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A NEW LOGISTIC GENERALIZATION ARISING FROM DISTRIBUTIONS OF ORDER STATISTICS: PROPERTIES AND APPLICATIONS

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ABSTRACT: *Distributions with the variable support $x \in R$ that exhibit strict symmetricity are versed in literature; and serve as model-fit for various forms of bell shaped outcomes; where normal and logistic distributions are renowned examples. This strictness, however, limits the application of these probability models to a particular kind of data; hence, its minimal utility. In this paper, therefore, a new generalization for the logistic distribution termed the Jones generalized logistic distribution is proposed. This new distribution is conditionally symmetric; which entails that the distribution attains symmetricity, only at equal parameter combinations. By implication, the proposed distribution serves the dual purpose of modeling both symmetric and asymmetric outcomes. Some properties of the proposed model have been derived. Finally, JGLD were fit to two different lifetime data as a demonstration to its relevance.*

KEYWORDS: Logistic Distribution, Symmetricity, Generalization, Entropy, Lifetime Data.

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INTRODUCTION

It is usually of interest to keep track of preceding events before a significant or landmark occurrence. This is referred to as lifetime data; where lifetime distribution is the model used in fitting for these outcomes. Another terminology for describing this phenomenon is "time to event data". Many classical distributions have been developed in both symmetric and asymmetric categories, for this modeling purpose; the likes of Gumbel, Normal, Arcsine, Hyperbolic Secant, Gamma, Exponential, Rayleigh, Logistic and several other distributions. The evolution of distribution extensions stem from the well-known fact that some of the classical distributions are not flexible enough to model some complex or wild data.

This limitation necessitates the development of generalized distributions (or generators); which is adds extra parameter(s) and consequently improve the flexibility of a baseline distribution for wild data accomodation. Some of the popular generators are Logistic-G due to Torabi and Montazeri (2014), Gamma-G due to Zografos and Balakrishnan (2009), Sin-G due to Souza et al. (2019), Kumaraswamy Transmuted-G due to Afifya et al. (2016), Logistic-X due to Tahir et al. (2016), Transformed-Transformer(T-X) family of distributions due to Alzaartreh and Famoye (2013) and Topp Leone-G due to Tahir et al. (2018), Normal-G due to Ekum et al. (2022), Marshal-Olkin TX due to Klakattawi et al. (2022), A new generalized-X family of distribution due to Roozegar et al. (2022), to mention a few.

Gupta and Balakrishnan (1992), and Johnson et al. (1995) captured in their compilations the logistic distribution, popularly known as the growth model; which is a symmetric distribution and defined as

$$f_1(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}\ ;\ -\infty < x < \infty,\ \mu \epsilon R,\ \ \beta > 0 \qquad (1)$$

However, equation (1) can be represented in standard form at $\mu = 0\ \ and\ \beta = 1$ to obtain:

$$f_2(x) = \frac{e^{-x}}{(1+e^{-x})^2} \qquad (2)$$

and their corresponding cumulative distribution functions are given as:

$$F_1(x) = \frac{1}{1+e^{-(x-\mu)/\beta}} \qquad (3)$$

$$F_2(x) = \frac{1}{1+e^{-x}} \qquad (4)$$

This classical distribution is widely applied in many areas of life, which made it a significant remark in the field of statistics. Logistic distribution was used to analyze economic and demographic data, and bio-assay data due to Verhust (1845) and Berkson (1953) respectively as captured in Balakrishnan (1991). In an attempt to improve on its flexibility, Balakrishnan et al. (1988) and Gupta and Kundu (2010) developed Type 1 generalized logistic distribution and generalized logistic distributions respectively. More so, Makubate et al. (2021) proposed Marshal-Olkin half logistic-G distribution, which served as a flexible fit for modeling in several fields such as survival analysis and hydrology and engineering.

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By investigations, most of the distributions with the variable support $x \in R$ is observed to exhibit either strict or conditional symmetricity; strict symmetry in the sense that at all parameter values or combinations (for distributions with $k > 1$ parameters), the distribution remains symmetric. For example, Normal distribution, Logistic distribution, Arcsine hyperbolic secant distribution etc. However, a distribution is judged conditionally symmetric if symmetry occurs only at a given value of parameters. The first case could be viewed in a way to be more flexible a distribution than the latter; because, in the derivation of mathematical or statistical properties for example, it yields definite and direct results. However, for the conditionally symmetric distributions, numerical evaluations are usually made to obtain some results for its various properties.

This limitation notwithstanding, conditionally symmetric distributions assume different shapes at different parameter values. This implies that it can serve the dual purpose of data modeling, both for the symmetric and asymmetric order; hence, having a different flexibility edge over the strict type. Consequently, the paper is aimed at proposing a generalized distribution arising from the distributions of order statistics according to Jones (2004). This compound development is intended to improve on the logistic distribution for various rising needs other than symmetric data modeling. Other sections involve the construction of the proposed model, its properties and then the application to various data.

CONSTRUCTION OF JONES-G LOGISTIC DISTRIBUTION (JGLD)

Jones developed a generalized model from distributions of order statistics, and the probability density function is given as:

$$f(x,a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} g(x)[G(x)]^{a-1}[1-G(x)]^{b-1}, \quad a > 0, b > 0 \tag{5}$$

where $G(x)$ is a symmetric distribution, and $f(x,a,b)$ assumes conditional symmetricity at parameter values $a = b$. By inserting equation (2) into (5) we obtain JGLD as follows:

$$f(x,a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{1+e^{-x}}\right]^{a-1} \left[1-\frac{1}{1+e^{-x}}\right]^{b-1} \tag{6}$$

And the cumulative distribution function is derived thus

$$F(x,a,b) = \int_{-\infty}^{x} f(t,a,b)dt$$

$$F(x,a,b) = \frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} (e^x)^a \left(\frac{1}{1+e^x}\right)^b (1+e^x)^b \aleph \tag{8}$$

$$= \frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} (e^x)^a \aleph$$

where $\aleph = \text{Hypergeometric2F1}[a, a+b, 1+a, -e^x]$

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PROPERTIES OF JONES-G LOGISTIC DISTRIBUTION

Infinite Series Representation (ISR)

