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# VIBRATION OF NON-UNIFORM BERNOULLI-EULER BEAM UNDER MOVING DISTRIBUTED MASSES RESTING ON PASTERNAK ELASTIC FOUNDATION SUBJECTED TO VARIABLE MAGNITUDE

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**ABSTRACT:** *This paper investigates the dynamic response of a clamped-clamped non-uniform Bernoulli-Euler beam resting on a Pasternak elastic foundation to variable magnitude moving distributed masses. The predicament is dictated by a partial differential equation of fourth order, which features coefficients that are both variable and singular. The primary aim is to derive an analytical solution for this category of a dynamic problem. To achieve this, we employ the method of Galerkin with a series representation of the Heaviside function to reduce the equation to second-order ordinary differential equations with variable coefficients. We simplify these transformed equations using (i) the Laplace transformation technique in conjunction with convolution theory for solving moving force problems, and (ii) finite element analysis in conjunction with the Newmark method for solving analytically unsolvable moving mass problems due to their harmonic nature. We first solve the moving force problem using the finite element method and compare it against analytical solutions as validation for its accuracy in solving analytically unsolvable moving mass problems. The numerical solution obtained from the finite element method is shown to be comparable favorably against analytical solutions of our moving force problem. Lastly, we calculate displacement response curves for both moving distributed force and mass models at various time t for our dynamical problem presentation purposes.*

**KEYWORDS**: Bernoulli-Euler Beam, Pasternak Elastic Foundation, Clamped-Clamped, Newmark Method, Moving Distributed Masses.

## INTRODUCTION

The study of force-induced vibrations in elastic bodies, such as stretched strings, spring-mass systems, and rods, has been subject to extensive research by numerous authors (Krylov, 1905; Timoshenko, 1921; Kenney, 1954; Stanisic *et al.*, 1968; Stanisic & Hardin, 1969; Stanisic *et al.*, 1974; Sadiku & Leipholz, 1987; Esmailzadeh & Ghorashi, 1995; Oni, 1997; Oni & Ogunyebi, 2008; Oni & Omolofe, 2010). These vibrations may arise from (i) a force or load that is solely dependent on the spatial coordinates. (ii) a force that exhibits spatial and temporal variability. The forces in question may possess a steady or fluctuating intensity. The focus of this study is on the impact of a varying force, moving at an unchanging velocity, upon an elastic entity. Specifically, we examine the behavior of a beam subjected to such conditions. It is noteworthy to mention that an elastic body, whether it be slender or stout in shape, is commonly regarded as a one-dimensional entity (Oni, 1997; Stanisic & Hardin, 1969; Stanisic *et al.*, 1974; Inglis, 1934). Its physical characteristics such as rigidity, weight and length are identified solely by its position along the elastic axis. Hence, the partial differential equation that characterizes the movement of an elastic body is composed solely of two autonomous variables: distance along the axis and time. When examining a finite beam in motion due to a moving force, scholars such as Timoshenko (1921), Inglis (1934), and Muscolino and Palmeri (2007) limited their discussions to analyzing transverse oscillations induced by a uniform velocity harmonic force. It was assumed that the beam was subjected to simple support while an evaluation of the impact of a dynamic force on the beam is provided. Steele (1971) conducted a study on the impact of a unit force moving at a constant velocity on beams. The analysis focused on both elastic and non-elastic foundation scenarios. Wu and Dai's (1987) previous research centered around dynamic responses of multi-span non-uniform beams under moving loads using the transfer matrix method. Dogush and Eisenberger (2002) conducted a comprehensive study on the dynamic behavior of non-uniform beams with multiple spans, which were subjected to moving loads at both constant and variable velocities. The author employed modal analysis and direct methods to investigate this phenomenon thoroughly. Similarly, Ahmadian *et al.* (2006) explored the analysis of a variable cross-section beam that was exposed to a moving concentrated force and mass using finite element method. Although the aforementioned research on both uniform and non-uniform beams is noteworthy, it should be noted that moving loads have been simplified as concentrated loads that are applied at specific points along a single line in space. In other words, the moving load is represented as a lumped load. In practical application, it is widely acknowledged that loads are distributed over either a small segment or the entire length of a structural member as they traverse through the structure. These moving loads are commonly referred to as uniform distributed loads. Concentrated forces, on the other hand, are purely mathematical abstractions and cannot be observed in real-world scenarios where surface forces act upon an area and body forces operate within a volume. It should also be noted at this point that only long thin uniform beams known as Euler's beam - that rest on one parametric foundation or bi-parametric foundation with non-harmonic characteristics were taken into consideration.

