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ON THE COMMUTATIVITY DEGREE OF FINITE GROUPS OF ORDER $p^a q^b$ VIA DEGREE EQUATION

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ABSTRACT: *Commutativity degree is a numerical derivation that carries a lot of information about the structure of finite groups. It measures the extent to which two randomly selected non-identity elements of a group commute. The upper bound for the order of the centre of a finite group were obtained by Cody (2010), while Anna (2010) determined same in terms of degree of commutativity; Jelten et al. (2021) worked on commutativity degree p(G) of finite groups via the class equations. In the present paper, we use the derived group of a group as input and the degree equation as a tool to derive a scheme for the commutativity degree of groups of order* $p^a q^b$ *which are essentially groups of order* $4n$ *with* $2 \le n \le 25$, *where* p *is an even prime,* q, *an odd prime such that* $1 < q < 20$; $2 \le a < 7$ *and* $0 \le b < 3$. *With this, we have that* $p(G) = (|G'| + 3)/|G|$ *as one of our results and discovered that 24 groups satisfy the restrictions given as outlined in our discussion in this paper.*

KEYWORDS: Representation, derived groups, prime, group ring, abelianization.

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PRELIMINARIES

Definition

A group consists of a set G together with a rule for combining any two elements g and h to form another element written gh, which must satisfy the following axioms:

(i) For all g, h and k in G, g (hk) = (gh) k

(ii) There exists an element e in G called the identity element such that for all g in G,

$$eg = ge = g$$

(iii) For all g in g, there exists an element g^{-1} in G called the inverse of g such that:

$$g g^{-1} = g^{-1} g = e.$$

The rule for combining the elements of G is called the product operation on g.

It is to be noted that the commutative law is not in general required to hold for all groups. It becomes necessary to distinguish between ab and ba. Hence, we have the next definition.

Definition

A group G with the property that ab = ba for every pair of elements $a,b \in G$ is said to be a commutative or abelian group. A group in which there exists a pair of elements $a,b \in G$ endowed with the property that $ab \neq ba$ is called a non-abelian or noncommutative group.

In the next definition we have that a group consists of smaller groups.

Definition

A non-empty subset N of a group G is said to be a subgroup of G written $N \leq G$ if N is a group under the operation inherited from G. The properties of the subgroup N determine whether it is proper or improper. We call G simple if the only subgroups of G are the trivial ones.

Definition

A group G is said to be cyclic if it is generated by a single element, say a and we write G = <a>, that is, $G = \{a^n : n \in Z\}$ in the finite case. In the infinite case, G is cyclic if the powers of $a \in G$ exhaust G.

For every even number, there is at least one non abelian group of order 2n. An important fact about cyclic groups is that they are generally abelian.

Definition

The number of elements in a group G written |G| is called the order or cardinality of the group. If G is finite of order *n*, we have |G| = n, otherwise $|G| = \infty$. If G has infinite order.

The least number n if it exists such that $a^n = 1$ for a in G is called the order of a and we write o(a) = n, that is, $o(a) = \min\{a > 0: a^n = 1\}$. If no such n exists, then $o(a) = \infty$. In the latter, we

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say that powers of a are distinct but not all are distinct in the former. An element of order two is said to be an involution.

Furthermore, the order of an element of G divides |G|. In particular, $a^{|G|} = 1$, where G is finite.

Definition

Let G be a group and H < G. For q ∈ G, the subset Hq = {Hq: h ∈ H} of G is called the right coset of H in G. Distinct right cosets of H in G form a partition of G, that is, every element of G is precisely in one of them. Left coset is similarly defined. The number of distinct right cosets of H in G written |G : H| is called the index of H in G. If G is finite, so is H and G is partitioned into |G:H| which coset each of order |H| and |H| and |G:H| divide |G|.

Definition

A subgroup N of G such that every left coset is a right coset and vice versa is called a normal subgroup of G, that is, Nx = xN or $x^{-1}Nx \leq N$ and we write $N \trianglelefteq G$. Subgroups of abelian groups are abelian.

The coset decomposition of G in 1.5 above leads to the following consequence:

Proposition

If H is a subgroup of G, then |G|=|G:H||H|. From this we derive a fundamental result in group theory: the Lagrange's Theorem which relates a group to its subgroups.

