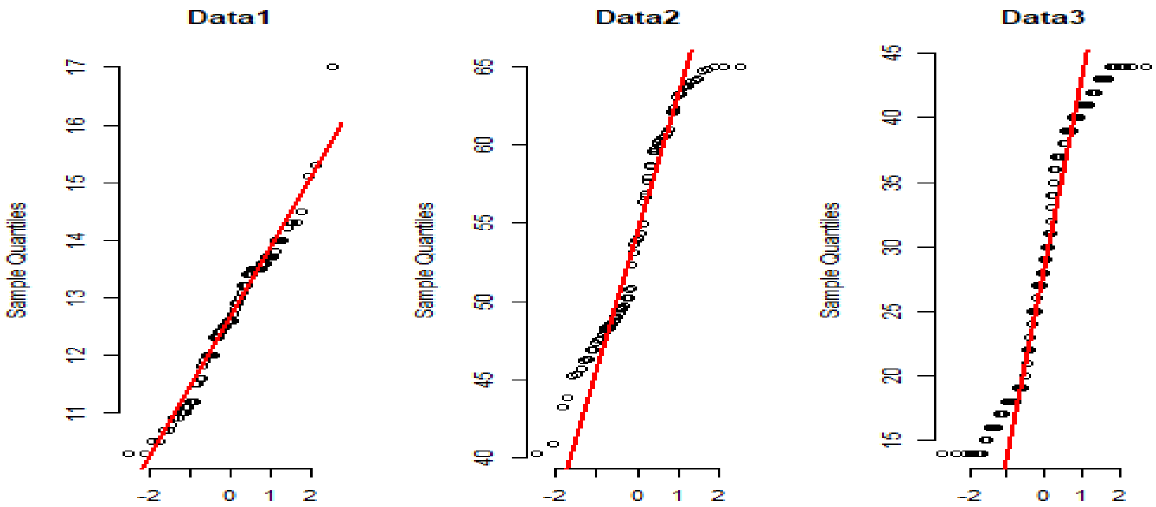

Figure 3: Q-Q plot data representation

Table 2: Performance Comparison for EGJD

Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
EGJD	$\theta = 0.17223$ $\alpha = 22.6685$ $\beta = 53.6811$	-148.74	303.49	311.02	1
EGLD	$\theta = 0.01455$ $\alpha = 379.147$ $\beta = 430.988$	-149.41	304.82	312.36	2
EGGuD	$\theta = 3.65438$ $\alpha = 0.36763$ $\beta = 0.13935$ $\gamma = 19.8501$	-190.64	389.27	399.31	4
EGFD	$\theta = 16.0947$ $\alpha = 8.04911$ $\beta = 1.92883$ $\gamma = 0.10928$	-153.49	314.97	325.02	3
EGED	$\theta = 2.13527$ $\alpha = 0.09748$ $\beta = 8.92881$	-238.56	483.12	490.65	5

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Table 3: Performance Comparison for KGJD

Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
KGJD	$\theta = 0.1089$ $\alpha = 13.0502$ $\beta = 6.2225$	- 269.06	544.12	551.27	1
KGLD	$\theta = 0.6134$ $\alpha = 2.6719$ $\beta = 0.0338$	-429.34	864.68	871.83	4
KGED	$\theta = 0.5265$ $\alpha = 0.1014$ $\beta = 0.0322$	-403.71	813.41	820.56	3
KGPD	$\theta = 6.0412$ $\alpha = 0.1874$ $\beta = 0.0281$ $\varphi = 4.3515$	-287.62	583.25	592.78	2

Table 4: Performance Comparison for MOGJD

Model	Parameter Estimate	lnL	AIC	BIC	Rank
MOGJD	$\theta = -0.04487$ $\alpha = 2.12097$	336.23	-668.46	-662.41	1
MOG-GED	$\theta = 2.9709$ $\alpha = 12.867$ $\beta = 0.1331$	-567.40	1140.80	1149.87	2
MOG-GWD	$\theta = 0.0216$ $\alpha = 1.6623$ $\beta = 0.0743$	-573.08	1152.17	1161.242	3
MOG-GLoD	$\theta = 116.98$ $\alpha = 57.335$ $\beta = 337.74$	-573.97	1153.95	1163.022	4

From tables 1, 2 and 3, we could deduce comparatively following the scientific criteria (AIC and BIC), that EGJD, KGJD and MOGJD show to fit better in the modeling of the data respectively.

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CONCLUSION

Conventionally, as studied, it is a common routine for researchers to develop models based on ease and convenience. Choice for tractability is usually a norm, and this stems from the fact that most G-Class or compound distributions are already complex or cumbersome in structure. So, to avoid much calculation or mathematical bottlenecks, tractability becomes a favorable factor option. Consequently, the study sought to project the ignored relevance of model development using intractable distributions. The distribution exemplified as intractable here is the Juchez probability distribution; this is due to its mathematical structure. Some properties were derived, for example the shape of the PDF, reliability, hazard and cumulative hazard function, parameter estimation, statistical ordering, inverse cumulative function, C-moments and L-moments. However, the difficulty in deriving some properties that depend on mathematical integration (for example: mean, moment, moment generating function, Characteristic function, MTTF, MRL etc.), even with the use of software, proves the intractability of the distribution. In more explicit terms, having enriched polynomial components in a distribution could limit mathematical efficiency. The fact notwithstanding, the intractable nature of the distribution does not affect its fitness in data modeling as seen in the study; hence, researchers are recommended to also embrace sophisticated distributions in model development.

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