The current study focuses on the vibration of a clamped-clamped non-uniform Bernoulli-Euler beam resting on a Pasternak elastic foundation, subject to varying magnitudes of moving distributed masses. The investigation takes into account critical aspects related to inertia terms, while also considering the beam's elastic properties such as its flexural rigidity and mass per unit length both assumed not constant due to the non-uniform cross-section of the beam. It is important to note that damping effects are negligible in this scenario.



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**Problem Formulation**

This study examines the problem of a clamped-clamped non-uniform Bernoulli-Euler beam that carries a mass $M$. The properties of the beam, including its moment of inertia $I$ and mass per unit length $\mu$, vary along its span length $L$.

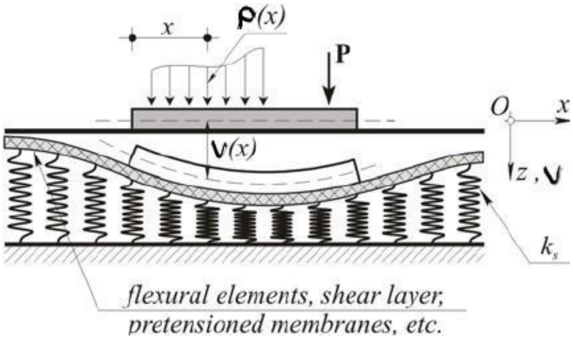


**Figure 1:** Geometry of a variable moving masses of non-uniform Bernoulli–Euler beam resting on Pasternak foundation.

**Figure 1** depicts the transverse displacement $V(x,t)$ of the beam as it moves at a constant speed. The equation of motion is given as

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right]+\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}-N_0\frac{\partial^2 V(x,t)}{\partial x^2}+K_0V(x,t)-G_0\frac{\partial^2 V(x,t)}{\partial x^2}=P(x,t), \quad (1)$$

In this problem, the time coordinate is represented by $t$, while $\mu$(x) denotes the variable mass per unit length of the beam. Additionally, $EI(x)$ refers to the variable flexural stiffness and $x$ represents the spatial coordinate. $K_0$ represents foundation stiffness, $G_0$ signifies shear modulus, $N_0$ indicates axial force, and $P(x,t)$ denotes uniform distributed load acting on the beam. It should be noted that in this particular scenario, the distributed load moving across the beam possesses a mass comparable to that of said beam. As such, it cannot be overlooked as its inertia plays a significant role in determining dynamical system behavior. Therefore, $P(x,t)$ will take on a specific form based on these factors as follow

$$\begin{aligned} P(x,t) &= \cos(\omega t)\sum_{i=1}^{j} M_i g H\left[x-f(t)\right]\left[1-\frac{1}{g}\frac{\mathrm{d}^2V(x,t)}{\mathrm{d}t^2}\right], \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} &= \frac{\partial^2}{\partial t^2}+2\frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{\partial^2}{\partial x\partial t}+\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2\frac{\partial^2}{\partial x^2}+\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}\frac{\partial}{\partial x}, \end{aligned} \quad (2)$$

where $g$ denotes the acceleration due to gravity, $\frac{d^2}{dt^2}$ is a convective acceleration operator, $\frac{\partial^2}{\partial t^2}$ is the support beam's acceleration at the point of contact with the moving mass, $\frac{df(t)}{dt}\frac{\partial^2}{\partial xdt}$ is is the well-known Coriolis acceleration, $\left(\frac{df(t)}{dt}\right)^2\frac{\partial^2}{\partial x^2}$ is the centripetal acceleration of the moving mass and $\frac{d^2f(t)}{dt^2}\frac{\partial}{\partial x}$ is the acceleration component in the vertical direction when the moving load is not constant.


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Similarly, for a consistent speed of *c*, the direction and distance covered by the load on the beam at any given moment in time *t* can be expressed as follows.

$$f(t) = c_i t, \tag{3}$$

Furthermore, it is postulated that the mobile weight bears a mass denoted by M and that time t is confined to the duration during which mass M rests on the beam. In other words,

$$0 \leq f(t) \leq L, \tag{4}$$

The function *H* [*x*-*f* (*t*)] is the Heaviside function, commonly used in engineering applications to measure functions that are binary in nature, i.e., either "on" or "off". Its definition reads as follows.

$$H(x) = \begin{cases} 1, & x > ct. \\ 0, & x \leq ct. \end{cases} \qquad H[x - f(t)] = \begin{cases} 1, & x \geq f(t). \\ 0, & x < f(t). \end{cases} \tag{5}$$

For instance, consider the variable moment of inertia denoted by *I* and the mass per unit length of the beam as defined in [18].

