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THE CHOICE OF MAXIMUM LIKELIHOOD ESTIMATE AMONG THE LOGNORMAL, WEIBULL AND MIXED-LOGNORMAL-WEIBULL DISTRIBUTIONS: AN EMPIRICAL EXAMINATION OF STOCK PRICE RETURNS

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ABSTRACT: *This paper compared the maximum likelihood estimates (MLE) of lognormal, Weibull and Mixed-lognormal-weibull distributions. The data for this study were Coca-cola stock price returns obtained from https://ng.www.investing.com/equities/cocacola-bottle-historical-data and the result with the help of Excel package shows that Weibull distributionhas the minimum Mean Squared Error* $(MSE_{\min})$ *among the lognormal and Mixed-lognormal-weibull distributions; hence, the maximum likelihood estimate of the Weibull distribution is the choice.*

KEYWORDS: Maximum Likelihood Estimate, Lognormal Distribution, Weibull Distribution, Mixed-Lognormal-Weibull Distribution(MLWD), Mean Squared Error (MSE).

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INTRODUCTION

The method of Maximum Likelihood Estimate was formed in early 1920's by R.A. Fisher, one of the foremost statisticians of our time. Fisher demonstrated the advantages of Maximum Likelihood Estimation (MLE) by showing that it yields sufficient estimators whenever they exist, and that the estimators are asymptotically minimum variance unbiased estimators. Therefore, the essential feature of the method of Maximum Likelihood Estimate is that we look at the values of a random sample and then choose as our estimate of the unknown population parameter the value for which is the probability of obtaining the observed data (Freund & Walpole, 1980).

Maximum Likelihood Estimate Method helps to find the estimator for the unknown population parameter. There are other methods of estimation also available such as Least Square Method, Bayesian Estimation Method, Method of Moments, but Maximum Likelihood Estimation is the widely used method for estimating parameters. See, for example, Renganathan (2017), Norden (1972), Westling et al. (2023), Kvan and Samaniego (1994), Eliason (1993), Ginos (2009), and Kozlov and Marysuradze (2019).

If the observed sample values are $X_1, X_{2,}..., X_n$, we can write,

$$P\left(X_1 = x_1, X_2 = x_2, ..., X_n = x_n\right) = f\left(x_1, x_2, ..., x_n : \theta\right) \tag{1}$$

Equation (1) is just the value of the joint probability distribution of the random variables $X_1, X_2, ..., X_n$ at the sample point $\left(x_1, x_2, ..., x_n\right)$. Since, the sample values have been observed and are thus, fixed numbers, we regard $f\left(x_1, x_2, ..., x_n : \theta\right)$ as the value of the function of the parameters θ, referred to as the likelihood function given by

$$L\left(\theta\right) = f\left(x_1, x_2, ..., x_n : \theta\right) \tag{2}$$

Hence, the method of maximum likelihood estimate consists of maximizing the likelihood function with respect to θ and it is referred to as the value of θ which maximizes the likelihood function as the minimum likelihood estimate of θ. See, for example, Freund and Walpole (1980).

Hence, in this paper, we consider the maximum likelihood estimate of 2-parameter lognormal, Weibull and Mixed-Lognormal-Weibull Distributions in the view of selecting the distribution with minimum variance as the choice among the considered distributions.

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MATERIALS AND METHODS OF ANALYSIS

Materials

The Maximum Likelihood Estimate of Lognormal Distribution

Lognormal distribution is very useful in modeling continuous random variables and data which would be considered normally distributed data except for the fact that it may be more or less skewed. See, for example, Limpert et al. (2001).

The likelihood function of the lognormal distribution for a series of $X_{i's}\left(i=1.2,...,n\right)$ is derived by taking the product of the probability densities of the individual $X_{i's}$;

$$L\left(x_i,\mu,\sigma^2\right)=\prod_{i=1}^{n} f\left(x_i,\mu,\sigma^2\right) \tag{3}$$

$$=\prod_{i=1}^{n}\left[\left(2\pi\sigma^2\right)^{-\frac{1}{2}}\frac{1}{x}\exp\left(-\frac{\ln\left(x\right)-\mu}{2\sigma^2}\right)^2\right]$$

$$=\left(2\pi\sigma^2\right)^{-\frac{n}{2}}\prod_{i=1}^{n}\frac{1}{x}\exp\left(\sum_{i=1}^{n}-\frac{\left(\ln\left(x\right)-\mu\right)^2}{2\sigma^2}\right) \tag{4}$$