The following forms of functions can be represented thus, as infinite series:

$$e^{s} = \sum_{i=0}^{\infty} \frac{s^{i}}{i!} \tag{10}$$

$$(a+s)^{n} = \sum_{j=0}^{\infty} \binom{n}{j} a^{n-j} s^{j} \tag{11}$$

$$\rightarrow \qquad (a+s)^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j} a^{-n-j} s^{j} \tag{12}$$

$$(1-s^{c})^{n} = \sum_{l=0}^{\infty} \binom{n}{l} (-1)^{l} s^{cl} \tag{13}$$

$$\text{Hypergeometric2F1}[i,j,k,z] = \sum_{p=0}^{\infty} \frac{i_p\, j_p}{k_p} \frac{z^{p}}{p!} \tag{14}$$

$$\text{HypergeometricPFQ}[u,v,z] = \sum_{p=0}^{\infty} \frac{u_p}{v_p} \frac{z^{p}}{p!} \tag{15}$$

where equations (10-15) suffice to express the PDF and CDF components as

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-x)^{i}}{i!}$$

$$(e^{x})^{a} = \sum_{i=0}^{\infty} \frac{(ax)^{i}}{i!}$$

$$\frac{1}{(1+e^{-x})^{2}} = (1+e^{-x})^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} \sum_{j=0}^{\infty} \frac{(-x\,j)^{j}}{j!}$$

$$\left[\frac{1}{1+e^{-x}}\right]^{a-1} = (1+e^{-x})^{1-a} = \sum_{k=0}^{\infty} \binom{1-a}{k} \sum_{k=0}^{\infty} \frac{(-x)^{k}}{k!}$$

$$\left[1-\frac{1}{1+e^{-x}}\right]^{b-1} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{b-1}{l} \binom{-l}{m} (-1)^{l} \sum_{q=0}^{\infty} \frac{(-x\,m)^{q}}{q!}$$

Hence, by implication of ISR, the PDF and CDF of JGLD following equations (6) and (8) are respectively given as:

$$f(x,a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{(-x)^{i}}{i!} \sum_{j=0}^{\infty} \binom{-2}{j} \sum_{j=0}^{\infty} \frac{(-x\,j)^{j}}{j!} \sum_{k=0}^{\infty} \binom{1-a}{k} \sum_{k=0}^{\infty} \frac{(-x)^{k}}{k!}$$
$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{b-1}{l} \binom{-l}{m} (-1)^{l} \sum_{q=0}^{\infty} \frac{(-x\,m)^{q}}{q!}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^{l} \frac{(-x)^{i}}{i!} \frac{(-x\,j)^{j}}{j!} \frac{(-x)^{k}}{k!} \frac{(-x\,m)^{q}}{q!} \tag{16}$$

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and $$F(x,a,b) = \frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{(ax)^i}{i!} \sum_{p=0}^{\infty} \frac{a_p\,(a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!}$$

$$= \frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\,(a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \quad (17)$$

JGLD PDF Properness

To test the validity of a distribution, it suffices to state that $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\therefore \quad \int_{-\infty}^{\infty} f(x,a,b)dx = \int_{-\infty}^{\infty} \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{1+e^{-x}}\right]^{a-1} \left[1 - \frac{1}{1+e^{-x}}\right]^{b-1} \right) dx \quad (18)$$

$$= \frac{\text{Gamma}[a+b]\left(\frac{\text{Hypergeometric2F1}[a,a+b,1+a,-1]}{a} + \frac{\text{Hypergeometric2F1}[b,a+b,1+b,-1]}{b}\right)}{\text{Gamma}[a]\text{Gamma}[b]} \quad (19)$$

Conventionally, the outcome of the integral in equation (18) should be one. In this case however, it does not appear directly due to the hyper-geometric and gamma functions that must be parametrically evaluated. Nonetheless, the results obtained equal one consistently, computing equation (19) with any evaluable gamma or hyper geometric parameters. This is to say that the choice of the parameters is subject to the domain of both gamma and hyper geometric functions, constrained within the distribution's parameter supports.

JGLD Shape Parametric Behavior

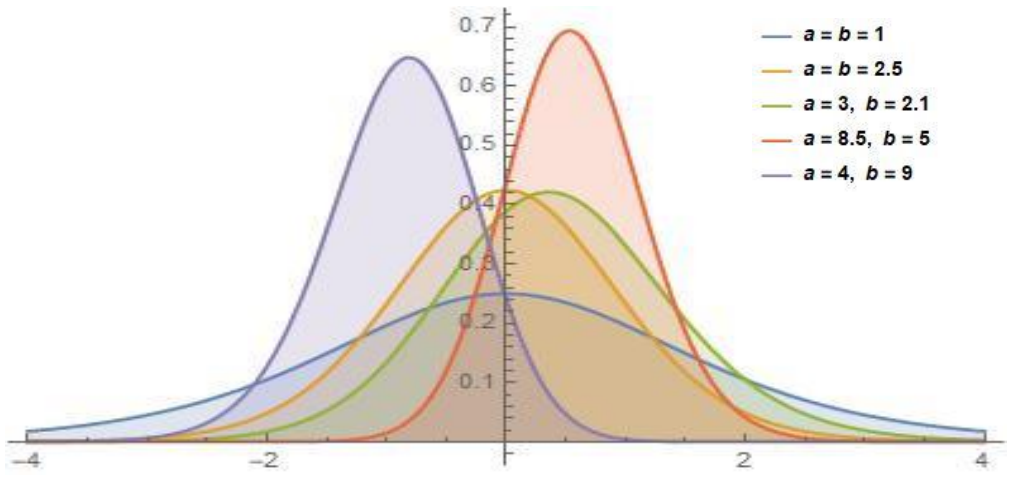

Figure 1: JGLD PDF Plots at $a = b, a < b$ and $a > b$

As observed in Figure 1, the distribution is symmetric at all parameter values $a = b$; and then, it is asymmetric at different unequal values of a and b (i.e. $a > b$ or $a < b$). This entails that JGLD can model data from both symmetric and asymmetric order (left and right skewed).