$$I(x) = I_0\left(1 + \sin\frac{\pi x}{L}\right)^3, \quad \mu(x) = \mu_0\left(1 + \sin\frac{\pi x}{L}\right), \tag{6}$$

where $I_0$ and $\mu_0$ represent the constant moment of inertia and mass per unit length, respectively, for the corresponding uniform beam. To achieve this, by substituting equations (2), (3) and (6) into equation (1) and conducting necessary simplification and rearrangement, we obtain the desired result as follow

$$\begin{aligned}
&\frac{EI_0}{4}\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^4 V(x,t)}{\partial x^4} + \frac{6\pi EI_0}{4L}\left(5\cos\frac{\pi x}{L} + 4\sin\frac{2\pi x}{L}\right. \\
&\left.- \cos\frac{3\pi x}{L}\right)\frac{\partial^3 V(x,t)}{\partial x^3} + \frac{3\pi^2 EI_0}{4L^2}\left(3\sin\frac{3\pi x}{L} + 8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2} \\
&+ \mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial t^2} - N_0\frac{\partial^2 V(x,t)}{\partial x^2} + K_0 V(x,t) - G_0\frac{\partial^2 V(x,t)}{\partial x^2} \\
&+ \cos(\omega t)\sum_{i=1}^{j} M_i H(x - c_i t)\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2c_i\frac{\partial^2 V(x,t)}{\partial x \partial t} + c_i^2\frac{\partial^2 V(x,t)}{\partial x^2}\right] = \sum_{i=1}^{j} M_i g \cos\omega t H(x - c_i t).
\end{aligned} \tag{7}$$

The boundary conditions of the aforementioned problem are considered to be arbitrary, meaning they can adopt any form of classical boundary conditions. Conversely, the initial conditions without sacrificing generality are provided as follows.


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$$V(x,0) = \frac{\partial V(x,0)}{\partial t} = 0. \quad (8)$$

Equation (7) constitutes the fundamental equation in the dynamic problem.

Solution Procedure

Equation (7) is a partial differential equation with variable coefficients that is non-homogeneous. It appears that the separation of variables method cannot be used due to a difficulty in obtaining separate equations whose functions are dependent on only one variable. As a result, we turn to an adapted version of the approximate method, which is best suited for solving various problems related to structural dynamics and commonly known as Galerkin's Method. In order to reduce the fourth order partial differential equation into a sequence of second order ordinary differential equations, we employ Galerkin's method as described by Oni and Awodola (2003, 2010). This approach leads us towards finding solutions in the form

$$V(x,t) = \sum_{m=1}^{n} Y_m(t)U_m(x), \quad (9)$$

The kernel function $U_m(x)$ is thoughtfully selected for the Galerkin's method in equation (9) to ensure that the specified boundary conditions are met. It should be noted that our analysis assumes general boundary conditions at $x = 0$ and $x = L$ for the beam in question. Therefore, we must carefully choose a suitable set of functions to represent the beam shapes in order to obtain the $m^{th}$ normal mode of vibration.

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L}, \quad (10)$$

is chosen such that the boundary conditions are satisfied. The kernel is chosen as

$$U_k(x) = \sin\frac{\lambda_k x}{L} + A_k \cos\frac{\lambda_k x}{L} + B_k \sinh\frac{\lambda_k x}{L} + C_k \cosh\frac{\lambda_k x}{L}, \quad (11)$$

In equations (10) and (11), $\lambda_m$ and $\lambda_k$ respectively denote the mode frequency. The constants $A_m$, $B_m$, $C_m$, $A_k$, $B_k$ and $C_k$ are determined by substituting equations (6) and (7) into the relevant boundary condition. Consequently, upon substitution of equation (9) into equation (7), we obtain:


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$$\sum_{m=1}^{n}\Bigg[\left(1+\sin\frac{\pi x}{L}\right)U_m(x)\ddot{Y}_m(t)+\frac{EI_0}{4\mu_0}\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)U_m''(x)Y_m(t)$$
$$+\frac{6\pi EI_0}{4\mu_0 L}\left(5\cos\frac{\pi x}{L}+4\sin\frac{2\pi x}{L}-\cos\frac{3\pi x}{L}\right)U_m'''(x)Y_m(t)\frac{3\pi^2 EI_0}{4\mu L^2}\left(3\sin\frac{3\pi x}{L}+8\cos\frac{2\pi x}{L}\right.$$
$$\left.-5\sin\frac{\pi x}{L}\right)U_m''(x)Y_m(t)-\frac{N_0}{\mu}U_m''(x)Y_m(t)+\frac{K_0}{\mu}U_m(x)Y_m(t)-\frac{G_0}{\mu}U_m''(x)Y_m(t)$$
$$+\sum_{i=1}^{j}M_i\cos\omega t\big[H(x-c_it)U_m(x)\ddot{Y}_m(t)+2c_iH(x-c_it)U_m'(x)\dot{Y}_m(t)$$
$$+c_i^2H(x-c_it)U_m''(x)\big]-\sum_{i=1}^{j}M_ig\cos\omega tH(x-c_it)\Bigg]=0. \tag{12}$$