The log-likelihood function of the lognormal for the series of $X_{i's}\left(i=1,2,...,n\right)$ is then derived by taking the natural log of the likelihood function.

$$\ell\left(x_i,\mu,\sigma^2\right)=\ln\left(2\pi\sigma^2\right)^{-\frac{n}{2}}\prod_{i=1}^{n}\frac{1}{x}\exp\left(\sum_{i=1}^{n}\left[\frac{\left(\ln\left(x\right)-\mu\right)^2}{2\sigma^2}\right]\right)$$

$$=-\frac{n}{2}\ln\left(2\pi\sigma^2\right)-\sum_{i=1}^{n}\ln\left(x_i\right)-\sum_{i=1}^{n}\left(\frac{\left(\ln\left(x\right)-\mu\right)^2}{2\sigma^2}\right)$$

$$=-\frac{n}{2}\ln\left(2\pi\sigma^2\right)-\sum_{i=1}^{n}\ln\left(x_i\right)-\sum_{i=1}^{n}\left(\frac{\ln\left(x_i\right)^2}{2\sigma^2}\right)+\sum_{i=1}^{n}\left(\frac{2\ln\left(x_i\right)\mu}{2\sigma^2}\right)-\frac{n\mu}{2\sigma^2} \tag{5}$$

We now find the estimate of μ and σ, which maximizes $\ell\left(x_i,\mu,\sigma^2\right)$. To do this, we then differentiate partially with respect to μ and σ and set it to equal to zero.

Now let $\theta=\ell\left(x_i,\mu,\sigma^2\right)$ such that,

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$$\frac{\partial\theta}{\mu} = \sum_{i=1}^{n} \frac{\ln(x_i)}{\hat{\sigma}^2} - \frac{2n\hat{\mu}}{2\sigma^2} = 0 \tag{6}$$

$$\frac{n\hat{\mu}}{\sigma^2} = \sum_{i=1}^{n} \frac{\ln(x_i)}{\hat{\sigma}^2}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln(x_i)}{n} \tag{7}$$

$$\frac{\partial\theta}{\partial\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} - \sum_{i=1}^{n} \frac{\left(\ln(x_i) - \mu\right)^2}{2} \left(-\hat{\sigma}^2\right)^{-2}$$

$$= -\frac{n}{2\hat{\sigma}^2} + \sum_{i=1}^{n} \frac{\left(\ln(x_i) - \mu\right)^2}{2\hat{\sigma}^4} = 0 \tag{8}$$

$$\therefore \hat{\sigma}^2 = \sum_{i=1}^{n} \frac{\left(\ln(x_i) - \sum_{i=1}^{n} \frac{\ln(x_i)}{n}\right)^2}{n} \tag{9}$$

The lognormal distribution has the following properties:

(i) Probability Density Function (PDF) is given by

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2}\frac{\left(\ln(x) - \mu\right)^2}{\sigma^2}\right\}, x \in (0, \infty) \tag{10}$$

(ii) The Cumulative Distribution Function is given as

$$F(x) = \Phi\left\{\frac{(\ln x - \mu)}{\sigma}\right\}, x \in (0, \infty) \tag{11}$$

(iii) The Reliability (Survival) function is given by

$$R_{\log}\left(X_t/\theta_i\right) = 1 - \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right)^2 \tag{12}$$

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(iv) The hazard function is given as:

$$h_{log}\left(X_t/\theta_i\right)=\frac{\frac{1}{\sqrt{2\pi\sigma x}}\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\}}{1-\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \tag{13}$$

And summarily, the central moments of lognormal distribution are given as:

(i) $$E\left(X_L\right)=\exp\left\{\mu+\frac{\sigma^2}{2}\right\} \tag{14}$$

(ii) $$Var\left(X_L\right)=\left[\exp\left\{2\mu+\sigma^2\right\}\left(\exp\left\{\sigma^2\right\}-1\right)\right] \tag{15}$$

(iii) $$Skew\left(X_L\right)=\left[\left(\exp\left\{\sigma^2\right\}+2\right)\sqrt{\left(\exp\left\{\sigma^2\right\}-1\right)}\right] \tag{16}$$

(iv) $$Kurt\left(X_L\right)=\left[\exp\left\{4\sigma^2\right\}+2\exp\left\{\sigma^4\right\}+3\exp\left\{\sigma^4\right\}-3\right] \tag{17}$$

Note that skewness and kurtosis do not depend on μ because μ is a location parameter.