THEOREM

If G is a finite group and H is a non-empty subgroup of G, then |H| divides |G|.

Remark

The right coset Hq is the equivalence class of q in G. These equivalence classes yield a decomposition of G into disjoint subsets. Consequently, two right cosets are either identical or disjoint. Cosets are also called 'shifted' or 'translated' subgroups. These arise as 'inhomogeneous solution spaces' to linear equations and differential equations. The index of a subgroup in a group counts the number of such subgroups in the group.

The lemma that follows provides us with the criterion for determining when subgroups are normal.

Proposition

If H is a subgroup of G such that H has only two right cosets itself and one other then, H is normal in G. In the finite case, this means that the order of H is one-half of the order of G. Equivalently all subgroups of index 2 are normal.

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Proof

Any element in G is either in H or in G. If $x \in H$, then xH = H = Hx. If $x \notin H$, then xH is the set of elements in G - H since $H \cap Hx = \varnothing$. Thus:

$$xH = G - H = Hx.$$

Therefore, by definition, H is a normal subgroup of G.

A consequence of Proposition 1.11 follows next.

Lemma

If H is a finite subgroup of G and if it is the only subgroup of order |H|, then H is normal in G.

Definition

Let V be a vector space over a field F. The general linear group GL(V) is the set of all automorphisms of V viewed as a group under composition.

If V has finite dimension n, then GL(V) = GL(n, F) which is the group of invertible n by n matrices with entries in F.

Definition

The centre Z(G) of a group G is the set of all elements z in G that commute with every element q in G. We write:

$$Z(G) = \{z \in G | zq = qz, \text{ for all } q \in G\}$$

Note that Z(G) is a commutative normal subgroup of G and G modulo; its centre Z(G) is isomorphic to the inner automorphism, inn(G) of G.

If Z(G) = {e}, then G is said to have a trivial centre. The centre of a group G is its subgroup of largest order that commutes with every element in the group and its characteristic.

We relate conjugacy class to the centre of the group.

Definition

Let a, $q \in G$. Then a is conjugate to q in G if there exists an element $g \in G$ such that $q = g^{-1}ag$. The set of elements in G that are conjugate to a in G is denoted by C(a). And as such, C(a) = $\{g^{-1} ag \mid g \in G\}$. This is called the conjugacy class of a in G. The order of the conjugacy class divides |G|.

Distinct conjugacy classes form a partition of the group and hence induces a decomposition of G into disjoint equivalence classes, as in Herstein (1964).

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Definition

The centralizer $C_G(q)$ of an element q in G is the set of all elements $g \in G$ that commute with q, that is: $C_G(q) = \{g \in G \mid gq = qg, \text{ for some } q \in G\}$. This is a subgroup of G.

The index of $C_G(q)$ in G is the size of the conjugacy class C(q) of q in, that is, $|C(q)| = |G : C_G(q)|$. In particular, $C_G(q)$ is a subgroup of G but not a normal subgroup in general. Consequently, the quotient of G by $C_G(q)$ is not a group.

Next is a corollary from James, G. and Martin, L. (2001) and Louis (1975).

Corollary

If G is a finite group, then:

(i) Every group is a union of its conjugacy classes and distinct conjugacy classes are disjoint;

(ii) A conjugacy class is an equivalence relation where the equivalence classes are the conjugacy classes.

(iii) If H is a subgroup of Z(G), then H is a normal subgroup of G. In particular, Z(G) is normal in G.

A relationship between the centre of G and the centralizer of the elements of G is given by:

Lemma

The centre Z(G) of a group G is the intersection of the centralizers $C_G(a)$ of elements a in G.

The fact that the centre of a group is the union of conjugacy classes containing one element gives rise to an important theorem, the class equation.

Definition

If $a \in G$, then $N_G(a)$ is the normalizer of a in G. It comprises precisely the set of those elements in G which commute with a. It is a subgroup of G.

Herstein, I. N. (1964) has it that if G is a finite group, then the number of elements conjugate to a in G is the index of the normalizer of a in G.

What follows is the definition of a special function—the structure preserving function in groups. It is the central concept common to most aspects of modern algebra.