To derive an expression for $Y_m(t)$, let us examine a mass $M$ that moves uniformly at velocity $c$ along the $x$-coordinate. The solution for any number of moving masses may be obtained by superimposing the individual solutions, as the governing equation is linear. In order to determine the expression for a single mass $M_1$, it is necessary that the left-hand side of Eq. (12) be orthogonal to function $U_k(x)$. Therefore, utilizing Equations. (10) and (11) in (12) produces

$$I_0^*\ddot{Y}_m(t)+I_1^*Y_m(t)+\frac{\cos\omega t}{\mu_0}M\left[I_2^*\ddot{Y}_m(t)+2cI_3^*\dot{Y}_m(t)+c^2I_4^*Y_m(t)\right]=\frac{g\cos\omega t}{\mu_0}MI_5^0, \tag{13}$$

where

$$I_0^*=\sum_{m=1}^{n}\int_0^L\left(1+\sin\frac{\pi x}{L}\right)U_m(x)U_k(x)\mathrm{d}x,\quad I_1^*=I_{1A}+I_{1B}+I_{1C}-I_{1D}+I_{1E}-I_F, \tag{14}$$

$$I_{1A}=\frac{EI_0}{4\mu_0}\sum_{m=1}^{n}\int_0^L\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)U_m^{iv}(x)U_k(x)\mathrm{d}x, \tag{15}$$

$$I_{1B}=\frac{6\pi EI_0}{4\mu_0 L}\sum_{m=1}^{n}\int_0^L\left(5\cos\frac{\pi x}{L}+4\sin\frac{2\pi x}{L}-\cos\frac{3\pi x}{L}\right)U_m'''(x)U_k(x)\mathrm{d}x, \tag{16}$$

$$I_{1C}=\frac{3\pi^2 EI_0}{4\mu_0 L^2}\sum_{m=1}^{n}\int_0^L\left(3\sin\frac{3\pi x}{L}+8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}\right)U_m''U_k(x)\mathrm{d}x, \tag{17}$$

$$I_{1D}=\frac{N}{\mu_0}\sum_{m=1}^{n}\int_0^L U_m''U_k(x)\mathrm{d}x,\ I_{1E}=\frac{K_0}{\mu_0}\sum_{m=1}^{n}\int_0^L U_mU_k(x)\mathrm{d}x,\ I_{1F}=\frac{G_0}{\mu_0}\sum_{m=1}^{n}\int_0^L U_m''U_k(x)\mathrm{d}x, \tag{18}$$

$$I_2^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_mU_k(x)\mathrm{d}x,\quad I_3^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_m'U_k(x)\mathrm{d}x, \tag{19}$$

$$I_4^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_m''U_k(x)\mathrm{d}x,\quad I_5^0=\int_0^L H(x-ct)U_k(x)\mathrm{d}x. \tag{20}$$

Using the property of Heaviside function, it can be expressed in series form given by

Adekunle *et al.* (2017) i.e.


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$$H(x-ct)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x\cos(2n+1)\pi ct}{2n+1}-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x\sin(2n+1)\pi ct}{2n+1}. \tag{21}$$

Thus, in view of (14)–(20) and (21), it can be shown that

$$\ddot{Y}_m(t)+\frac{I_1^*(m,k)}{I_0^*(m,k)}Y_m(t)+\frac{\varepsilon_0\cos\omega t}{I_0^*(m,k)}\Bigg\{\Bigg[L\psi_{1A}(m,k)+\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ct}{2n+1}I_5^*(m,k)$$
$$-\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ct}{2n+1}\times I_6^*(m,k)\Bigg]\ddot{Y}_m(t)+2c\Bigg[L\psi_{2A}(m,k)+\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ct}{2n+1}I_7^*(m,k)$$
$$-\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ct}{2n+1}I_8^*(m,k)\Bigg]\dot{Y}_m(t)+c^2\Bigg[L\psi_{3A}(m,k)+\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi ct}{2n+1}I_9^*(m,k)$$
$$-\frac{L}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi ct}{2n+1}I_{10}^*(m,k)\Bigg]Y_m(t)\Bigg]\Bigg\}=\frac{MgL\cos\omega t}{\mu\lambda_k I_0^*(m,k)}\times\Big[-\cos\lambda_k x+A_k\sin\lambda_k x$$
$$+B_k\cosh\lambda_k x+C_k\sinh\lambda_k x+\cos\frac{\lambda_k ct}{L}-A_k\sin\frac{\lambda_k ct}{L}-B_k\cosh\frac{\lambda_k ct}{L}-C_k\sinh\frac{\lambda_k ct}{L}\Big], \tag{22}$$