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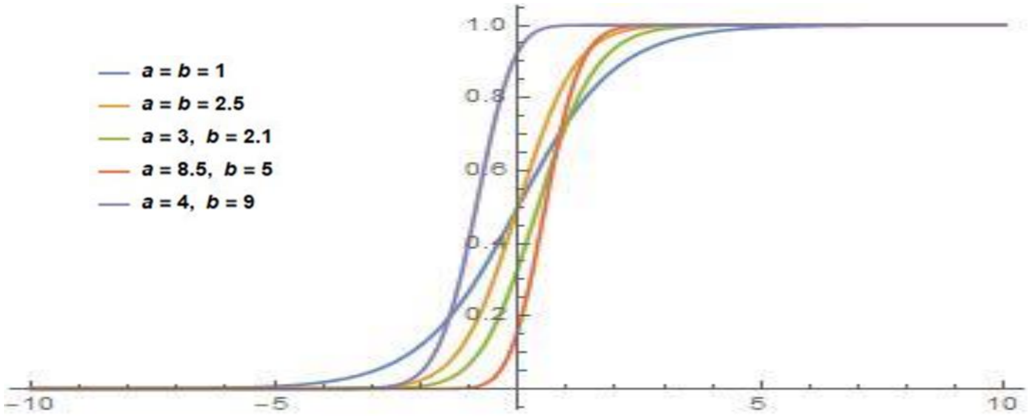

Figure 2: The plot of the CDF

This convergence at *1.0* as seen in Figure 2 validates the properness of the distribution at different combinations of the parameter for $a, b > 0$; regardless of the inexplicit nature of the PDF. More so, the derivation of a probability table for the PDF of JGLD across the variables and different parameter values will consolidate this validation insight.

Moments And Moment Generating Function for JGLD

The moment and moment generating function of JGLD are derived given that $f(x)$ is the function in equation (6), thus:

$$\mu^r = E(X)^r = \int_{-\infty}^{\infty} x^r f(x,a,b)dx$$

$$= \int_{-0}^{\infty} x^r \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l \frac{(-x)^i}{i!}\frac{(-x\,j)^j}{j!}\frac{(-x)^k}{k!}\frac{(-x\,m)^q}{q!}\right)dx$$

$$= \int_{-0}^{\infty} x^r \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l e^{-x-x-xj-xm}\right)dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l \int_{-0}^{\infty} x^r e^{-2x-xj-xm}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l (2+j+m)^{-(1+r)}\Gamma(1+r),$$
(20)

$$r > -1,\ j+m > -2$$

Evaluating equation (20) at $r = 1,2,3\ \&\ 4$, we obtain the raw moments about the origin:

$$\mu_1' = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l (2+j+m)^{-2} = \mu$$

$$\mu_2' = \frac{2\,\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j}\binom{1-a}{k}\binom{b-1}{l}\binom{-l}{m}(-1)^l (2+j+m)^{-3}$$

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$$\mu'_3 = \frac{6\,\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \, (2+j+m)^{-4}$$

$$\mu'_4 = \frac{24\,\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \, (2+j+m)^{-5}$$

The central moment about the mean of the JGLD is:

$$\mu_n = E[\,(X - E[X])^n\,] = \sum_{j=0}^{n} \binom{n}{j} \, (-1)^{n-j} \, \mu'_j \mu^{n-j} \quad (21)$$

So that $\mu_2 = \mu'_2 - \mu^2 = \sigma^2$ (22)

$$\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3 \quad (23)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4 \quad (24)$$

More so, the coefficient of variation CV $= \frac{\sigma}{\mu'_1}$, the coefficient of skewness $CS = \frac{\mu_3}{\mu_2^{3/2}}$, the coefficient of kurtosis CK $= \frac{\mu_4}{\mu_2^2}$, and the variance-to-mean-ratio $VMR = \frac{\sigma^2}{\mu'_1}$ can further be derived following equations (21-24).

The moment generating function of JGLD Distribution is derived as:

$$M_x(\mathrm{t}) = E(\, e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \, f(x) \; dx \quad (25)$$

$$M_x(\mathrm{t}) = \int_0^{\infty} e^{tx} \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \; \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{1+e^{-x}}\right]^{a-1} \left[1 - \frac{1}{1+e^{-x}}\right]^{b-1} \right) dx \quad (26)$$

$$= \int_0^{\infty} e^{tx} \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \, \frac{(-x)^i}{i!} \frac{(-x\,j)^j}{j!} \, \frac{(-x)^k}{k!} \frac{(-x\,m)^q}{q!} \right) dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \int_{-0}^{\infty} e^{tx} \, e^{-2x-xj-xm} \, dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \int_{-0}^{\infty} \, e^{-2x-xj-xm+tx} \, dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty} \binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \, \frac{1}{2+j+m-t}$$

$$j + m - t > -2$$

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Quantile simulating function

The mathematical expression for quantile forecast model is derived thus:

$$F(x) = \theta \quad \rightarrow \quad x = F^{-1}(\theta);\, 0 \leq \theta \leq 1.$$

Now, the JGLD quantile function is follows from the equating the CDF as obtained in equation (8) by θ.

$$\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ (e^{x})^{a}\ \aleph = \theta$$

$$\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ (e^{x})^{a}\ \aleph - \theta = 0$$

