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WELL-CONDITIONED AND ILL CONDITIONED LINEAR FIRST ORDER INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT: *We construct a linear first order ordinary differential equation with the parameter λ. We show that our constructed equation is well conditioned for $\lambda < 0$ and ill conditioned for $\lambda > 0$. We also state and prove some related theorems.*

KEYWORDS: Well-conditioned, Ill conditioned, Initial value problem, Differential equation, Linear differential equation.

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INTRODUCTION

Let us consider initial value problem of a linear first order ordinary differential equation:

$$y^{\iota} + p(t)y = r(t) \; ; \quad y(t_0) = y_0 \, , \quad t_0 \; \leq t \; \leq b \tag{1}$$

The consideration of linear differential equations of the form (1) is because most models in science, engineering, social sciences, management, etc can be formulated or reduced to the form of (1). See [2], [3], [4], [5] and [6].

Linear differential equation (1) has various applications in engineering where $r(t)$ is frequently called the input and $y(t)$ is called the output or response to the input, see [4]. For instance, in electrical engineering, the differential equation may govern the behavior of an electric circuit and the output $y(t)$ is obtained as the solution of that equation corresponding to the input $r(t)$, see [4]. Also, certain nonlinear equations can be reduced to linear form. For instance, the Verhulst equations otherwise called the logistic population model can be reduced to the form of (1)

Considering the various applications and usefulness of a linear differential equation (1), it therefore becomes imperative to study the property of its well-conditioned and ill conditioned respectively.

Well-Conditioned and Ill Conditioned Linear First Order Differential Equation

The words well-conditioned and ill conditioned are used extensively for matrices and systems of linear equations. See [1] and [7]. It is worthy of mention that a well conditioning/ill conditioning of a problem which is called respectively mathematical stability/instability is different from a numerical stability/instability of a method, See [1], [3], [4] and [7].

[1] defines the mathematical stability of a first order initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{2}$$

as an effect on the solution $y(t)$ of a perturbation in the initial value y_0 of (2).

For instance, when solving (2), we generally assume that the solution $y(t)$ is being sought on a given interval $t_0 \leq t \leq b$. In that case, it is possible to obtain the following stability. Make a small change in the initial value for the initial value problem, changing y_0 to $y_{0+\epsilon}$. Call the resulting solution $y_\epsilon(t)$.

Thus, our perturbed problem will be:

$$y'_{\epsilon}(t) = f(t, y_\epsilon(t)), \quad t_0 \leq \; t \leq \; b, \; y_\epsilon(t_0) = y_{0+\epsilon} \tag{3}$$

It can be shown that for all small values of ϵ, see [1], $y(t)$ and $y_\epsilon(t)$ exist on the interval $[t_0, b]$ and moreover,

$$\| \, y_\epsilon - \; y \, \|_\infty = max \, |y_\epsilon(t) - y(t)| \leq c\epsilon \tag{4}$$

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for some $c > 0$ that is independent of ϵ. Thus, small changes in the initial value y_0 will lead to small changes in the solution $y(t)$ of the initial value problem. If the maximum error in (4) is much larger than ϵ, then the initial value problem (2) is said to be ill conditioned, otherwise it is said to be well conditioned, see [1].

RESULTS AND DISCUSSION

Let us reformulate (1):

$$y' + p(t)y = r(t)\,; \qquad y(t_0) = y_0, \qquad t_0 \le t \le b$$

It follows that,

$$y' = -\,p(t)y + r(t)$$

If we set $-p(t) = \lambda$ and $r(t) = \lambda t$,

then we obtain our constructed equation as:

$$y' = \lambda y + \lambda t = \lambda[y + t] \tag{5}$$

Equation (5) is called, Ntekim linear first order differential equation. In particular, let our initial condition be given for example as

$$y(0) = 1,\ 0 \le t \le b$$

(5) Can also be rewritten as:

$$y' - \lambda y = \lambda\, t\,; \qquad y(0) = 1, \qquad 0 \le t \le b \tag{6}$$

Rewriting (6) in its differential form, we've :

$$(-\lambda y - \lambda t)\, dt + dy = 0 \tag{7}$$

Comparing with $Mdt + Ndy = 0$

$$M = -\lambda y - \lambda t \quad \text{and} \quad N = 1$$

$$\frac{\partial M}{\partial y} = -\,\lambda \quad \text{and} \quad \frac{\partial N}{\partial t} = 0$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} = -\,\lambda$$

Theorem 1: The differential equation $y' = \lambda[y + t]$ admits an integrating factor that is dependent only on t.