where

$$\varepsilon_0=\frac{M}{\mu L}. \tag{23}$$

Equation (22) stands as the fundamental governing equation for the dynamic problem. This coupled, non-homogeneous second-order ordinary differential equation applies to all variants of classical boundary conditions. Consequently, two distinct cases emerge from Equation (22): the moving force and moving mass problems.

**Non-uniform Bernoulli–Euler Beam Traversed by Moving Distributed Force for Clamped-Clamped End Condition**

In this segment, we derive an approximate model for the differential equation that characterizes the reaction of the elastic structure. This is achieved by disregarding inertia terms, specifically setting $\varepsilon_0$ to zero. Furthermore, we will focus solely on the clamped-clamped end condition as our example. Under these circumstances, both displacement and bending moment are negligible and vanish entirely.

$$V_m(0,t)=0=V_m(L,t),\quad \frac{\partial^2 V_m(0,t)}{\partial x^2}=0=\frac{\partial^3 V_m(L,t)}{\partial x^3}, \tag{24}$$

and hence for normal modes

$$U_m(0)=0=U_m(L),\quad \frac{\partial^2 U_m(0)}{\partial x^2}=0=\frac{\partial^3 U_m(L)}{\partial x^3}, \tag{25}$$


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which implies

$$U_k(0) = 0 = U_k(L), \quad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^3 U_k(L)}{\partial x^3}. \tag{26}$$

It is easily shown that

$$A_m = \frac{\sinh\lambda_m - \sin\lambda_m}{\cos\lambda_m - \cosh\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = -C_m \text{ and } B_m = -1, \tag{27}$$

and

$$\cos\lambda_m \cosh\lambda_m = 1, \tag{28}$$

substituting equations (24)–(28) into equation (22), yields

$$\begin{aligned}\ddot{Y}_m(t) + \alpha_f^2 Y_m(t) = P_o \cos\omega t\,[-\cos\lambda_k x + A_k \sin\lambda_k x + B_k \cosh\lambda_k x \\ +C_k \sinh\lambda_k x + \cos\theta_k t - A_k \sin\theta_k t - B_k \cosh\theta_k t - C_k \sinh\theta_k t],\end{aligned} \tag{29}$$

where

$$\alpha_f^2 = \frac{I_1^*(m,k)}{I_o^*(m,k)}; \quad \theta = \frac{\lambda_k c}{L}; \quad P_o = \frac{MgL}{\mu\lambda_k I_o^*(m,k)}, \tag{30}$$

Thus, by applying the Laplace transform technique and convolution theory with the given initial conditions (8), we can obtain a solution to equation (29) as follow

$$\begin{aligned}V(x,t) = \textstyle\sum_{m=1}^{n} \frac{P_o}{2\alpha_f} \Big\{ \frac{\alpha_f}{(\alpha_f^2-\omega^2)(\alpha_f^2-\Omega_1^2)(\alpha_f^2-\Omega_2^2)} \big[ 2\theta_{mc}\left(\alpha_f^2 - \Omega_1^2\right)\left(\alpha_f^2 - \Omega_2^2\right)(\cos\omega t - \cos\alpha_f t) \\ + \left(\alpha_f^2 - \omega^2\right)\left(\alpha_f^2 - \Omega_2^2\right)(\cos\Omega_1 t - \cos\alpha_f t) + \left(\alpha_f^2 - \omega^2\right)\left(\alpha_f^2 - \Omega_1^2\right)(\cos\Omega_2 t - \cos\alpha_f t)\big] \\ - \frac{1}{\left(\alpha_f^2-\Omega_i^2\right)\left(\alpha_f^2-\Omega_j^2\right)\left(\Omega_j^2+\theta_2^2\right)\left(\Omega_i^2+\theta_2^2\right)} \Big\{ A_k(\Omega_3^2 + \theta_2^2)(\Omega_4^2 + \theta_2^2) \big[ \left(\alpha_f^2 - \Omega_2^2\right)(\alpha_f \sin\Omega_1 t \\ -\Omega_1 \sin\alpha_f t) + \left(\alpha_f^2 - \Omega_1^2\right)(\alpha_f \sin\Omega_2 t - \Omega_2 \sin\alpha_f t)\big] - 2\left(\alpha_f^2 - \Omega_1^2\right)\left(\alpha_f^2 - \Omega_2^2\right)\Big\{ B_k \big[\theta\left(\alpha_f^2 + \omega^2 + \theta^2\right) \\ (\cosh\theta t \sin\omega_t t \sin\alpha_f t) + \big[\alpha_f\left(\alpha_f^2 + \omega^2 + \theta^2\right) - 2\alpha_f\omega\big](\cosh\theta t \cos\omega t - \cos\alpha_f t)\big] \\ +C_k \big[\theta_f\left(\alpha_f^2 + \omega^2 + \theta^2\right)\sinh\theta t \sin\omega t + \big[\alpha_f\left(\alpha_f^2 + \omega^2 + \theta^2\right) - 2\alpha_f\omega\big]\sinh\theta t \cos\omega t\big]\Big\}\Big\}\Big\} \times \\ \sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L},\end{aligned}$$

