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TRANSMUTED TOPP-LEONE EXPONENTIAL DISTRIBUTION: THEORY AND APPLICATION TO REAL DATASET

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ABSTRACT: *The main aim of this study is to add to the existing literature on probability distributions. In this study, the transmutation map approach proposed by Shaw and Buckley (2007) was used to develop a probability distribution called Transmuted Topp-Leone Exponential (TTLE) distribution. The moment, moment generating function and entropy are among the statistical properties of the distribution that were derived. The maximum likelihood approach was used to estimate the parameters of the novel distribution. The TTLE distribution was applied to a real-world data set and compared to other well-known standard distributions; the result of the analysis revealed that the newly developed distribution is more superior than the competing models.*

KEYWORDS: Exponential distribution, Transmuted Map, Maximum likelihood estimation method, Entropy, Moments.

INTRODUCTION

The exponential distribution is one of the popular probability distributions used in modeling real-world data in reliability engineering. However, this distribution does not have a bath tub or upside-down bathtub shaped hazard rate function, which is why it cannot be utilized to model the lifetime of certain systems. To overcome this shortcoming, several generalizations of the classical exponential distribution have been discussed by different authors in recent years. Many authors introduced flexible distributions for modeling complex data and obtaining a better fit. Some of the flexible distributions introduced recently include: transmuted Topp-Leone Weibull proposed by Ibrahim and Yousof (2020), transmuted exponential Topp-Leone developed by Mohammed and Ugwuowo (2021), Transmuted Topp-Leone Power Function Distribution introduced by Hassan *et al.* (2021), Topp-Leone odd log-logistic exponential distribution proposed by Afify *et al.* (2021), transmuted Topp-Leone Flexible Weibull proposed by Alobaidi et al. (2021), Gompertz inverse rayleigh developed by Halid and Sule (2022), and sine-exponential distribution introduced by Isa *et al.* (2022).

Recently, the use of the classical models in Biomedical Sciences, engineering and other fields revealed that some of the data sets cannot adequately be modeled using existing classical distributions. As a result, many researchers proposed new generalizations of the existing distributions in order to provide a better fit for these data sets. In this article, we propose a new distribution called the Transmuted Topp-Leone Exponential Distribution by using the transmutation map approach proposed by Shaw and Buckley (2007).

Definition: A random variable X is said to have transmuted distribution if its distribution function is given by:

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \qquad (1)$$

with corresponding pdf given by

$$f(x) = g(x)[(1+\lambda) - 2\lambda G(x)] \qquad (2)$$

where $g(x)$ and $G(x)$ are the cdf and pdf of the base distribution. It is important to note that at $\lambda = 0$; we have the distribution of the base random variable (Shaw *et al.*, 2009).

Transmuted Topp-Leone G Family

A positive random variable *x* is to be Transmuted Topp-Leone G Family if it is given by:

$$f(x) = 2\alpha g(x)[1-G(x)]\{1-[1-G(x)]^2\}^{\alpha-1}\{1+\lambda - 2\{1-[1-G(x)]^2\}^{\alpha}\} \qquad (3)$$

and the corresponding cdf is given by:

$$F(x) = (1+\lambda)\{1-[1-G(x)]^2\}^{\alpha} - \lambda\{1-[1-G(x)]^2\}^{2\alpha}, x \geq 0 \qquad (4)$$

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where $G(x)$ is the cumulative distribution function of the baseline distribution, and $\alpha > 0$ and $|\lambda| \leq 1$ are the two additional shape parameters. The Transmuted Topp-Leone G is a wider class of continuous distributions. It includes the Transmuted (TM) and Topp Leone (TL) families of distributions.

Transmuted Topp Leone-Exponential Distribution

A positive random variable *x* is to be Transmuted Topp-Leone exponential (TTLE) distribution if its probability density function is given by:

$$f(x;\alpha,\gamma,\lambda) = 2\alpha\gamma e^{-2\gamma x}[1-e^{-2\gamma x}]^{\alpha-1}\{1+\lambda - 2[1-e^{-2\gamma x}]^{\alpha}\} \quad (5)$$

and the corresponding cdf is given by:

$$F(x;\alpha,\gamma,\lambda) = (1+\lambda)[1-e^{-2\gamma x}]^{\alpha} - \lambda[1-e^{-2\gamma x}]^{2\alpha} \quad (6)$$