Definition

Let G and G^* be groups, the function from G to G^* which preserves the structure of the groups is called homomorphism. Equivalently the function

$\varphi : G \to G^*$ with the property that:

$$(gh)^{\varphi} = (g^{\varphi})(h^{\varphi}) \text{ for all } g,h \in G \text{ is called a homomorphism.}$$

Next is Grun's Lemma proved by Sandra, S. et al (2011).

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Lemma

If a group G has centre Z(G) other than {1} and if the centre of G/Z(G) is also different from {1} then, there exists a homomorphism map of G unto a subgroup of Z(G) other than {1}.

Now we define a kind of homomorphism with a unique property.

Definition

A representation of a group G is a homomorphism ϕ of G into another group A.

That is: $\phi : G \to A$, for all $s,t \in G$ such that:

$$(st)^{\phi} = (s)^{\phi}(t)^{\phi}$$

Note that $\ker\phi = \{s \in G : s^{\phi} = 1\}$ and $\mathrm{im}\phi = \{s^{\phi} : s \in G\}$ are respectively the kernel and image of ϕ. If in particular A = GL(n, F), the set of n by n invertible matrices with entries in the field F, we define the representation of G as the homomorphism

$$\phi : G \to GL(n, F).$$

Definition

Let V be a representation. We say W is a sub representation of V if W is a subspace of V that is invariant under G. That is for all w in W, gw is in W.

Here, V is said to be irreducible if and only if the only sub representations of V are V and {0}.

Definition

Suppose that with each element x in G, there is associated an n by n non-singular matrix M(x)=(aij(x)), i,j =1,2,...,n with coefficients in the field F in such a way that for x, y in G, M(x)M(y) = M(xy); then M(x) is called a matrix representation of G of dimension n over F. Matrix representation is also called an F - representation.

Remark

In representation theory, the group G is considered as a group of matrices where we replace each element of an abstract group with a matrix. The entire theory of group representation arises from matrix groups.

Next is Lagrange's theorem.

Theorem

If a group G is finite and H is a subgroup of G, then the order of H divides the order of G.

The theorem that follows is from Mark, R. (2011):

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Theorem

If a finite group G has a centre Z(G) and G/Z(G) is cyclic; then G is abelian.

From Houshang, B. and Hamid, M. (2009), we have the next proposition which is an important property of p - groups.

Proposition

If the order of a finite group G is a power of a prime p; then G has a non trivial centre. Equivalently the centre of a p group contains more than one element.

A theorem for our consideration in this paper is due to Cody, C. (2010) on commutativity of finite non-abelian groups, where a bound for the size of the centre of a non-abelian group is given.

Theorem

If G is a finite non abelian group, then the maximum possible order of the centre of G is ¼|G|.That is, $|Z(G)| \leq 1/4|G|$.

Proof

Let $z \in Z(G)$, since G is non abelian, $Z(G) \neq G$. Thus there exists an element

$q \in G$ such that q is not in the centre. This implies that $C_G(q) \neq G$ and

$C_G(q) \neq Z(G)$. Since $z \in Z(G)$, every element in G commutes with z, so qlz = zq. It follows that $z \in C_G(q)$. Since $q \in C_G(q)$, we have that Z(G) is a proper subset of $C_G(q)$. Since a group that is a subset of a subgroup under the same operation is itself a subgroup of the subgroup, we find that Z(G) is a proper subgroup of $C_G(q)$. By Theorem 1.26 and corollary 1.28, it follows that:

$$|Z(G)| \leq 1/2|C_G(q)|.$$

Now, since we assumed $C_G(q) \neq G$, then $C_G(q)$ is a proper subset of G. Therefore, by Theorem 1.26 and the fact that the centralizer of any group element is a subgroup of G, we find that $|C_G(q)| \leq 1/2|G|$. That is:

$$\begin{aligned} |Z(G)| &\leq 1/2|C_G(q)| \\ &\leq 1/2(1/2|G|) \\ &\leq 1/4|G|. \end{aligned}$$

Anna, C. (2010) has the next definition.

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Definition

The commutativity degree of a finite group G is the probability p(G) that two elements of G selected at random (with replacement) commute. That is:

p(G)={(x,y):xy=yx, for any x and y in G}.