Hazard and Other Related Functions of JGLD

The hazard function of the distribution from JGLD family is derived from the survival function given as:

$$s(x,a,b) = 1 - F(x,a,b)$$

$$s(x,a,b) = 1 - \left(\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ (e^{x})^{a}\ \aleph\right)$$

$$= 1 - \left(\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ \sum_{i=0}^{\infty}\frac{(ax)^{i}}{i!}\sum_{p=0}^{\infty}\frac{a_p\,(a+b)_p}{(1+a)_p}\frac{(-e^{x})^{p}}{p!}\right)$$

$$= 1 - \left(\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ \sum_{i=0}^{\infty}\sum_{p=0}^{\infty}\frac{(ax)^{i}}{i!}\frac{a_p\,(a+b)_p}{(1+a)_p}\frac{(-e^{x})^{p}}{p!}\right)$$

where the hazard function is

$$h(x,a,b) = \frac{f(x,a,b)}{s(x,a,b)}$$

$$= \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{e^{-x}}{(1+e^{-x})^{2}}\left[\frac{1}{1+e^{-x}}\right]^{a-1}\left[1-\frac{1}{1+e^{-x}}\right]^{b-1}}{1-\left(\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ (e^{x})^{a}\ \aleph\right)}$$

$$= \frac{a\,\Gamma(a+b)\left[\frac{1}{1+e^{-x}}\right]^{a}\left[\frac{1}{1+e^{x}}\right]^{b}}{a\,\Gamma(a)\Gamma(b) - (e^{x})^{a}\,\Gamma(a+b)\aleph}$$

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JGLD Entropy Study

The measure of randomness of a system, say, a random variable X, is termed entropy, Renyi (1961). The Renyi entropy of X with density function $p(x)$, following JGLD is defined by,

$$N_e = \frac{1}{1-c} \, log\{\int_{-\infty}^{\infty} p^c(x)dx\}, \; c > 0, \; c \neq 1$$

$$N_e = \frac{1}{1-c} \, log\left\{\int_{-\infty}^{\infty} \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \; \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{1+e^{-x}}\right]^{a-1} \left[1 - \frac{1}{1+e^{-x}}\right]^{b-1}\right]^c dx\right\}$$

$$= \frac{1}{1-c} \mathrm{Log}\left\{\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^c (\Gamma(ac)\aleph R_{ac} + \Gamma(bc)\aleph R_{bc})\right\}, \; bc > 0, \; ac > 0$$

where $\aleph R_{ac} = \text{Hypergeometric2F1Regularized}[ac, (a+b)c, 1+ac, -1]$

$\aleph R_{bc} = \text{Hypergeometric2F1Regularized}[bc, (a+b)c, 1+bc, -1]$

Furthermore, Tsalli (1988), defined entropy for a given random variable X as:

$$Ts_e = \frac{1}{s-1} \left\{1 - \int_{-\infty}^{\infty} f^s(x)dx\right\}, \; s \in R, \; s \neq 1$$

To obtain the Tsalli's entropy for a random variable which follows JGLD, we have thus:

$$Ts_e = \frac{1}{s-1} \left\{1 - \int_{-\infty}^{\infty} \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \; \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{1+e^{-x}}\right]^{a-1} \left[1 - \frac{1}{1+e^{-x}}\right]^{b-1}\right]^s dx\right\}$$

$$= \frac{1}{s-1}\left\{1 - \left(\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^s [\Gamma(as)\aleph R_{as}] + \Gamma(bs)\aleph R_{bs}]\right)\right\}, \; bs > 0, \; as > 0$$

where $\aleph R_{as} = \text{Hypergeometric2F1Regularized}[as, (a+b)s, 1+as, -1]$

$\aleph R_{bs} = \text{Hypergeometric2F1Regularized}[bs, (a+b)s, 1+bs, -1]$

Table 1: Empirical Analysis on Renyi Entropy

C	$a = 1$, $b = 3.5$	$a = 1.8$, $b = 4$	$a = 3$, $b = 3.8$	$a = 3.9$, $b = 5.6$
0.5	3.1890	2.3056	1.7428	1.3674
0.7	3.3105	2.8831	2.3550	2.5238
0.75	3.6320	3.3593	2.7856	3.3130
0.8	4.1987	4.1460	3.4780	4.6375
0.87	5.9066	6.4438	5.4551	8.6782
0.9	7.4444	8.4981	7.2016	12.4264
1***?	∞	∞	∞	∞
1.35	-2.3973	-5.8368	-4.0392	-32.196
2	-5.0198	-32.466	-11.935	-1478.82
2.75	-73.848	-1466.2	-258.38	-1091274
3	-228.46	-6733.2	-930.39	-13716122
3.6	-4896.257	-387436.2	-29563.2	-10061882097

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Table 2: Empirical Analysis on Tsalli Entropy

S	$a = 1$, $b = 3.5$	$a = 1.8$, $b = 4$	$a = 3$, $b = 3.8$	$a = 3.9$, $b = 5.6$
0.5	-3.3303	-3.4628	-4.3628	-8.7643
0.7	-3.4623	-6.3126	-11.099	-42.015
0.75	-3.8486	-8.2517	-15.432	-69.169
0.8	-4.5516	-11.482	-22.617	-120.89
0.87	-6.7466	-21.214	-44.235	-306.79
0.9	-8.7676	-30.147	-64.091	-499.66
1***?	$-\infty$	$-\infty$	$-\infty$	$-\infty$
1.35	6.7390	57.077	130.43	8454.1
2	38.182	1022.16	2119.5	5678233
2.75	1245.05	168524.9	320253.4	92542980715
3	4981.50	1191238	2203492	3.26e+12
3.6	197779.1	192988461	331196826	2.84e+16

From Tables 1 and 2, it is a clear observation that entropy can assume either a positive and or a negative value. More so, for any two preceding or consecutive values of the parameters c_i and s_i, say (c_1 and c_2) or (s_1 and s_2), the JGLD Renyi (N_e) and Tsalli Ts_e entropy, conditionally satisfies the entropy behavioral proposition as given by Golshani and Pasha (2010):