(31)

The aforementioned (31) illustrates the transverse displacement reaction to a distributed force that moves at a constant velocity of a non-uniform Bernoulli-Euler beam, which is clamped at both ends and rests on an elastic foundation known as Pasternak.


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**Non-uniform Bernoulli–Euler Beam Traversed by Moving Distributed Mass for Clamped-Clamped End Condition**

In this section, we seek the solution to equation (22) in its entirety without neglecting any terms of the coupled differential equation. It is clear that an exact solution to this equation cannot be attained through conventional means. Even Struble's widely-used technique Struble (1962) fails to handle it due to the fluctuating magnitude of the moving load. Therefore, we turn to employing finite element method (FEM) for modeling the structure and subsequently utilize Newmark numerical integration method for solving the resulting semi-discrete time-dependent equation in order to obtain our desired responses.

***Finite Element Method (FEM)***

The finite element method postulates that the indeterminate transverse deflection of the non-uniform beam, $V(x, t)$, can be approximated by a collection of piecewise continuous functions defined over discrete sub-regions known as elements. These elements consist of numerical values representing the unknown deflection within each region. Consequently, the initial step in implementing this technique involves partitioning the spatial solution domain of said non-uniform beam, which happens to be its length in this case, into a finite number of sub-domains designated as finite elements. These elements are interconnected at specific points called nodes. Subsequently, the weak or variational form corresponding to governing equation (1) is constructed thus:

Let us take into account a customary segment of dimension L, with its domain $\lambda_e = (0, L)$. By inserting equations (2) and (3) into equation (1), we obtain.

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right] + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - N_0\frac{\partial^2 V(x,t)}{\partial x^2} + K_0 V(x,t) - G_0\frac{\partial^2 V(x,t)}{\partial x^2}$$
$$= \cos(\omega t)\sum_{i=1}^{j} M_i H(x - c_i t)\left[g - \left(\frac{\partial^2 V(x,t)}{\partial t^2} + 2c_i\frac{\partial^2 V(x,t)}{\partial x \partial t} + c_i^2\frac{\partial^2 V(x,t)}{\partial x^2}\right)\right]. \quad (32)$$

To address the resolution of equation (32), we will examine a mass $M$ that moves uniformly at a velocity $c$ along the $x$-coordinate. As the governing equation is linear, finding solutions for any number of moving masses can be achieved through superposition of individual solutions. For the single mass $M_1$, let Galerkin's weight function $W(x)$ be utilized. By multiplying equation (32) with this weight function and integrating over the domain $\lambda_e$, simplification and rearrangement lead to its solution.


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$$\int_0^{L^e} EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\frac{\partial^2 W(x)}{\partial x^2}\mathrm{d}x + \int_0^{L^e}\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x - \int_0^{L^e}(N_0+G_0)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x$$
$$+\int_0^{L^e} K_0 V(x,t)W(x)\mathrm{d}x - Mg\cos(\omega t)\int_0^{L^e} H(x-ct)W(x)\mathrm{d}x + M\cos(\omega t)\int_0^{L^e} H(x-ct)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x$$
$$+2Mc\cos(\omega t)\int_0^{L^e} H(x-ct)\frac{\partial^2 V(x,t)}{\partial x\partial t}W(x)\mathrm{d}x + Mc^2\cos(\omega t)\int_0^{L^e} H(x-ct)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x$$
$$+W(L^e)B_3^e - W(0)B_1^e - \frac{\partial W}{\partial x}\Big|_{x=L^e}B_4^e + \frac{\partial W}{\partial x}\Big|_{x=0}B_2^e = 0, \quad (33)$$

where

$$\lambda = \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right],\ \phi = EI(x)\left[\frac{\partial^2 V(x,t)}{\partial x^2}\right],\ B_k^e = \left[\lambda W(x)\right]\Big|_0^{L^e} - \left[\phi\frac{\partial W(x)}{\partial x}\right]\Big|_0^{L^e}, \quad (34)$$