Proof: $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} = -\lambda = p(t).$$

We can say that, $\frac{dln\mu}{dx} = -\lambda.,$ It follows that, $\mu = e^{-\lambda t}$

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Let us multiply (6) through by $e^{-\lambda t}$

$$e^{-\lambda t}[y' - \lambda y] = e^{-\lambda t}\lambda t$$

$$\frac{d}{dt}(ye^{-\lambda t}) = \lambda t e^{-\lambda t}$$

It follows that,

$$ye^{-\lambda t} = \lambda \int \quad te^{-\lambda t}dt + c$$

$$ye^{-\lambda t} \quad = \quad \lambda\left[\frac{-te^{-\lambda t}}{\lambda} + \frac{1}{\lambda}\int \quad e^{-\lambda t}dt\right] + c \qquad ye^{-\lambda t} = -te^{-\lambda t} - \frac{e^{-\lambda t}}{\lambda} + c$$

$$y = -t - \frac{1}{\lambda} + ce^{\lambda t}, \qquad t \geq 0$$

Applying the initial condition, we've:

$$y(0) = 1 = -\frac{1}{\lambda} + c \Rightarrow c = 1 + \frac{1}{\lambda}$$

It follows that,

$$y(t) = -t - \frac{1}{\lambda} + (1 + \frac{1}{\lambda})e^{\lambda t}, \qquad t \geq 0 \tag{8}$$

Theorem 2: $\mu = e^{-\lambda t}$ is an integrating factor for the differential equation $y' = \lambda[y + t]$

Proof:

Multiply both sides of (7) through by $e^{-\lambda t}$

$$(-\lambda ye^{-\lambda t} - \lambda te^{-\lambda t})\, dt + e^{-\lambda t}dy = 0 \tag{9}$$

Comparing with $Mdt + Ndy = 0$

$$M = -\lambda ye^{-\lambda t} - \lambda te^{-\lambda t}, \qquad N = e^{-\lambda t}$$

$$\frac{\partial M}{\partial y} = -\lambda e^{-\lambda t}, \qquad \frac{\partial N}{\partial t} = -\lambda e^{-\lambda t}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, it follows that (9) is an exact differential equation and hence, $\mu = e^{-\lambda t}$ is an integrating factor for the given differential equation. Being motivated by the definition of mathematical stability / instability by [1], let us consider the perturbed problem for our constructed initial value problem (5) which is given by :

$$y'_{\epsilon}(t) = \lambda[y_{\epsilon}(t) + t], \quad 0 \leq t \leq b, \quad y_{\epsilon}(0) = 1 + \epsilon \tag{10}$$

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This perturbed problem has a solution given by:

$y_\epsilon(t) = -t - \frac{1}{\lambda} + (1 + \epsilon + \frac{1}{\lambda})e^{\lambda t},\ t \geq 0$
(11)

For the error,

$y(t)$ - $y_\epsilon(t)$ = - $\epsilon e^{\lambda t}$
(12)

$0 \leq t \leq b$ $\quad max\ |y(t) - y_\epsilon(t)| = \{|\epsilon|,\ \lambda < 0\ |\epsilon| e^{b\lambda},\ \lambda > 0$
(13)

Observe that, if $\lambda < 0$, the error $|y(t) - y_\epsilon(t)|$ decreases as t increases. Thus, we say that (5) is stable or well-conditioned when $\lambda < 0$. In contrast, for $\lambda > 0$, the error $|y(t) - y_\epsilon(t)|$ increases as t increases. And for $b\lambda$ moderately large, say $b\lambda \geq 10$, the change in $y(t)$ is quite significant at $t = b$. The problem (5) is increasingly unstable or ill conditioned as λ increases, that is, when $\lambda > 0$.

Theorem 3: The initial value problem

$y' = \lambda[y + t],\ \ y(0) = 1,\ 0 \leq t \leq b$ is well conditioned for $\lambda < 0$ and ill conditioned for $\lambda > 0$.

Proof: This follows from the result above.

CONCLUSION

We have constructed a linear first order ordinary differential equation with the parameter λ. We have shown that our constructed problem is stable for $\lambda < 0$ and unstable for $\lambda > 0$. We have also propounded and proved three related theorems.

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