A result which Sarah, M. et al. (1991) said is woven from elementary results on subgroups, centralizer, Lagrange's theorem and conjugacy classes.

Commutativity degree is a property of groups which is determined by the nature of the centre and measures the probability P(G) that pairs of elements of a finite non abelian group G selected at random commute. In the case of a non-abelian group, the probability is 5/8.

Anna, C. (2010) proves the theorem that follows:

Theorem

If G is a finite non - abelian group, then the commutativity degree is 5/8 if and only if $G/Z(G) \cong V_4$, the Klein 4- group.

This theorem has a corollary which describes the structure of a group having commutativity degree of $(p^2+p-1)/p^3$ as follows:

Corollary

(a) Let p be the smallest prime dividing the order of G/Z(G) then:

$P(G) = (p^2+p-1)/p^3$ if and only if $|G/Z(G)|=p^2$;

(b) If the commutativity degree of G is $(p^2+p-1)/p^3$ where p is the smallest prime dividing the order of G/Z(G), then $G \cong P \times A$ such that P is a p - group and A is abelian.

Remark

The group product in (b) above is described in our result as preserving the maximal property of the centre of finite non abelian group with $|Z(G)| \leq 1/4|G|$.

In 1991, Sarah, M. and Gary, J. on counting centralizers in finite groups have it that: the basic classification scheme for groups reflects the importance of the notion of commutativity in understanding group structure. This is because labelling a group as abelian, nilpotent, super nilpotent, soluble or simple indicates in a sense the degree of commutativity that such groups enjoy.

Theorem

Let G be a finite group. Then the degree of commutativity p(G) of G is p(G) = |C|/|G|.

Consequently, the commutativity degree of a finite group is the same as counting the number of conjugacy classes of G.

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From James Gordon and Martin (2001), we have the definition of one of the inputs in this paper.

Definition

The subgroup $G^/$ of a group G generated by the elements of the form $sts^{-1}t^{-1}$, for all s, t $\in$ G is called the derived group or commutator subgroup of G. We write $[s,t] = sts^{-1}t^{-1}$ and call this the commutator of s and t. Thus:

$$G^/ = \{ [s,t] : s, t \in G \}.$$

The commutator [s,t] is an element of G that measures the failure of the elements s and t to commute. The derived subgroup $G^/$ is normal in G and the quotient $G/G^/$ is called the abelianization of G. It is the largest abelian quotient of G.

We can use $G^/$ to determine whether a group is abelian or non abelian in the following sense:

A finite group G is abelian if and only if $G^/=\{1\}$.

From Keith, C. (2010), we have:

Remark

Commutativity of G is equivalent to both Z(G)=G and $G^/=\{1\}$. The conditions Z(G) = {1} and $G^/$ = G are not equivalent. The commutativity of G is also equivalent to G/Z(G) and $G^/$being trivial. However, if $G^/$ is finite, G/Z(G) need not be finite. Any subgroup of G that is contained in Z(G) is normal in G. Since the centre is abelian. Furthermore, any subgroup of G that contains [G,G] is normal in G since $G/ G^/$ is abelian .

For a non-abelian group, a measure of how close the group is to being abelian is based on how close the commutator subgroup is to the identity. The larger $G^/$ is the less abelian G is since if G is abelian, $G^/=\{1\}$. In general G' is a normal subgroup of G. By Lagrange's theorem without loss of generality, we have that $|G'|$ divides $|G|$.

From Ledermann, W. & Weir, A.J. (1996) and Louis, S. (1975) and Baumslag, B. and Chandler, B. (1968) more properties of the derived group include:

Theorem

If G is a finite group then:

(i) the derived group $G^/$ is a normal subgroup of G and $G/G^/$ is abelian

(ii) if H is any normal subgroup of G such that G/H is abelian then $G^/\leq H$. The quotient of G by $G^/$ is the 'largest' quotient group of G which is abelian.

We outline the following for reference from Herstein, I. N. (1964).

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Theorem

The derived subgroup $G^{/}$ of a finite group G is the unique minimal normal subgroup of G such that $G/G^{/}$ is abelian. That is, G/N is abelian implies that $G^{/} \leq N$, and $G/G^{/}$ is also abelian.