$\forall\ C < 1: \quad c_1 < c_2 \rightarrow N_{e1} \leq N_{e2}$

$\forall\ C > 1: \quad c_1 < c_2 \rightarrow N_{e1} \geq N_{e2}$

$\forall\ S < 1: \quad s_1 < s_2 \rightarrow Ts_{e1} \geq Ts_{e2}$

$\forall\ S > 1: \quad s_1 < s_2 \rightarrow Ts_{e1} \leq Ts_{e2}$

Parameter Estimation of the JGL Distribution

Let $x_1, x_2, x_3, \ldots, x_n$ be random samples given a PDF $f(x)$; then the log-likelihood function for the random variable is given as:

$$lnLf(x,\omega) = \sum_{i=0}^{n} lnf(x_i,\omega);$$

where, $Lf(x,\omega) = \prod_{i=0}^{n} f(x_i,\omega)$

If $X \sim JGLD(x,\omega)$, $\omega = a, b$, then

$$Lf(x,a,b) = \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n \frac{e^{-\sum_{i=1}^{n} x_i}}{\sum_{i=1}^{n}(1+e^{-x_i})^2} \sum_{i=1}^{n}\left[\frac{1}{1+e^{-x}}\right]^{a-1} \sum_{i=1}^{n}\left[1-\frac{1}{1+e^{-x}}\right]^{b-1}$$

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Taking the natural logarithm of both sides,

$$lnLf(x,a,b) = nln[\Gamma(a+b)] - nln[\Gamma(a)\Gamma(b)] - \sum_{i=1}^{n} x_i - 2ln\sum_{i=1}^{n}(1+e^{-x_i})$$
$$+(a-1)ln\sum_{i=1}^{n}\left[\frac{1}{1+e^{-x}}\right] + (b-1)ln\sum_{i=1}^{n} 1 - \left[\frac{1}{1+e^{-x}}\right]$$

Maximizing at $\frac{\partial\, lnLf(x,a,b)}{\partial a} = \frac{\partial\, lnLf(x,a,b)}{\partial b} = 0$

$$\frac{\partial\, lnLf(x,a,b)}{\partial a} = n\psi[a\,*] + ln\sum_{i=1}^{n}\left[\frac{1}{1+e^{-x}}\right]$$

$$\rightarrow \quad n\psi[a\,*] + ln\sum_{i=1}^{n}\left[\frac{1}{1+e^{-x}}\right] = 0 \qquad (27)$$

$$\frac{\partial\, lnLf(x,a,b)}{\partial b} = n\psi[*\,b] + ln\sum_{i=1}^{n}\left[1 - \left[\frac{1}{1+e^{-x}}\right]\right]$$

$$\rightarrow \quad n\psi[*\,b] + ln\sum_{i=1}^{n}\left[1 - \left[\frac{1}{1+e^{-x}}\right]\right] = 0 \qquad (28)$$

Where

$\psi[a\,*] = (\psi[0, a+b] - \psi[0, a])$, and $\psi[*\,b] = (\psi[0, a+b] - \psi[0, b])$.

$\psi[**]$, implies poly-gamma function given by the simplification of the outcome of the derivative $\frac{\partial\, ln\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]}{\partial *}$. To obtain the estimates of the parameters, equations (27) and (28) can be solved numerically, since they are implicitly structured.

C-Moments of G-Class distribution

The skewness and kurtosis as introduced by Kenney and Keeping (1962), and Moor (1986) respectively are obtained by appropriately making substitutions in:

$$\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)}\ (e^x)^a\ \aleph = \theta \qquad (29)$$

which represents the inverse cumulative distributions $Q(\theta, a, b)$. Now, at $a = b = 2$, equation (29) becomes:

$$\frac{3e^{2x}}{(1+e^x)^4} = \theta$$

By implication the skewness and kurtosis are derived thus:

$$X_{sk} = \frac{Q\left[\frac{9}{12}, a, b\right] + Q\left[\frac{5}{20}, a, b\right] - 2Q\left[\frac{9}{18}, a, b\right]}{Q\left[\frac{15}{20}\right] - Q\left[\frac{6}{24}\right]}$$

$$X_k = \frac{Q\left[\frac{14}{16}, a, b\right] - Q\left[\frac{15}{24}, a, b\ \right] - Q\left[\frac{12}{32}, a, b\right] + Q\left[\frac{3}{24}, a, b\right]}{Q\left[\frac{3}{4}, a, b\right] - Q\left[\frac{5}{40}, a, b\right]}$$

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Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from JGLD Distribution. Let $X_1 < X_2 <, \dots, < X_n$ denote the corresponding order statistics. The pdf and the cdf of the kth order statistics say $Y = X_k$ is given by:

$$f_Y(\mathrm{y}) = \frac{\mathrm{n}!}{(k-1)!(n-k)!} f(y)\, F^{k-1}(y)\, \{1 - F(y)\}^{n-k}$$

But

$$\{1 - F(y)\}^{n-k} = \sum_{l=0}^{\infty} \binom{n-k}{l} (-1)^l [F(y)]^l$$

$$\therefore \qquad f_Y(\mathrm{y}) = \frac{\mathrm{n}!}{(k-1)!(n-k)!} \{ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{i=j=k=l=m=q=0}^{\infty}$$

$$\binom{-2}{j} \binom{1-a}{k} \binom{b-1}{l} \binom{-l}{m} (-1)^l \frac{(-x)^i}{i!} \frac{(-x\, j)^j}{j!} \frac{(-x)^k}{k!} \frac{(-x\, m)^q}{q!} \left[\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\, (a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \right]^{k-1}$$

$$\sum_{l=0}^{\infty} \binom{n-k}{l} (-1)^l \left[\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\, (a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \right]^l \qquad (30)$$