$\lambda$ represents the shear force, while $\varphi$ denotes the bending moment. The four boundary terms, $B^e$ (where k ranges from 1 to 4), are crucial and mandatory for both end nodes of the element. Additionally, it can be easily demonstrated that

$$\int_0^{L^e} H(x-ct)f(x)\mathrm{d}x = \int_{ct}^{L^e} f(x). \quad (35)$$

Thus, equation (33) becomes

$$\int_0^{L^e} EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\frac{\partial^2 W(x)}{\partial x^2}\mathrm{d}x + \int_0^{L^e}\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x - \int_0^{L^e}(N_0+G_0)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x$$
$$+\int_0^{L^e} K_0 V(x,t)W(x)\mathrm{d}x - Mg\cos(\omega t)\int_{ct}^{L^e} W(x)\mathrm{d}x + M\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x$$
$$+2Mc\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial x\partial t}W(x)\mathrm{d}x + Mc^2\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x$$
$$+W(L^e)B_3^e - W(0)B_1^e - \frac{\partial W}{\partial x}\Big|_{x=L^e}B_4^e + \frac{\partial W}{\partial x}\Big|_{x=0}B_2^e = 0. \quad (36)$$

The weak form of the variable magnitude moving distributed masses of the non-uniform Bernoulli-Euler beam, which is resting on an elastic foundation, can be found in equation (36). In order to obtain an approximate solution for the element being analyzed and develop its corresponding shape function, we assume that the unknown deflection $V(x,t)$ can be expressed approximately.

$$V(x,t) \approx V_n(x,t) = H_1(x)V_1(t) + H_2(x)V_2(t) + H_3(x)V_3(t) + H_4(x)V_4(t)$$
$$= \sum_{k=1}^{4} H_k(x)V_k(t) = \{H\}\{V(t)\}, \qquad j = 1,2,3,4 \quad (37)$$


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where $H_j(x)$ are called Hermite cubic shape functions and $V_k(t)$ are the modal deflection functions and $H$ is a row vector defined as

$$[H] = [H_1(x), H_2(x), H_3(x), H_4(x)]. \quad (38)$$

Employing the methodologies entailed in formulating the Hermite-cubic interpolation functions as delineated by Junkins and Kim (1993), produces.

$$H_1 = 1 - \frac{3x^2}{h^2} + \frac{2x^3}{h^3}, \quad H_2 = x - \frac{x^2}{h} + \frac{x^3}{h^2}, \quad H_3 = \frac{3x^2}{h^2} - \frac{2x^3}{h^3}, \quad H_4 = -\frac{x^2}{h} + \frac{x^3}{h^2}, \quad (39)$$

where $x$ is the spatial coordinate. Now substituting equations. (37)–(39) into the weak form (36), after some simplification and rearrangement gives

$$[K^e]\{V(t)\} + [C^e]\{\dot{V}(t)\} + [M^e]\{\ddot{V}(t)\} + \{f^e\} + \{Q^e\} = 0. \quad (40)$$

The matrix equation (40) serves as the governing equation that characterizes the behavior of a typical finite element within a non-uniform beam subjected to a harmonic moving load. $[K^e]$ denotes the stiffness matrix of the element, $[M^e]$ represents its mass matrix, $[C^e]$ signifies its centripetal matrix, $\{f^e\}$ is indicative of the force vector and $\{Q^e\}$ reflects the element boundary term vector.

Subsequently, the next step involves assembling these aforementioned equations. The process for amalgamating various matrices and vectors for multiple beam elements that form a mesh has been extensively discussed by Wu (2005) and Irvine (2010). Henceforth, this culminates in an assembled governing equation of motion which describes the dynamic behavior exhibited by problems involving moving loads with Pasternak foundation.

$$[K]\{V(t)\} + [C]\{\dot{V}(t)\} + [M]\{\ddot{V}(t)\} = \{F\}, \quad (41)$$

where $[K]$, $[M]$ and $[C]$ are the assembled (global or overall) stiffness, mass, centripetal and load vector.

To acquire a comprehensive and distinctive resolution (41), it is imperative to enforce the specified boundary conditions on both the deflection/slopes and shear force/bending moments. Ultimately, in a free vibration system that lacks the centripetal matrix, (41) diminishes into a harmonic form.

$$\left([K] - \omega_i^2[M]\right)\{V(t)\} = 0, \quad (42)$$

The natural frequency is represented by $\omega^2$ while the system's corresponding mode shape is denoted by $V(t)$. Several techniques can be employed to determine both the eigenvalue $\omega^2$ and its corresponding $V(t)$. The dynamic response of a non-uniform beam subjected to a partially distributed moving load can be derived through the direct solution of equation (41) using the Newmark method.