The Maximum Likelihood Estimate of Weibull Distribution

The method of maximum likelihood estimation is also commonly used for estimating the parameters of Weibull distribution. See, for example, Cohen (1965), Harter and Moore (1965), Al-fawazan (2000), Cheng and Chen (1988), Johnson et al. (1994), and Nwobi and Ugomma (2014).

A random variable, X_t, is Weibull distributed if its probability density function is given as:

$$f\left(x_t\right)=\begin{cases}\frac{\beta}{\alpha}\left(\frac{x_t-\nu}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{x_t-\nu}{\alpha}\right)^{\beta}\right\}\\ 0\end{cases},\ x_t\ge 0,\alpha>0,\beta>0 \text{ and } \nu=0 \tag{18}$$

Here, α and β are positive constants, while ν is nonnegative. The two constants α and β are the parameters of the Weibull distribution; when α , β and ν are known, the distribution whose probability density function (pdf) is given by Equation (18) and is referred to as a 3-parameter Weibull Distribution (Nwobi & Ugomma, 2014).

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Equation (18) is known as a 2-parameter Weibull when $\nu = 0$ and it is given by:

$$f(x) = \left\{ \left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{\beta-1} \exp\left\{ -\left(\frac{X_t}{\alpha}\right)^{\beta} \right. \right. \quad x_t > 0, \beta > 0, \alpha > 0, \nu = 0$$

(19)

Let $X_1, X_2, ..., X_n$ be a random sample of size, n, drawn from a population with probability density function, $f(x, \theta)$; where $\underline{\theta} = (\beta, \alpha)$ is an unknown vector of parameters. The likelihood function is defined by:

$$L = f(x_i, \beta, \alpha) = \prod_{i=1}^{n} f(x_i, \beta, \alpha) \tag{20}$$

The maximum likelihood of $\theta = (\beta, \alpha)$, maximizes L or equivalently, the logarithm of L when

$$\frac{\partial \ln L}{\partial \theta} = 0 \tag{21}$$

Consider the Weibull distribution pdf given in Equation (19). Then, its likelihood function is given by:

$$L(x_1, x_2, ..., x_n; \beta, \alpha) = \prod_{i=1}^{n} \left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{X_t}{\alpha}\right)^{\beta} \right]$$

$$= \left(\frac{\beta}{\alpha}\right)\left(\frac{X_t}{\alpha}\right)^{n\beta-n} \sum_{i=1}^{n} x_i^{(\beta-1)} - \ln\left(\alpha^{\beta-1}\right) \exp\left[-\sum_{i=1}^{n} \left(\frac{X_t}{\alpha}\right)^{\beta} \right] \tag{22}$$

Taking the natural logarithm of both sides, we obtain:

$$\ln L = n \ln\left(\frac{\beta}{\alpha}\right) + (\beta - 1)\sum_{i=1}^{n} x_i - \ln\left(\alpha^{\beta-1}\right) - \sum_{i=1}^{n} \left(\frac{X_t}{\alpha}\right)^{\beta} \tag{23}$$

Differentiating, Equation (23) partially w.r.t β and α, in turn equating to zero, we obtain the following estimating equations as:

$$\frac{\partial}{\partial \beta} \ln L = \tfrac{n}{\beta} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\alpha} \sum_{i=1}^{n} x_i^{\beta} \ln x_i = 0 \tag{24}$$

and

$$\frac{\partial}{\partial \alpha} \ln L = -\tfrac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} x_i^{\beta} = 0 \tag{25}$$

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From Equation (25), we obtain an estimator of α as:

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\beta} \tag{26}$$

and on substitution of Equation (25) in Equation (26) we obtain:

$$\frac{1}{\beta} + \frac{1}{\alpha}\sum_{i=1}^{n}\ln x_i - \frac{\sum_{i=1}^{n} x_i^{\beta}\ln x_i}{\sum_{i=1}^{n} x_i^{\beta}} = 0 \tag{27}$$

$$\hat{\beta} = \frac{1}{\ln x_i - \frac{\sum_{i=1}^{n} x_i}{n}} \tag{28}$$

The Weibull distribution has moments of all orders and the following properties:

(i) The Mean is given by:

$$E(X_t) = \mu$$

$$\Rightarrow E(X_t) = \alpha\Gamma\left(1 + \frac{1}{\beta}\right)$$

(29)

(ii) The variance is given by:

$$Var(X_t) = E(X_t^2) - \left[E(X_t)\right]^2$$

$$= \alpha\Gamma\left(1+\frac{1}{\beta}\right)^2 - \left[\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)\right]^2$$

$$Var(X_t) = \sigma^2\Gamma\left(\frac{2}{\beta}+1\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)$$

(30)

(iii) Skewness is given by:

$$Skew(X_t) = E\left[(X_t) - E(X_t)\right]^3 = 2\mu_1^3 - 3\mu_2\mu_1 + \mu_3$$

$$= 2\frac{1}{\alpha^{\frac{3}{\beta}}}\left[\Gamma\gamma_1\right]^3 - 3\frac{1}{\alpha^{\frac{2}{\beta}}}\Gamma\left(1+\frac{2}{\beta}\right)\frac{1}{\alpha^{\frac{1}{\beta}}}\Gamma\gamma_1 + \frac{1}{\alpha^{\frac{3}{\beta}}}\Gamma\left(1+\frac{3}{\beta}\right)$$

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$$= \frac{1}{\alpha^{\frac{1}{\beta}}}\left[\left[2\gamma_1\right]^3 - 3\gamma_1\Gamma\left(1+\frac{2}{\beta}\right) + \Gamma\left(1+\frac{3}{\beta}\right)\right] \quad (31)$$

(iv) Kurtosis is given by:

$$kurt\left(X_t\right) = \frac{\Gamma\left(1+\frac{4}{\beta}\right) - 4\Gamma\gamma_1\Gamma\left(1+\frac{3}{\beta}\right) + 6\Gamma^2\gamma_1\Gamma\left(1+\frac{2}{\beta}\right) - 3\Gamma^4\gamma_1}{\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\gamma_1^2\right]}$$

$$\frac{\Gamma\left(1+\frac{4}{\beta}\right) - 4\gamma_1\Gamma\left(1+\frac{3}{\beta}\right) + 6\gamma_1^2\Gamma\left(1+\frac{2}{\beta}\right) - 3\gamma_1^4}{\gamma_2}$$

(32)

where $\gamma_1 = \Gamma\left(1+\frac{1}{\beta}\right)$ and $\gamma_2 = \left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)^2\right]$

(v) The survival and failure (hazard) rates of 2-parameter Weibull Distribution are, respectively, given as:

$$\bar{F}_{web}\left(X_t\right) = 1 - F\left(X_t\right) = \exp\left[-\left(\frac{X_t}{\alpha}\right)^{\beta-1}\right], X_t > 0 \quad (33)$$

and

$$h_{web}\left(X_t\right) = \frac{f\left(X_t\right)}{\bar{F}\left(X_t\right)} = \frac{\beta}{\alpha}\left(\frac{X_t}{\alpha}\right)^{\beta-1}, X_t > 0 \quad (34)$$

The Maximum Likelihood Estimate of MLWD

The mixture of two or more component distributions is the newest area of concern in modeling life data and reliability studies. In the course of the development of this distribution, it was assumed that the population consists of a mixture of two independent sub-populations with zero correlation and each population has its unique properties.

The distribution for the mixed population can be expressed as:

$$f\left(x_i\right) = \sum_{i=1}^{n} w_i g_i\left(x_i;\theta_i\right) \quad (35)$$

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where, $0 \le w_i \le 1; \sum w_i = 1, i = 1, 2, ..., n; \theta_i$ are the parameters representing the mixed distribution, and w_i are mixing parameters, which represent the proportion of combining a number of distributions. See, for example, Razali et al. (2008), Kollu et al. (2012), Sultan and Al-Moisher, 2015.