Proposition

(i) The number of the irreducible representations of any group G is equal to the number of conjugacy classes of G;

(ii) Every irreducible representation of an abelian group G over the set of complex numbers is one dimensional.

Proof

(i) The class functions are determined by their values on the conjugacy classes of G. These are complex vector spaces. They have dimensions equal to the number of conjugacy classes.

The irreducible representations form a basis for the same vector space. Thus, the number of conjugacy classes and the number of irreducible representations are the same.

(ii) Since G is an abelian group, it has |G| conjugacy classes. From (i) above, it shows that G has |G| number of irreducible representation and we have that: $|G| = n_1^2 + n_2^2 + \ldots + n_{|G|}^2$.

It clearly shows that this can be satisfied only when $n_i = 1$ for all i.

Next, we count the number of inequivalent irreducible representations in terms of Z(G) proven by Jelten B. Naphtali (2015).

Theorem

Let G be a finite non abelian group with size p^n. Then, G has

$|C| \leq |Z(G)| + 3$ inequivalent irreducible representations if

$|G^{/}| \leq |Z(G)|$.

Definition

The homomorphism $\phi : G \to Gl_n(C)$ is a complex representation of the group G where n is the degree of the representation. The structure of $C[G]$ is further described by theorems of Masches and Wedderburn whose combined theorem according to Anna (2010), is as follows:

Theorem: (Masche and Wedderburn)

Let G be a finite group. Then the group ring $C[G]$ can be written as

$C[G] = C \times M_{n_2}(c) \times M_{n_3}(c) \times \cdots \times M_{n_l}(c)$ for some positive integer $n_i \geq 1$.

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Any finite dimensional algebra described in the form above is said to be semi-simple. Over a single finite dimensional algebra, every module is a direct sum of single modules each of which is isomorphic to a simple left-ideal.

In the light of the above theorem, we have the next remark.

Remark

$C[G]$ has finitely many non-isomorphic single modules which can be explicitly expressed as $C^{n_1}, C^{n_2}, C^{n_3}, \ldots, C^{n_l}$, $n \geq 1$.

This implies that G has l non-equivalent irreducible representation of degrees $1, n_2, n_3, \ldots, n_l$.

The factor C corresponds to the map $\phi : G \to C^*$ defined as $\phi(g) = 1$ for all $g \in G$. Hence, we have from proposition 1.39 the following equation called the degree equation.

Theorem

$|G| = 1, + n_2^2 + n_3^2 + \ldots + n_l^2$ where l = number of conjugacy classes.

The irreducible representation of a group is intimately tied with the conjugacy classes of the group as in preposition 1.39.

Definition

A representation that does not have proper sub representation is said to be irreducible otherwise reducible.

Remark

An irreducible representation of a finite abelian group G must be one dimensional and there are |G| distinct irreducible representations.

From Louis (1975) and Herstein (1964), we have a relationship between the order of a finite group and its irreducible representations—a compressed form of theorem 1.44.

Corollary

Equivalent representation consists of $n_i \times n_i$ $1 \leq i \leq |C|$ matrices such that $|G| = \sum n_i^2$ where $|C|$ is the number of conjugacy classes of G. The n_i are the degrees of the irreducible representation. Hence, $|G : G'|$ or $\left| G/G' \right|$ is the number of n_i such that n_i=1.

Next proposition gives some group theoretic properties of the derived subgroup from Gordon and Martin (2001).

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Proposition

Assume that $N\Delta G$. Then:

i. $G'\Delta G$

ii. $G' \leq N$ if and only if G/N is abelian. In particular, G/G' is abelian.

Proof:

i. Note that for all $a, b, x \in G$, we have

$x^{-1}(a,b)x = (x^{-1}ax)(x^{-1}bx)$ and

$x^{-1}a^{-1}x = (x^{-1}ax)^{-1}$

Now, G' consists of elements of the form $[g,h]$.

Next, we prove that $x^{-1}[g,h]x \in G'$ for all $g, h, x \in G$.

