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CALCULATION OF A CLASS OF CONFLUENT HYPERGEOMETRIC EQUATION AND ANALYSIS OF ITS ROLES IN OPTION PRICING MODELS

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ABSTRACT: *The confluent hypergeometric equation is one of the most important differential equations in physics, chemistry, finance and many more. This work deals with the power series solution of a class of confluent hypergeometric equation with α, a real constant and z, an independent variable. The confluent hypergeometric function of the first kind $M(\alpha, \alpha+2, z)$ is derived together with the second power series solution, $\widetilde{M}(\alpha, \alpha+2, z)$. The analysis of the roles of the derived function in option pricing models are given.*

KEYWORDS: Confluent Hypergeometric Equation, Power Series, Option Pricing, Regular Singular Point.

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INTRODUCTION

The confluent hypergeometric equation is an important differential equation that arises in optics [1,2], general relativity [3,4], finance and many other areas. Its solution depends in an essential way on whether or not $\alpha, \alpha + 2$ and $\alpha - (\alpha + 2)$ are integers. Solutions of confluent hypergeometric equations are defined by the confluent hypergeometric functions [5]. [6] acquired new generalized entropies by using the confluent hypergeometric function of the first type. Okkes [7] used fractional calculus theory to solve some classes of singular differential equations and fractional order differential equations. [8] obtained the particular solutions of the confluent hypergeometric equation by using the nabla fractional calculus operator. [9] expressed some Voltera-type fractional integro-differential equations with a multivariable confluent hypergeometric function as their kernel. [6,9] considered the general solution of the stationary state Schrodinger equation in terms of confluent hypergeometric functions. [12,13] employed confluent hypergeometric functions in a discussion of the bound and continuum states of the hydrogen atom and other problems in quantum mechanics. [14] considered the use of confluent hypergeometric functions in determining the bound states of the attractive Coulomb potential. Confluent hypergeometric functions have also been used in some areas of finance. Hence, [15] used the function in an option-pricing approach to investment. [16] used it in pricing of callable bonds. Also, [17] showed that this function is necessary to represent densities associated with the minimal sufficient functionals of Ornstein-Uhlenbeck processes, and all the related processes satisfying an invariance principle. The reason for the success of confluent hypergeometric function in these applications is that it includes as special cases the incomplete-gamma and the normal distribution functions, in addition to mixtures of the function, which makes certain classes of these functions closed under such operations. Based on these features, a confluent hypergeometric function can be seen as a natural tool to model option prices and more generally, functionals of densities. In this work, we present the power series solution of a class of confluent hypergeometric equations and its uses in option pricing models.

Power Series

An expression of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is said to be a power series about $z = z_0$. If $z_0 = 0$, the power series reduces to $\sum_{k=0}^{\infty} a_k z^k$, which is a power series about zero. The series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is said to converge at a point z if $\sum_{k=0}^{n} a_k(z - z_0)^k$ exists. Furthermore, the series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges absolutely if $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges. There is a number $r \geq 0$, called the radius of convergence such that $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges for $|z - z_0| < r$ and diverges otherwise. The series that converges only at $z = z_0$ is said to have the radius of convergence $r = 0$ while when it converges for all values of z, r is infinite.

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Ordinary and Singular Points

Consider the linear differential equation

$$P(z)w'' + Q(z)w' + R(z)w = f(z), \tag{1}$$

with polynomial coefficients. The point $z = z_0$ is called an ordinary point of equation (1) if $P(z_0) \neq 0$. A singular point of equation (1) is any point $z = z_1$, for which $P(z_1) = 0$. To be able to classify a singular point of a linear differential equation of the type of equation (1) as regular or irregular, we have to divide equation (1) by $P(z)$ to have

$$w'' + \frac{Q(z)}{P(z)}w' + \frac{R(z)}{P(z)}w = \frac{f(z)}{P(z)}. \tag{2}$$

If the point $z = z_0$ is a singular point of equation (1), then $\frac{Q(z_0)}{P(z_0)}$ and $\frac{R(z_0)}{P(z_0)}$ do not exist. By redefining

$$\frac{Q(z)}{P(z)} = \frac{r_1(z)}{(z-z_0)^m} \text{ and } \frac{R(z)}{P(z)} = \frac{r_2(z)}{(z-z_0)^n}, \tag{3}$$

we immediately see that if $m \leq 1$ and $n \leq 2$,

$$(z - z_0)\,\frac{Q(z)}{P(z)} = (z - z_0)\,\frac{r_1(z)}{(z-z_0)^m} = r_1(z_0) \tag{4}$$

or zero, which is finite. Also,

$$(z - z_0)^2\,\frac{R(z)}{P(z)} = (z - z_0)^2\,\frac{r_2(z)}{(z-z_0)^n} = r_2(z_0) \tag{5}$$

or zero, which is finite. This shows that the singularity at $z = z_0$ is removable and such a singular point is called a regular singular point. A singular point that is not a regular singular point is called an irregular singular point.