The probability density function (pdf) of the mixture distribution in Equation (35) is expressed as:

$$f\left(x_i; w, \theta_i\right) = wf_1\left(x_i; \theta_1\right) + \left(1 - w\right) f_2\left(x_i; \theta_2\right) \tag{36}$$

where, w and (1 – w) are the mixing parameters whose sum is equal to 1.

The pdf of lognormal and Weibull distributions are given, respectively, as:

$$f\left(x_1\right) = \frac{1}{\sqrt{2\pi\sigma^2 x}} \exp\left\{-\frac{1}{2}\frac{\left(\ln x - \mu\right)^2}{\sigma^2}\right\}, x \ge 0, \sigma > 0 \tag{37}$$

and

$$f\left(x_2\right) = \frac{\beta}{\sigma}\left(\frac{x}{\alpha}\right)^{\beta - 1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}, 0 \le x \le \infty, \beta, \sigma > 0 \tag{38}$$

And the respective cumulative function of Equation (37) and Equation (38) are respectively, given as:

$$F\left(x_1\right) = \frac{\left(\ln x - \mu\right)}{\sigma}, x \ge 0, -\infty < \mu < \infty, \sigma > 0 \tag{39}$$

and

$$F\left(x_2\right) = \exp\left\{-\frac{x}{\alpha}\right\}^{\beta}, x \ge 0, \beta, \alpha > 0 \tag{40}$$

So, the joint Pdf of Equation (37) and Equation (38) can be expressed as:

$$g_1\left(x_i; \theta_1\right) + g_2\left(x_i; \theta_2\right) = f\left(x_i; \theta_i\right) \tag{41}$$

Substituting Equation (35) and Equation (36) into (37), we obtain the joint pdf of the mixing distributions as:

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$$f\left(x_i, w, \mu, \sigma^2, \beta, \alpha\right) = w\left(\begin{array}{l}\frac{1}{\sqrt{2\pi\sigma^2 x}}\exp\left\{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma^2}\right)^2\right\} \\ +(1-w)\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left\{-\frac{x}{\alpha}\right\}^{\beta}\end{array}\right) \quad (42)$$

And the joint CDF in Equation (42) can be given as:

$$F\left(x_i, \mu, \sigma^2, \beta, \alpha\right) = \frac{w(\ln x-\mu)}{\sigma} + (1-w)\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}$$

(43)

The Maximum Likelihood Estimate for the scale and mixing parameters were obtained by Menden and Harter in 1958, where the shape parameter was assumed to be known. A number of authors have found the Maximum Likelihood Estimation method very useful in obtaining the parameters of mixture distributions. Ssee, for example, Ashour (1987), Ahmad and Abdurahman (1994), Sultan and Al-Mosheer (2015), Razali et al. (2008), Elmahdy (2007), Neuman (1998), Kacecilogu and Wang (1998), Ugomma and Nwobi (2023).

The maximum likelihood approach considered for this study for the estimation of the parameters of the mixed distribution density function in Equation (42) is based on a random sample of size n. The MLE $\hat{\theta}$ is obtained as the solution of the likelihood equation as:

$$\frac{\partial\theta}{\partial\theta_i} = 0$$

(44)

Or equivalently, the partial derivative of the log likelihood function is given as:

$$\frac{\partial \log L(\theta)}{\partial\theta_i} = 0$$

$$= \sum_{i=1}^{n} \log\left(f\left(x\ \right)\right)$$

(45)

where

$$L(\theta) = \prod_{i=1}^{n} f\left(x_i;\theta_i\right)\ ;\ i = 1,2,3,4,5 \quad (46)$$

Therefore, the likelihood function corresponding to the mixture density in Equation (42) is then expressed as:

$$L(\theta) = \prod_{i=1}^{n}\left[w\left(f_1\left(x_i;\theta_1\right)\right) + (1-w)\left(f_2\left(x_i;\theta_2\right)\right)\right] \quad (47)$$

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where, $\theta_1 = (\mu, \sigma)$ and $\theta_2 = (\alpha, \beta)$.