But $x^{-1}[g,h]x = x^{-1}g^{-1}h^{-1}ghx$

$$= (x^{-1}gx)^{-1}(x^{-1}hx)^{-1}(x^{-1}gx)(x^{-1}hx)$$

$$= [x^{-1}gx, x^{-1}hx]$$

Hence, $G'\Delta G$.

ii. Let $g, h \in G$, we have that

$ghg^{-1}h^{-1} \in N \Leftrightarrow Ngh = (Nhg)(Nh) = (Nh)(Ng)$

Hence, $G' \leq N$ if and only if G/N is abelian; since we have proved $G'\Delta G$, we deduce that G/G' is abelian.

Given G is finite, $|G:G'|$ is the number of $n_i = 1$.

The next is the degree equation from Louis, S. (1975).

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Theorem

Let G be a group and the degrees of inequivalent irreducible representations of G be r_i , for $1 \leq i \leq |C|$, then:

$|G| = \sum r_i^{\,2}$; where

$|C|$ is the number of conjugacy classes of G, $|G/G^{/}|$ is the number of $r_i = 1$, and

each r_i divides $|G/Z(G)|$.

From proposition 1.39 and remark 1.43 and corollary 1.4.7, we get that:

Theorem

$$|G| = |G:G^{/}| + \sum_{i=|G:G^{/}|+1}^{k(G)} n_i^2$$

where $n_i>2$. This is the form of the degree equation used in this paper.

The degree equation is expressed in terms of the derived subgroup and the sum of the irreducible representations of degree (or equivalently conjugacy class) greater than one. It's one of the primary tools beside the class equation used in determining the commutativity degree of a group G. However, the degree equation derives its construction from group representation and the group ring which we denoted by $C[G]$ earlier in this work.

Remark

Louis (1975) in a proposition involving direct products of derived group says, if $G = H \times K$ then $G' = H' \times K'$. This displays a relationship between the direct product of groups and their derived subgroups.

Before we prove our results, for reference, next, we have in Table 1 the table for the commutativity degrees of finite groups of order less than 101 from Anna (2010). The groups in our work are groups of order 4n based on the restrictions stated in the abstract

1.52 TABLE: Commutativity Degrees of Groups of Order Less Than 101

|G|	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**
	½	5/8	2/5	5/8	5/8	5/8	½	5/8	7/22	5/8		
	1/3	7/16	¼	½	5/14	17/32	1/3	2/5		½		
				3/8		7/16	¼	13/40		7/16		
				1/3		11/32	1/6	¼		3/8		
				7/24						1/3		
				5/24						5/16		
										7/24		
										¼		
										5/24		
										1/6		

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|G|	**60**	**64**	**68**	**72**	**76**	**80**	**84**	**88**	**92**	**96**	**100**
	½	5/8	5/17	5/8	11/38	½	½	5/8	13/46	5/8, 17/32	2/5
	2/5	17/32	2/17	½		7/16	5/14	7/22		½, 7/16	7/25
	1/3	7/16		3/8		2/5	1/3	25/88		3/8, 11/32	¼
	3/10	25/64		1/3		13/50	2/7			1/3, 5/16	4/25
	¼	11/32		7/4		23/80	5/21			7/24, 9/32	13/100
	1/5	19/64		¼		¼	5/28			¼, 7/32	1/10
	1/12	1/4		5/24		17/80	1/6			5/24,18/86	
		13/64		1/6		7/4	1/7			3/16, 1/6	
		1/32		1/8						7/48,13/96	
				1/12						1/8, 11/96	
										5/48	

RESULTS

Theorem

Let G be a group such that $|G| < \infty$, then the commutativity degree of G is given by $P(G) \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right)$

Proof

From theorem 1.50, the degree equation $|G| = |G:G'| + \sum_{i=|G:G'|+1}^{|C|} (n_i)^2$ where $n_i \geq 2$ $n_i \geq 2$, $|G:G'| + 1 \leq i \leq |C|$; $|G'| \neq 1$ for each i

Then $|G| = |G:G'| + \sum_{i=|G:G'|+1}^{|C|} (n_i)^2$

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$$\geq |G:G'| + 4(|c| - |G:G'|)$$

$$\geq |G:G'| + 4|c| - 4|G:G'|$$

$\geq 4|C| - 3|G:G'|$, given

$|G| + |G:G'| \geq 4|C|$

That is, $|C| \leq \dfrac{|G| + 3|G:G'|}{4}$

$|C| \leq \dfrac{1}{4}\left(|G| + \dfrac{3|G|}{|G'|}\right)$ since $|G| \leq \infty$, then $|G'| \leq \infty$