where $|\lambda| \leq 1$. The survival function $S(x)$, the hazard function $h(x)$, reverse hazard function $r(x)$, cumulative hazard function $H(x)$ and quantile function $Q(u)$ are presented as follows:

$$S(x) = 1 - \{(1+\lambda)(1-e^{-2\gamma x})^{\alpha} - \lambda(1-e^{-2\gamma x})^{2\alpha}\} \quad (7)$$

$$h(x) = \frac{2\alpha\gamma e^{-2\gamma x}[1-e^{-2\gamma x}]^{\alpha-1}\{1+\lambda-2\lambda[1-e^{-2\gamma x}]^{\alpha}\}}{1-[(1+\lambda)[1-e^{-2\gamma x}]^{\alpha} - \lambda[1-e^{-2\gamma x}]^{2\alpha}]} \quad (8)$$

$$r(x) = \frac{2\alpha\gamma e^{-2\gamma x}[1-e^{-2\gamma x}]^{\alpha-1}\{1+\lambda-2\lambda[1-e^{-2\gamma x}]^{\alpha}\}}{[(1+\lambda)[1-e^{-2\gamma x}]^{\alpha} - \lambda[1-e^{-2\gamma x}]^{2\alpha}]} \quad (9)$$

$$H(x) = -\, ln\, ln\, \{1 - [(1+\lambda)[1-e^{-2\gamma x}]^{\alpha} - \lambda[1-e^{-2\gamma x}]^{2\alpha}]\} \quad (10)$$

$$Q(u) = F^{-1}\left\{\frac{1}{2\lambda}\, ln\, ln\left[\left(\frac{(1+\lambda)+[(1+\lambda^2)-4\lambda u]^{\frac{1}{2}}}{2\lambda}\right)^{\frac{1}{\alpha}} - 1\right]\right\} \quad (11)$$

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Mixture Representation

The pdf of the Transmuted Topp-Leone exponential distribution can be expressed as follows:

$$f(x;\alpha,\gamma,\lambda) = 2\alpha\gamma e^{-2\gamma x}[1-e^{-2\gamma x}]^{\alpha-1}\{1+\lambda-2\lambda[1-e^{-2\gamma x}]^{\alpha}\}$$

Using power series expansion, $e^{-2\gamma x}$ can be written as:

$$e^{-2\gamma x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (2\lambda x)^i$$

$$[1-e^{-2\gamma x}]^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} e^{-2j\gamma x}$$

$$e^{-2j\gamma x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (2jkx)^k$$

$$[1-e^{-2\gamma x}]^{\alpha} = \sum_{l=0}^{\infty} (-1)^l \binom{\alpha}{l} e^{-2l\gamma x}$$

$$e^{-2l\gamma x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (2l\gamma x)^m$$

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{2(1+\lambda)j^k l^m \gamma^{i+k+m}(-1)^{i+j+k+l+m}}{i!\,k!\,m!} \binom{\alpha-1}{j}\binom{\alpha}{l} x^{i+k+m}$$

Therefore, the pdf can be written as:

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \Psi\, x^{i+k+m} \qquad (12)$$

where

$$\Psi = \frac{2j^k l^m \gamma^{i+k+m}(-1)^{i+j+k+l+m}}{i!\,k!\,m!} \binom{\alpha-1}{j}\binom{\alpha}{l}$$

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Some Mathematical Properties

Moments

In this subsection, we discuss the r^{th} moment for Transmuted Topp-Leone exponential (TTLE) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). We start with the well-known definition of the r^{th} moment of the random variable X with probability density function $f(x)$ given by:

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi \int_0^{\infty} x^{r+i+k+m}\, dx$$

$$E(X^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi\, \aleph \qquad (13)$$

where

$$\aleph = \int_0^{\infty} x^{r+i+k+m}\, dx$$

Moment Generating Function

The moment generating function of a random variable X is the expected value of e^{tx}, that is:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

We say that the moment generating function exists if there exists a positive constant a such that $M_x(t)$ is finite for all $t \in [-a, a]$. Therefore, the moment generating function for the Transmuted Topp-Leone exponential distribution is given by:

$$M_x(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi \int_0^{\infty} e^{tx} x^{r+i+k+m}\, dx$$

$$M_x(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi\, H \qquad (15)$$

where

$$H = \int_0^{\infty} e^{tx} x^{r+i+k+m}\, dx$$

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Entropy

Entropy is used as a measure of information or uncertainty which is present in a random observation of its actual population. There will be a greater uncertainty in the data if the value of entropy is large. The entropy for the true continuous random variable X is defined as:

$$I_y(x) = \frac{1}{1-\theta} log\, log \int_{-\infty}^{\infty} f(x)^{\theta}\, dx$$