This implies that:

$$f(x_i, w, \mu, \sigma^2, \beta, \alpha) = \prod_{i=1}^{n} \begin{bmatrix} w(2\pi\sigma^2)^{-\frac{1}{2}} \frac{1}{x} \exp\left\{ -\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma^2} \right)^2 \right\} \\ +(1-w)\frac{\beta}{\alpha}\left(\frac{x}{\sigma}\right)^{\beta-1} \exp\left\{ -\frac{x}{\alpha} \right\}^{\beta} \end{bmatrix}$$

$$= w\left[(2\pi\sigma^2)^{-\frac{1}{2}} \prod_{i=1}^{n} \frac{1}{x} \exp\left\{ \sum_{i=1}^{n} -\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma^2} \right)^2 \right\} \right] +$$

$$\left[(1-w)\frac{\beta}{\alpha}\left(\frac{x}{\sigma}\right)^{n\beta-n} \sum_{i=1}^{n} x_i^{\beta-1} - \ln(\alpha^{\beta-1}) \exp\left\{ -\sum_{i=1}^{n} \frac{x}{\alpha} \right\}^{\beta} \right]$$

(48)

Taking the log likelihood function for the mixture distribution in Equation (48), we obtain:

$$\ell(x_i; \theta_1, \theta_2) = \left(\log\left(w\left(-\frac{n}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{n} \ln x - \sum_{i=1}^{n} \ln x^2 + \sum_{i=1}^{n}\left(\frac{2\ln x - \mu}{2\sigma^2} \right) - \frac{n\mu}{2\sigma^2} \right) \right) \right)$$

$$+\left(\log\left((1-w)n\ln\left(\frac{\beta}{\alpha}\right) + (\beta-1)\sum_{i=1}^{n} x_i - \ln(\alpha^{\beta-1}) - \sum_{i=1}^{n}\left(\frac{x}{\alpha}\right)^{\beta} \right) \right)$$

(49)

Let Q, be the function of the log likelihood, such that:

$$Q = (w, \mu, \sigma^2, \alpha, \beta) = \sum_{i=1}^{n} \left[\log\begin{pmatrix} w\left(-\frac{n}{2}\ln(2\pi\sigma^2) - \ln x - \frac{\ln x^2}{2\sigma^2} + \frac{\ln x - \mu}{2\sigma^2} - \frac{n\mu}{2\sigma^2} \right) \\ +\left((1-w)n\ln\left(\frac{\beta}{\alpha}\right) + (\beta-1)x_i - \ln(\alpha^{\beta-1}) - \left(\frac{x}{\alpha}\right)^{\beta} \right) \end{pmatrix} \right]$$

(50)

Taking the partial derivative of the log likelihood function of Equation (50), with respect to the parameters and in turn equating to zero yields the following equations:

$$\frac{\partial Q}{\partial \mu} = \sum_{i=1}^{n} \frac{\ln x}{\sigma^2} - \frac{n\mu}{\sigma^2} = 0$$

$$\hat{\mu}_{mix} = \sum_{i=1}^{n} \frac{\ln x}{n} \quad (51)$$

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$$\frac{\partial Q}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \sum_{i=1}^{n}\left(\frac{\ln x - \mu}{2}\right)^2 \left(-\sigma^2\right)^{-2}$$

$$= -\frac{n}{2\sigma^2} - \sum_{i=1}^{n}\left(\frac{\ln x - \mu}{2\sigma^4}\right)^2 = 0$$

$$= \hat{\sigma}^2_{mix} = \sum_{i=1}^{n}\left(\frac{\ln x - \hat{\mu}}{n}\right)^2$$

(52)

$$\frac{\partial Q}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} x_i - \frac{1}{\alpha}\sum_{i=1}^{n} x_i^{\beta} - \frac{1}{\alpha} = 0$$

$$= \hat{\beta}_{mix} = \frac{1}{\ln x_i - \frac{1}{n}\sum_{i=1}^{n} x_i}$$

(53)

$$\frac{\partial Q}{\partial \alpha} = -\frac{n}{2} + \frac{1}{\alpha^2}\sum_{i=1}^{n} x_i^{\beta} = 0$$

$$\Rightarrow \hat{\alpha}_{mix} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\beta} \tag{54}$$

$$\frac{\partial Q}{\partial w} = \frac{f_1\left(x_i;\theta_1\right) - f_2\left(x_i;\theta_2\right)}{wf_1\left(x_i;\theta_1\right) + \left(1-w\right) f_2\left(x_i;\theta_2\right)} = \frac{f_1\left(x_i;\mu,\sigma^2\right) - f_2\left(x_i;\alpha,\beta\right)}{wf_1\left(x_i;\mu,\sigma^2\right) + \left(1-w\right) f_2\left(x_i;\alpha,\beta\right)} \tag{55}$$