And

$$F_Y(\mathrm{y}) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\, \{1 - F(y)\}^{n-j}$$

$$= \sum_{j=k}^{n} \binom{n}{j} \left[\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\, (a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \right]^j \sum_{l=0}^{\infty} \binom{n-k}{l} (-1)^l$$

$$\left[\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\, (a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \right]^l$$

$$= \left[\frac{\Gamma(a+b)}{a\,\Gamma(a)\Gamma(b)} \right]^{j+l} \sum_{j=k}^{n} \binom{n}{j} \sum_{l=0}^{\infty} \binom{n-k}{l} (-1)^l \left[\sum_{i=p=0}^{\infty} \frac{(ax)^i}{i!} \frac{a_p\, (a+b)_p}{(1+a)_p} \frac{(-e^x)^p}{p!} \right]^{j+l}$$

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JGLD Probability Numerical Simulation

In distribution development, it is usually important to investigate the conformity of a proposed distribution to the probability axiom $0 \leq p(x) \leq 1$. This is a numerical approach to validating the different derivational claims about the properness and or symmetricity of a distribution, with respect to different parameter support or domain.

Table 3: Variable and Parameter Numerical Evaluation of JGLD for $a \neq b$

	$p(-x,a,b)$					$p(x,a,b)$			
-X	$a = 0.1$ $b = 1.5$	$a = 0.5$ $b = 2.7$	$a = 1.25$ $b = 3.25$	$a = 2.3$ $b = 5.3$	X	$a = 0.1$ $b = 1.5$	$a = 0.5$ $b = 2.7$	$a = 1.25$ $b = 3.25$	$a = 2.3$ $b = 5.3$
-1	0.05809	1.97e-01	3.52e-01	4.74e-01	1	1.43e-02	2.18e-02	4.77e-02	2.36e-02
-2	0.07082	2.17e-01	2.33e-01	1.96e-01	2	4.31e-03	2.66e-03	4.27e-03	4.86e-04
-3	0.07264	1.69e-01	9.51e-02	3.57e-02	3	1.09e-03	2.30e-04	2.36e-04	4.40e-06
-4	0.06900	1.13e-01	3.12e-02	4.51e-03	4	2.55e-04	1.70e-05	1.05e-05	2.77e-08
-5	0.06359	7.11e-02	9.43e-03	4.92e-04	5	5.79e-05	1.18e06	4.28e-07	1.51e-10

Table 4: Variable and Parameter Numerical Evaluation of JGLD for $a = b$

	$p(-x,a,b)$					$p(x,a,b)$			
-X	$a = 0.1$ $b = 0.1$	$a = 1.5$ $b = 1.5$	$a = 3.25$ $b = 3.25$	$a = 5.3$ $b = 5.3$	X	$a = 0.1$ $b = 0.1$	$a = 1.5$ $b = 1.5$	$a = 3.25$ $b = 3.25$	$a = 5.3$ $b = 5.3$
-1	0.04311	2.22e-01	2.24e-01	1.78e-01	1	0.04311	2.22e-01	2.24e-01	1.78e-01
- 2	0.04049	8.66e-02	2.92e-02	6.38e-03	2	0.04049	8.66e-02	2.92e-02	6.38e-03
-3	0.03721	2.45e-02	1.88e-03	7.32e-05	3	0.03721	2.45e-02	1.88e-03	7.32e-05
-4	0.03387	5.98e-03	8.89e-05	5.04e-07	4	0.03387	5.98e-03	8.89e-05	5.04e-07
-5	0.03072	1.38e-03	3.72e-06	2.84e-09	5	0.03072	1.38e-03	3.72e-06	2.84e-09

By careful examination, the probability values in Tables 3 and 4 agree with the axiom $0 \leq p(x,a,b) \leq 1$ at both different and equal parameter values $\forall\, x \in R$. More so, at different parameter combinations $\forall\, a \neq b, p(-x,a,b) \neq p(x,a,b)$ and at equal parameter combinations $\forall\, a = b, p(-x,a,b) = p(x,a,b)$. This gives more insight on the conditionality of the distribution's symmetricity.

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DATA APPLICATION

In this section, presentations are made for two sets of lifetime data to objectively give essence to the proposed distribution. One of them is an extract from the application of generalized logistic distribution; which represents the strength measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows. See Gupta and Kundu (2010) for more details. More so, the second data was primarily collected from the maintenance department of First Independent Power Limited Company, Afam, River State, Nigeria. The lifetime data relates to 80 days Oil/winding temperature measurement (in Celsius). This calibration indicates system (or transformer) normalcy; and the system is observed to shut down once the temperature reads above 65℃; by implication it means that the power process has gone wrong.

Table 5: Carbon fiber strength measured in GPA (Data1)

1.901, 2.132, 2.203, 2.228, 2.257, 2.35, 2.361, 2.396, 2.397, 2.445,2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.74, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.03, 3.125, 3.139, 3.145, 3.22, 3.223, 3.235, 3.243,3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225,4.395,1.52,1.8,1.5,1.01

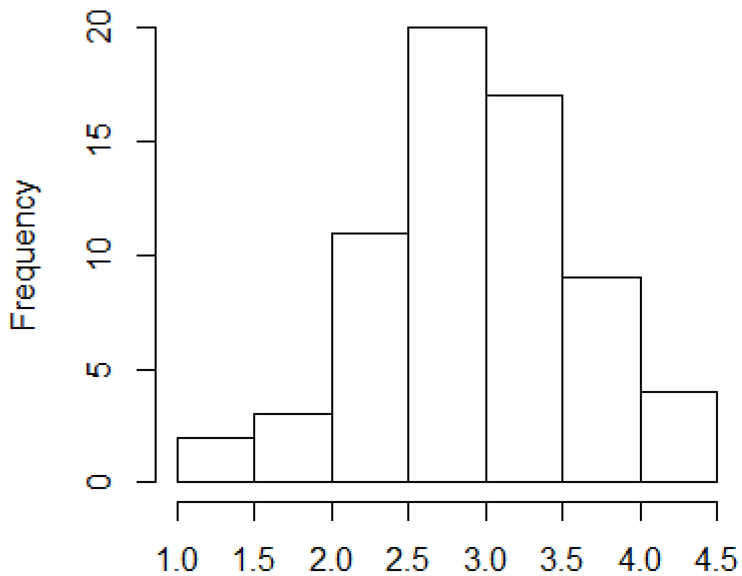

Figure 3: Histogram Chart for Data1

Table 6: Oil/winding temperature measurement (Data2)