Solutions near an Ordinary Point

Consider the equation

$$P(z)w'' + Q(z)w' + R(z)w = 0, \tag{6}$$

where P, Q and R are polynomial coefficients. The solution of equation (6) exists by the existence theorem since P, Q and R are continuous functions. However, we are interested in finding the solution of the equation in the neighborhood of a point, say z_0, at which $P(z_0) \neq 0$. That is, z_0 is an ordinary point since P is continuous, more so, $P(z_0) \neq 0$, there is some interval about the point z_0 in which P does not vanish in this interval where $\frac{Q}{P}$ and $\frac{R}{P}$ are continuous functions. It is this continuity condition that guarantees the existence of a unique solution of equation (6) satisfying the initial conditions $w(z_0) = w_0$, $w'(z_0) = w_0'$ for an arbitrary choice of w_0 and w_0'. The power series method involves the expression of the solution of a given differential equation as a power series. Theorem 2.1 furnishes the form of the solution of equation (6).

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Theorem 2.1: If z_0 is an ordinary point of equation (6), then the solution is given as

$$w = \sum_{k=0}^{\infty} a_k (z - z_0)^k; if\ z_0 \neq 0, \quad (7a)$$

or

$$w = \sum_{k=0}^{\infty} a_k z^k; if\ z_0 = 0. \quad (7b)$$

The task now is to determine the coefficient, a_k such that the power series (7) satisfies the given equation and ensures that the series actually converges. If we can show that the series does converge for $|z - z_0| < r\,; r > 0,$ then all the formal procedure such as differentiation term by term can be justified, and we would have constructed a solution of equation (6) that is valid for $|z - z_0| < r$. In order to obtain a_k, we substitute equation (7) and its derivatives w' and w'' into the given equation from where we determine a_k so that equation (7) satisfies equation (6).

Series Solution near a Regular Singular Point

Consider equation (6) in the neighborhood of a regular singular point $z = z_0$. Let $z_0 = 0$. The fact that $z = 0$ is a regular point means that $z\frac{Q(z)}{P(z)}$ and $z^2\frac{R(z)}{P(z)}$ have finite limits as $z \to 0$. Hence, they have power series of the form

$$z\frac{Q(z)}{P(z)} = \sum_{k=0}^{\infty} \lambda_k z^k, \quad (8)$$

$$z^2\frac{R(z)}{P(z)} = \sum_{k=0}^{\infty} \mu_k z^k, \quad (9)$$

which are convergent for some interval $|z| < r\,; r > 0$ about the origin. Divide equation (6) by $P(z)$ and multiply each term by z^2 to have

$$z^2 w'' + z^2\frac{Q(z)}{P(z)} w' + z^2\frac{R(z)}{P(z)} w = 0. \quad (10)$$

Equation (10) can be re-written as

$$z^2 w'' + z\left(z\frac{Q(z)}{P(z)}\right) w' + \left(z^2\frac{R(z)}{P(z)}\right) w = 0. \quad (11)$$

Substituting equations (8) and (9) into equation (11) gives

$$z^2 w'' + z\left(\sum_{k=0}^{\infty} \lambda_k z^k\right) w' + \left(\sum_{k=0}^{\infty} \mu_k z^k\right) w = 0. \quad (12)$$

We then assume a solution of the form

$$w = z^r \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k z^{k+r}. \quad (13)$$

At this point, the given problem reduces to obtaining the following: the indicial equation from where the values of r for which equation (13) is a solution of equation (6); the recurrence relation for a_k and the radius of convergence of the series, $\sum_{k=0}^{\infty} a_k z^k$.

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Applications

Consider the equation [18]

$$zw'' + \{(\alpha + 2) - z\}w' - \alpha w = 0, \tag{14}$$

with α constant. Equation (14) is a form of confluent hypergeometric equation, it is then solved by the power series method to obtain a confluent hypergeometric function.

Solution at $z = 0$: let

$$P_0(z) = -\alpha,$$
$$P_1(z) = (\alpha + 2) - z, \tag{15}$$
$$P_2(z) = z.$$

Hence, $P_2(0) = 0$ makes $z = 0$ a singular point. Next, we check whether $z = 0$ is a regular singular point, by studying the following limits:

$$\frac{(z-z_0)P_1(z)}{P_2(z)} = \frac{(z-0)\{(\alpha+2)-z\}}{z} = \alpha + 2, \tag{16}$$

$$\frac{(z-z_0)^2 P_0(z)}{P_2(z)} = \frac{(z-z_0)^2(-\alpha)}{z}$$

$$\frac{z^2(-\alpha)}{z} = 0\,. \tag{17}$$

Since both limits exist, $z = 0$ is a regular singular point.

Therefore, we assume the solution of the form

$$w = \sum_{k=0}^{\infty} \quad a_k z^{k+r}, \tag{18}$$

with $a_0 \neq 0$.