And some of the properties of MLWD are given as:

(i) The Mean:

$$E\left(X_{mix}\right) = w\left(\exp\left\{\mu + \frac{1}{2}\sigma^2\right) + \left(1-w\right)\alpha\Gamma\left(1 + \frac{1}{\beta}\right) \tag{56}$$

(ii) The Variance:

$$Var(X_{mix}) = E\left(X\right) - \left[EX\right]^2$$

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$$= w\left(\exp\left\{-2\mu+\sigma^{2}\right)\left(\exp\left\{-\left(\sigma^{2}-w\right)\right)+(1-w)\alpha^{2}\Gamma\left(1+\frac{2}{\beta}\right)\right.\right.$$

$$-(1-w)\Gamma^{2}\left(1+\frac{1}{\beta}\right)-2w(1-w)\left(\exp\left\{-\left(\mu+\frac{1}{2}\sigma^{2}\right)\Gamma\left(1+\frac{2}{\beta}\right)\right)\right.$$

(57)

(iii) The Skewness:

$$skew\left(X_{mix}\right)=w\left(\exp\left\{\sigma^{2}+2\right)\sqrt{\left(\exp\left\{\sigma^{2}-1\right)\right.}\right.$$

$$+(1-w)\frac{1}{\alpha^{\frac{1}{\beta}}}\left[\left(2\gamma_{1}\right)^{3}-3\gamma_{1}\Gamma\left(1+\frac{2}{\beta}\right)+\Gamma\left(1+\frac{3}{\beta}\right)\right]$$

(58)

(iv) The Kurtosis:

$$kurt\left(X_{mix}\right)=w\left(\exp\left\{4\sigma^{2}\right)+2\left(\exp\left\{\sigma^{4}\right)+3\exp\left(\sigma^{4}\right)-3\right)\right.\right.$$

$$+(1-w)\Gamma\left(1+\frac{4}{\beta}\right)-4\gamma_{1}\Gamma\left(1+\frac{4}{\beta}\right)+6\gamma_{1}^{2}\Gamma\left(1+\frac{2}{\beta}\right)-\frac{3\gamma_{1}^{4}}{\gamma_{2}} \qquad (59)$$

where $\gamma_{1}=\Gamma\left(1+\frac{1}{\beta}\right),\ \gamma_{2}=\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^{2}\left(1+\frac{1}{\beta}\right)^{2}\right]$

(vi) Reliability (Survival) function:

$$R_{mix}\left(X_{t}/\theta_{i}\right)=\sum_{i=1}^{n}\left[w\left(1-\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)^{2}\right)+(1-w)\left(1-\exp\left\{\frac{X_{t}}{\alpha}\right)^{2}\right]\right. \qquad (60)$$

(vii) Hazard function:

$$h_{mix}\left(X_{t}/\theta_{i}\right)=\sum_{i=1}^{n}\left(w\left[\frac{\frac{1}{\sqrt{2\pi\sigma x}}\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^{2}\right.}{1-\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)^{2}}\right]+(1-w)\left(\frac{\beta}{\sigma}\right)\left(\frac{X_{t}}{\alpha}\right)^{\beta-1}\right. \qquad (61)$$

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DATA DESCRIPTION AND METHOD OF ANALYSIS

Data Description

The data used for this study is the monthly stock prices collected from Coca-Cola Bottling Company plc from 2013 to 2022. Coca-Cola Bottling Company is one of the public liability companies enlisted in the Nigerian Stock Exchange and the data were gotten from https://ng.www.investing.com/equities/cocacola-bottle-historical-data

Method of Analysis

Let $S_1, S_2, \ldots, S_N$ be a random sample of size N from a population. Define $X_t = \ln\left(\frac{S_t}{S_{t-1}}\right), X_t \in (-\infty, \infty)$ as returns of the stock prices, say, $\{S_t : S_t > 0\}$. Hereinafter, X_t is referred to as our data set.

Comparison of Maximum Likelihood Estimate of the Distributions

To compare the distributions with the view of choosing the best among them, the method of Mean Squared Error (MSE) will be employed.