For the Transmuted Topp-Leone exponential distribution, the entropy is given by:

$$f(x)^{\theta} = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi\, x^{i+k+m} \right)^{\theta}$$
$$= \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi \right)^{\theta} (E)^{\theta}$$

where $E = x^{i+k+m}$

$$I_y(x) = \frac{1}{1-\theta} \left[\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Psi \right)^{\theta} log\, log \int_{0}^{\infty} \mathcal{E}^{\theta}\, dx \right] \quad (16)$$

Parameter Estimation

Method of Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the Transmuted Topp-Leone exponential distribution with pdf given in Equation (6), the log-likelihood function $l(\alpha,\, \gamma, \lambda)$ of the Transmuted Topp-Leone exponential distribution is given by:

$$l = n\, log\, log\, 2 + n\, log\, log\, \alpha + n\, log\, log\, \gamma - 2\gamma \sum_{i=1}^{n} x + (\alpha - 1) \sum_{i=1}^{n} log\, log\, [1 - e^{-2\gamma x}]$$
$$+ \sum_{i=1}^{n} log\, log\, [1 + \lambda - 2(1 - e^{-2\gamma x})^{\alpha}] \quad (17)$$

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Differentiating the log likelihood with respect to α, γ and λ gives:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \quad log\, log\, (1 - e^{-2\gamma x}) - \sum_{i=1}^{n} \quad \frac{(1 - e^{-2\gamma x})^{\alpha}\, ln\, ln\, (1 - e^{-2\gamma x})}{[1 + \lambda - 2(1 - e^{-2\gamma x})^{\alpha}]} \tag{18}$$

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n} \quad x + 2(\alpha - 1)\sum_{i=1}^{n} \quad x\, \frac{e^{-2\gamma x}}{(1 - e^{-2\gamma x})} - \sum_{i=1}^{n} \quad \frac{4\alpha\lambda x\, e^{-2\gamma x}(1 - e^{-2\gamma x})^{\alpha - 1}}{[1 + \lambda - 2(1 - e^{-2\gamma x})^{\alpha}]} \tag{19}$$

$$\frac{\partial l}{\partial \lambda} = [1 + \lambda - 2(1 - e^{-2\gamma x})^{\alpha}] \tag{20}$$

Application

This section presents the application of the new distribution using real data sets. We shall compare the fit of the new distribution with the baseline distribution (exponential), Weibull Inverse Weibull (W-IW), Exponentiated Inverse Weibull (E-IW) and Transmuted Inverse Weibull (T-IW). The pdfs of the competitive model are available in statistical literature. The unknown parameters of the above pdfs are all positive real numbers except for the T-IW distribution for which $|\lambda| \leq 1$. The data set consists of 100 observations of breaking stress of carbon fibers given by Nichols and Padgett (2006):

0.920, 0.9280, 0.997, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.814, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.3060

The result of the analysis of the breaking stress of carbon fibers was analyzed using the newly developed Transmuted Topp-Leone exponential distribution and its performance was compared with some competitors based on their AIC, BIC, CAIC and HGIC.

The summary of the analysis is presented in Table 1 below:

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Table 1: The Statistics AIC, BIC, CAIA and HQIC for Breaking Stress of Carbon Fibers Data

Distribution	AIC	$CAIC$	BIC	$HQIC$
Transmuted Topp-Leone Exponential	116.2	116.5	119.5	119.5
Exponential	303.1	303.1	310.3	304.2
Weibull Inverse Weibull	294.5	294.9	304.9	298.7
Exponentiated Inverse Weibull	295.7	296.0	303.5	298.9
Transmuted Inverse Weibull	350.5	351.6	358.3	353.6

Table 1 compares the Transmuted Topp-Leone exponential model with other important competitive distributions. The T-TLE model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics among all fitted models to these data. So, it may be considered as the best model among them.

CONCLUSION

A new extension of the Transmuted Topp-Leone model is proposed and studied. Some of its fundamental statistical properties such as the moment, moment generating function and entropy were derived. The parameters of the new model were estimated using Maximum Likelihood Method. The practicability and the usability of the model was shown using a real life data set.

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