60.5, 40.3, 53.2, 72, 54.6, 59.6, 55.4, 49.3, 43.2, 43.2, 40.9, 53.1, 56.6, 42.6, 57.2, 55.7, 54.1, 55.4, 54.8, 40.9, 68.6, 52.7, 55.7, 70.3, 71.9, 45.2, 72.3, 55.7, 52.3, 42.6, 54.3, 67.7, 60.2, 56.1, 54.4, 65.4, 53.5, 54.5, 57.9, 52.8, 53.8, 57.3, 71.5, 58.3, 54.8, 49.6, 50.2, 57.6, 66.1, 63.8, 64.1, 63.7, 68.1, 45.7, 67.6, 63.0, 62.1, 54.9, 54.0, 66.7, 53.9, 48.3, 56.9, 49.7, 63.8, 64.1, 63.8, 60.0, 45.3, 67.0, 63.2, 64.2, 46.2, 48.3, 46.3, 48.3, 57.9, 59.6, 53.9, 56.3, 64.5

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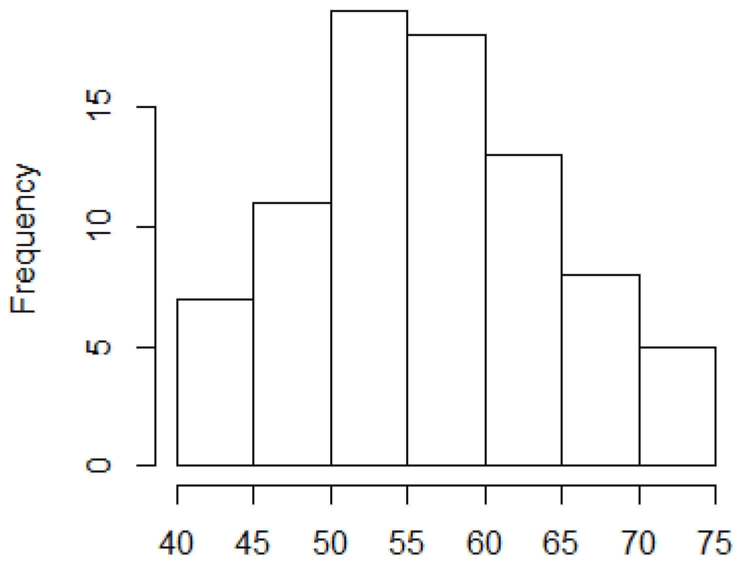

Figure 4: Histogram Chart for Data2

Table 7: Probability Model Comparative Case-Study

Distribution	PDF-Model
	Asymmetric Type
EGGuD	$\frac{cd}{b} e^{-e^{-\frac{x-a}{b}}} e^{-\frac{x-a}{b}} \left[1 - e^{-e^{-\frac{(x-a)}{b}}}\right]^{c-1} \left[1 - \left(1 - e^{-e^{-\frac{(x-a)}{b}}}\right)^{c}\right]^{d-1}$
EFD	$a^b bc\, x^{-b-1}\, e^{-\left(\frac{a}{x}\right)^b} \left[1 - e^{-\left(\frac{a}{x}\right)^b}\right]^{c-1}$
EGFD	$a^b cdb\, x^{-b-1} e^{-\left(\frac{a}{x}\right)^b} \left[1 - e^{-\left(\frac{a}{x}\right)^b}\right]^{c-1} \left[1 - \left(1 - e^{-\left(\frac{a}{x}\right)^b}\right)^{c}\right]^{d-1}$
	Symmetric Type
GLD Type1	$\frac{a\, e^{-x}}{(1+e^{-x})^{a+1}}$
GLD Type2	$\frac{a\, e^{-ax}}{(1+e^{-x})^{a+1}}$
Logistics	$\frac{e^{-\left(\frac{x-a}{b}\right)}}{b\left(1+e^{-\left(\frac{x-a}{b}\right)}\right)^{2}}$

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GGLD	$abc\frac{e^{-bx}}{(1+e^{-bx})^{a+1}}\left[1-\frac{1}{(1+e^{-bx})^{a}}\right]^{c-1}$

The compared distributions are categorized under asymmetric order; which are Exponentiated Generalized Gumbel Distribution (EGGuD), Exponentiated Frechet Distribution (EFD), Exponentiated Generalized Frechet Distribution (EGFD); whereas, Generalized Logistic Distribution (GLD) and Gamma Generalized Logistic Distribution (GGLD) are for the symmetric order.

Table 8: Application of JGLD distribution to Data1

	Model	Parameter	$-\log(L)$	AIC	BIC	Rank
Data 1	**JGLD**	$a = 37.566$ $b = 2.4535$	-74.41	154.83	161.97	1
	EGGuD	$a = 0.9684$ $b = 0.0965$ $c = 0.0885$ $d = 3.0437$	-86.651	181.30	190.06	4
	EFD	$a = 126.86$ $b = 0.5632$ $c = 3040.4$	-76.058	156.12	160.49	2
	EGFD	$a = 5.6249$ $b = 1.9025$ $c = 10.441$ $d = 0.4635$	-74.307	156.61	165.37	3

Table 9: Application of JGLD distribution to Data2

	Model	Parameter	$-\log(L)$	AIC	BIC	Rank
Data2	**JGLD**	$a = 37.566$ $b = 2.4535$	-70.5737	145.147	149.52	1
	GLDType1	$a = 17555$	-3792.48	7586.96	7589.35	6
	GLDType2	$a = 0.0176$	-407.908	817.816	820.21	4
	Logistic	$a = 56.549$ $b = 4.7412$	-286.695	577.391	582.18	3
	Normal	$a = 8.1633$ $b = 56.586$	-284.984	573.969	578.75	2
	GGLD	$a = 0.2647$ $b = 0.4594$ $c = 0.0384$	--411.499	828.999	836.18	5