$|C| \leq \dfrac{1}{4}|G|\left(1 + \dfrac{3}{|G'|}\right)$. From theorem 1.34, we get

$$P(G) = \frac{|C|}{|G|} \leq \frac{1}{4}\left(1 + \frac{3}{|G'|}\right)$$

Theorem

Let G be a group such that $|G| < \infty$, with G/ ≤ Z(G), then we deduce from theorem 2.1 above that the commutativity degree of G is

(i). p(G) = 1 if the group is abelian (ii) p(G) = 5/8 for |G/| ≥ 2 given that G is non-abelian and |G/| takes its lower bound. (iii) $P(G) \leq \dfrac{|G'| + 3}{|G|}$ for |G/| ≥ 2 with G/ ≤ Z(G) and G/ takes its upper bound.

Proof

(i) Recall that in the abelian case $|G'| = 1$, accordingly from theorem 2.1 above, we have

$P(G) = \dfrac{1}{4}\left(1 + \dfrac{3}{1}\right) = 1$ which is trivial.

(ii) Since G is non-abelian, $|G'| \geq 2$, $\min|G'| = 2$ is the lower bound for G/. So, from theorem 2.1, we have $|C| \leq \dfrac{|G|}{4}\left(1 + \dfrac{3}{2}\right) = \dfrac{5}{8}|G|$. Combining theorem 1.34 and theorem 1.29, we get p(G) = 5/8.

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(iii) For $\max|G|$, we have that $|G'| > 2$, at best $|G'| = Z(G) \leq \frac{|G|}{4}$ the upper bound of G/ as G/ ≤ Z(G), with theorem 1.29; we have from above that

$$|C| \leq \frac{|G|}{4}\left(1 + \frac{3}{|G'|}\right)$$

$$\leq \frac{4|G'|}{4}\left(1 + \frac{3}{|G'|}\right) \leq |G'|\left(1 + \frac{3}{|G'|}\right) = |G'|\left(\frac{|G'| + 3}{|G'|}\right) = |G'| + 3$$

Consequently, the commutativity degree becomes

$$P(G) \leq \frac{|G'| + 3}{|G|}$$ as required.

DISCUSSION

In this paper, we determine a scheme for computing the commutativity degree for $|G| \leq \infty$, in particular $|G| \leq 100$. We applied our scheme to commutative and non-commutative groups. We also considered the commutativity degrees taking the lower bound and upper of the derived group.

For $b = 0$ and 2 ≤ a ≤ 6, five groups satisfy the restrictions outlined in the abstract: All the five groups are p-groups in fact 2 - groups in particular of order p^a with p as defined and $|G| = p^a$

Now, when $b = 1$, $2 \leq a \leq 5$ and sixteen groups all of order $p^a q$ satisfy the underlying conditions, if we consider $b = 2$, then the order of the group becomes $p^a b^2$. Here, we have $2 \leq a \leq 3$ and three groups in all hold for our scheme. This gives a total of 24 groups which in all satisfy our restrictions for $|G| = p^a q^b = 4n$.

The groups $|G| = p^a q$ are essentially groups of the form $G \approx A \times B$, described om corollary 1.32(b) where A is a 2-group and B an abelian group while $|G'|$ assumes the upper bound.

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CONCLUSION

The results we obtained are highly simplified and are expressed in terms of the derived group, a major building block in the degree equation which is the main instrument in our paper. The numerical results from our paper give the same result as obtained by Anna (2010).

For instance, for $a = 2$, $q = 3$, $b = 1$, then $|G| = 12$ gives $p(G) = \frac{1}{2}$, which implies that there is a group of order 12 with probability $\frac{1}{2}$ of the likelihood that any two elements selected at random from the group commute. For $a = 2$, $b = 2$ and $q = 5$, $|G| = 100$; this gives a commutativity degree of $\frac{7}{25}$, meaning there is a group of order 100 such that the probability that two randomly selected elements commute is $\frac{7}{25}$. This corresponds with Anna's (2010) as seen in Table 1.52

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