The Mean Squared Error criterion for this study is given by:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left[\hat{F}(X_t) - F(X_t)\right]^2 \qquad (62)$$

where

$\hat{F}(X_t)$ is obtained by substituting the parameters of each distribution in their respective cumulative distribution functions (CDF) while $F(X_t) = \frac{i}{N}, i = 1, 2, \ldots, N$ is the empirical distribution function. The method with the minimum Mean Squared Error $(MSE_{\min})$ becomes the best MLE for the stock price returns (Nwobi & Ugomma, 2014).

The test statistic for this test is given by:

$$\chi_c^2 = \frac{\frac{1}{N}\sum_{i=1}^{N}\left[\hat{F}(X_t) - F(X_t)\right]^2}{F(X_t)} \qquad (63)$$

with $N-1$ degrees of freedom.

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EMPIRICAL INVESTIGATION

Table 1: Descriptive Statistics of Absolute Price Returns of Coca-Cola Stock Price

Sample Size	Sample Mean	Standard Dev	Skewness	Kurtosis
119	0.9893	0.0098	0.2992	1.4456

Table 1 displays the descriptive statistics of the stock price returns of Coca-Cola Bottling Company for the period of study. From the output, we observed that the data is positively skewed showing that the right tail of the distribution is longer than the left tail of the distribution and the kurtosis also suggests that the distribution is perfectly peaked, hence indicating that the stock price returns are normally distributed.

Table 2 : Descriptive Statistics of Absolute Log Returns of Coca-Cola Stock Price

Sample Size	**Sample Mean**	**Standard Dev**	**Skewness**	**Kurtosis**
119	-0.0166	0.1083	-0.2013	1.1001

Table 2 also shows the descriptive statistics of absolute log returns of Coca-Cola Bottling Company for the period of study. From the result, we observed that log returns are negatively skewed, showing that the left tail of the distribution is longer than the right tail of the distribution and the kurtosis also suggests peakedness in the distribution, therefore showing that the log returns of the stock price is normally distributed.

Table 3 : The Maximum Likelihood Estimate of the Parameters of the Distributions

Distribution	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$
Lognormal	-0.0166	0.1083		
Weibull			1.0108	-1.0059
MLWD	-0.0166	0.1083	1.0234	-1.0059

Table 3 is a display of the Maximum Likelihood Estimates of the parameters of lognormal distribution, Weibull distribution and Mixed-Lognormal-Weibull distribution (MLWD).

Table 4: Estimated Values of the Properties of the Distributions

Distribution	**Mean**	**Variance**	**Skewness**	**Kurtosis**
Lognormal	2.7047	2.0969	4.5658	3.1975
Weibull	0.0059	2.0097	-1.0091	-1.9920
MLWD	2.1626	1.8948	1.6555	3.4922

Table 4 shows that lognormal distribution has the highest variance, skewness and kurtosis compared to both Weibull distribution and MLWD. Lognormal exhibits positive and high peaked skewness and kurtosis (leptokurtic), meaning that the right tail of the distribution is longer than the left tail. The Weibull distribution has a negative skewness and kurtosis while MLWD has the minimum variance with positive skewness and positive high kurtosis. The result showed that both Weibull distribution and MLWD have better estimates using MLE than lognormal distribution.

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Table 5: Comparison of the MLE Estimate of the Distributions

Distribution	Mean	Variance	Skewness	Kurtosis
Lognormal	2.7047	2.0969	4.5658	3.1975
Weibull	0.0059	2.0097	-1.0091	-1.9920
MLWD	2.1626	1.8948	1.6555	3.4922

From Table 5, we observed that the Weibull distribution has the minimum mean squared error $(MSE_{\min})$. We therefore chose the MLE of the Weibull distribution as the better estimate among the estimates of the lognormal distribution and MLWD.

CONCLUSION

In this study, we compare the maximum likelihood estimates (MLE) of lognormal, Weibull and Mixed-lognormal-weibull distributions and the result shows that Weibull distribution has the minimum Mean Squared Error $(MSE_{\min})$ among the lognormal, Weibull and Mixed-lognormal-weibull distributions; hence, the maximum likelihood estimate of the Weibull distribution is chosen to be the best estimate.

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