Hence,

$$w' = \sum_{k=0}^{\infty} \quad a_k (k + r) z^{k+r-1}, \tag{19}$$

$$w'' = \sum_{k=0}^{\infty} \quad a_k (k + r)(k + r - 1) z^{k+r-2}. \tag{20}$$

Substituting equations (18), (19) and (20) into equation (14) gives

$$z \sum_{k=0}^{\infty} \quad a_k (k + r)(k + r - 1) z^{k+r-2}$$
$$+(\alpha + 2) \sum_{k=0}^{\infty} \quad a_k (k + r) z^{k+r-1} \tag{21}$$
$$-z \sum_{k=0}^{\infty} \quad a_k (k + r) z^{k+r-1}$$
$$-\alpha \sum_{k=0}^{\infty} \quad a_k z^{k+r} = 0.$$

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Simplifying equation (21) gives

$$\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)z^{k+r-1}$$
$$+(\alpha+2)\sum_{k=0}^{\infty} a_k(k+r)z^{k+r-1} \quad (22)$$
$$-\sum_{k=0}^{\infty} a_k(k+r)z^{k+r}$$
$$-\alpha\sum_{k=0}^{\infty} a_k z^{k+r} = 0.$$

In order to simplify equation (22), we need all powers of z to be equal to

$k+r-1$, which is the smallest power of z. Hence, we simplify the indices as follows.

$$\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)z^{k+r-1}$$
$$+(\alpha+2)\sum_{k=0}^{\infty} a_k(k+r)z^{k+r-1} \quad (23)$$
$$-\sum_{k=1}^{\infty} a_{k-1}(k+r-1)z^{k+r-1}$$
$$-\alpha\sum_{k=1}^{\infty} a_{k-1} z^{k+r-1} = 0.$$

Isolating the first terms of the sums starting from 0 gives

$$a_0\{r(r-1)+(\alpha+2)r\}z^{r-1}$$
$$+\sum_{k=1}^{\infty} a_k(k+r)(k+r-1)z^{k+r-1}$$
$$+(\alpha+2)\sum_{k=1}^{\infty} a_k(k+r)z^{k+r-1} \quad (24)$$
$$-\sum_{k=1}^{\infty} a_{k-1}(k+r-1)z^{k+r-1}$$
$$-\alpha\sum_{k=1}^{\infty} a_{k-1} z^{k+r-1} = 0.$$

From the linear independence of all powers of z, that is, of the functions $1, z, z^2$ and so forth, the coefficients of z^r vanish for all r. We therefore have from the first term

$$a_0\{r(r-1)+(\alpha+2)r\}, \quad (25)$$

which is the indicial equation.

Since $a_0 \neq 0$, we have

$$r(r-1)+(\alpha+2)r = 0. \quad (26)$$

Hence, the solutions of the indicial equation above are given below:

$$r^2 - r + r\alpha + 2r = 0$$
$$r(r+1+\alpha) = 0$$
$$\Rightarrow r = 0 \; or \; r = -(1+\alpha).$$

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That is

$$r_1 = 0\,;\; r_2 = -(1+\alpha). \tag{27}$$

Also, from the rest of the terms, we have

$$(k+r)\{(k+r-1)+(\alpha+2)\}a_k = \{(k+r-1)+\alpha\}a_{k-1}. \tag{28}$$

$$a_k = \frac{\alpha+(k+r-1)}{(k+r)\{(\alpha+2)+(k+r-1)\}}a_{k-1}, \tag{29}$$

for $k \geq 1$.

For $r = 0$,

$$a_k = \frac{\alpha+(k-1)}{k\{(\alpha+2)+(k-1)\}}a_{k-1}, k \geq 1. \tag{30}$$

Equation (30) is the required recurrence formula. We then simplify this relation by giving a_k in terms of a_0 instead of a_{k-1}. Hence, our assumed solution takes the form

$$w = a_0 \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\frac{z^k}{(\alpha+2)_k}. \tag{31}$$

Therefore, the first power series solution of the confluent hypergeometric equation often referred to as the confluent hypergeometric function of the first kind becomes

$$M(\alpha, \alpha+2, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\alpha+2)_k}\frac{z^k}{k!}, \tag{32}$$

where $(\alpha)_0 = 1$; $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ are the Pochhammer symbols. $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$, where $\Gamma(z)$ is the Gamma function.

Next, we find the second power series solution of the confluent hypergeometric function [19]. Let $\alpha \notin Z$, where Z is the set of integers and $\alpha - (\alpha+2) = -(1+q)$, where $q \in Z^{\geq 0}$, then $(\alpha+2) = 1+q+\alpha$. Therefore the second power series solution becomes

$$\widetilde{M}(\alpha, \alpha+2, z) = \frac{\Gamma(-1-\alpha)}{\Gamma(-1)}M(\alpha, \alpha+2, z) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}z^{-1-\alpha}M(-1, -\alpha, z), \alpha \notin Z \tag{33}$$

where $\Gamma(\cdot)$ is a gamma function.

CONCLUSION

In this work, we obtained the power series solutions of a class of confluent hypergeometric equations. Equation (32) is the power series definition of a confluent hypergeometric function of the first kind. It has some good features. The function includes as special cases the incomplete-gamma and the normal distribution functions. In addition to mixtures of it which makes certain classes of these functions closed under such operations. These features make the function a natural tool to model option prices and more generally, functionals of densities. Hence, the function can represent a variety of density-related functions. Equation (33) is the second linearly independent power series solution of equation (14).

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