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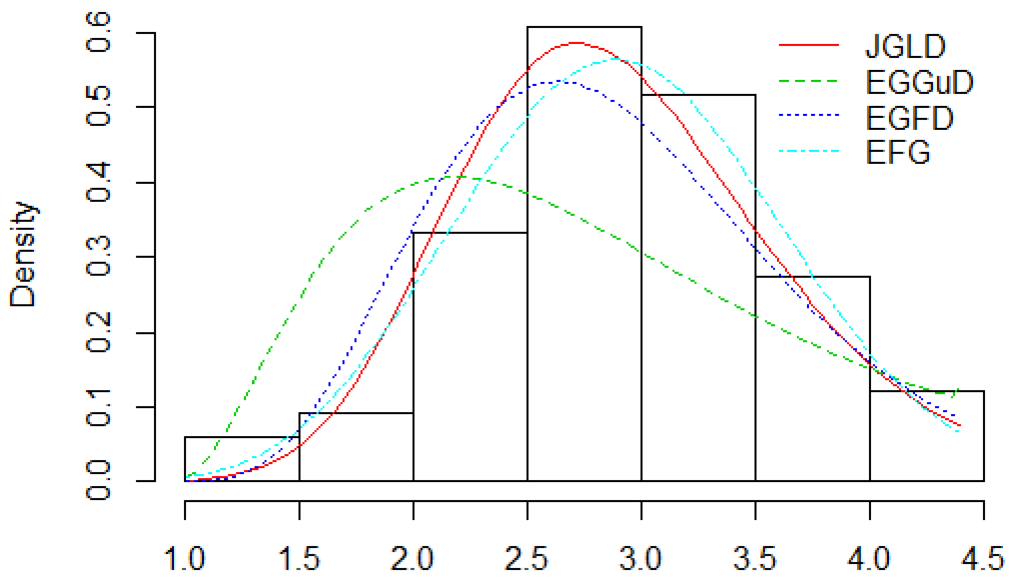

Figure (5): Density fit for the Data Set 1

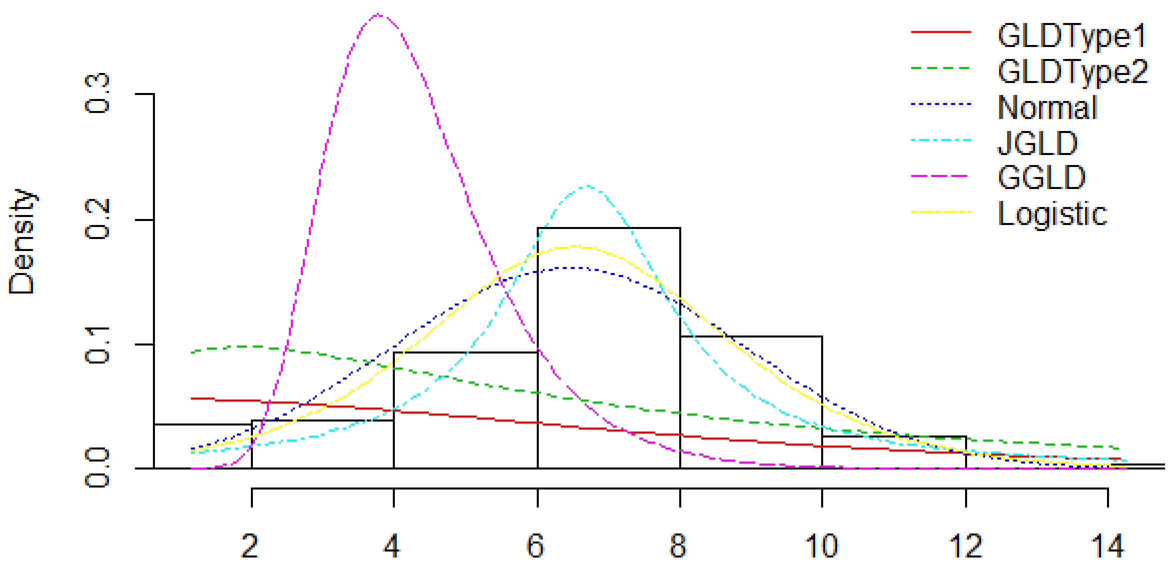

Figure (6): Density fit for the Data Set 2

The $-\log(L)$, AIC and BIC inferential results from Tables 8 and 9, show that JGLD is a better fit for data 1 and 2. Of course, the distributions with the highest values in $-\log(L)$ imply to be the best fit; likewise the distributions with the smallest values in AIC and BIC. However, the density fit plot for dataset 1 and 2 as seen in Figures 5 and 6 support the findings in Tables 8 and 9 respectively.

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CONCLUSION

The study engaged the development of symmetric-asymmetric model, combining logistic distribution and Jones G family of distributions. The density and distribution functions have been developed, and in their series representations as well; where the density function indirectly showed to be a proper probability distribution function. Other derived properties are the moments and the related measures; the moment generating function and other similar generating functions; survival, hazard, reverse hazard and cumulative hazard functions; the measures of central tendency, variance, skewness and kurtosis; Renyi and Tsalli entropy; maximum likelihood estimators, quantile simulating function and the probability numerical simulation; and order statistics. JGLD was fitted to oil/winding temperature data and carbon fiber data which were primarily and secondarily obtained respectively. The different inferential criteria showed that the JGLD outperformed both of the asymmetric and symmetric distributions compared. By implication, the research recommends JGLD as a dual purpose model for fitting skewed-symmetric (asymmetric) data and symmetric outcomes.

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Conflict of Interest

The authors have declared conflict of no interest.

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