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## COMMENTS ON THE CLOSED FORM SOLUTIONS

In theory, the deflections of a non-uniform Bernoulli-Euler beam have the potential to exceed reasonable limits. In practice, this phenomenon indicates that the beam is in a state of resonance. The velocity at which a load induces such resonance within the system is referred to as its critical speed. As demonstrated by (31), when subject to a moving distributed force and supported by a Pasternak foundation with clamped-clamped supports, said beam inevitably reaches such resonant states when

$$\omega_f = \omega \quad \text{or} \quad \omega_f = \Omega_1 \quad \text{or} \quad \omega_f = \Omega_2. \tag{43}$$

Equation. (30) shows that, the dynamic system will attain the state of resonance whenever velocity is

$$c = \frac{L}{m\pi}(\omega - \omega_f) \quad \text{or} \quad c = \frac{L}{m\pi}(\omega_f - \omega). \tag{44}$$

## ANALYSIS OF RESULT AND DISCUSSION

To illustrate the presented analysis, a non-uniform beam with a length of 5 meters is examined. The load velocity is set at 50 meters per second, while the Young modulus amounts to 2.10924 x $10^9$ Newtons per square meter and the moment of inertia measures at 0.00287698 cubic meters to the fourth power. The value of $\pi$ is equal to approximately 22 divided by seven, and the mass per unit length of the beam equals 2758.291 kilograms per cubic meter; furthermore, the ratio between load mass and beam mass stands at 0.25.

The transverse deflection of this beam can be calculated for various values of axial force N, foundation stiffness K as well as shear modulus G, which are all subject to variation in this study: N varies between four times ten raised to three and nine times ten raised to eight units (4 x $10^3$ - 4 x $10^9$), K ranges from four times ten raised to three and nine times ten raised to eight units (4 x $10^3$ - 4 x $10^9$), whereas G varies from four times ten raised to three up until nine times ten raised to eight newtons per cubic meter cubed (N/$m^3$).

These calculations result in several graphs displayed across figures number two through seven that showcase our findings on these variables' impact on transverse deflection over time.


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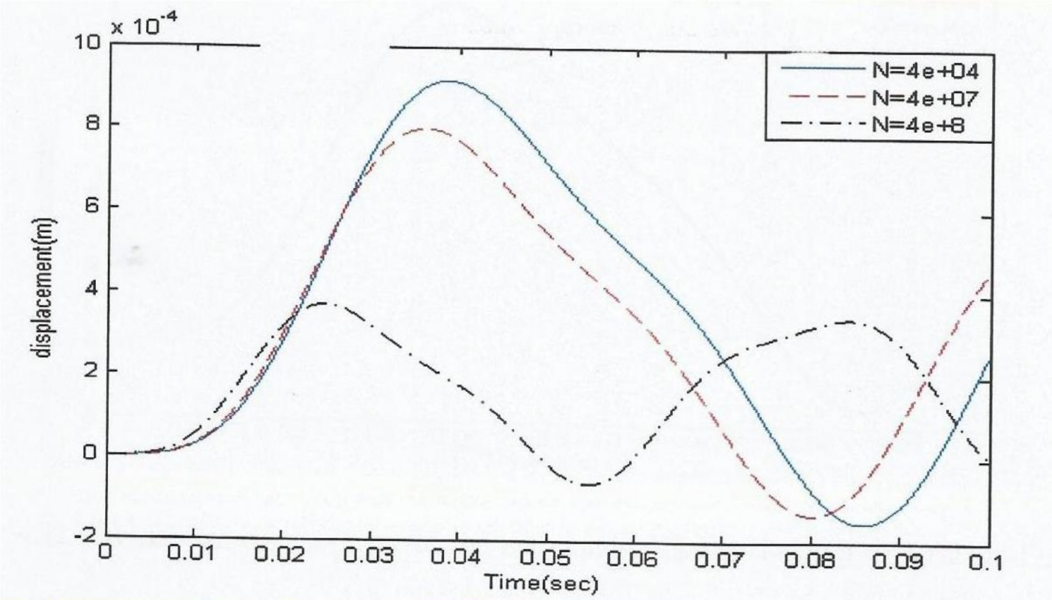


**Figure 2: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of axial force N and fixed values of K(4000) and G(4000) that traversed by moving distributed force.**

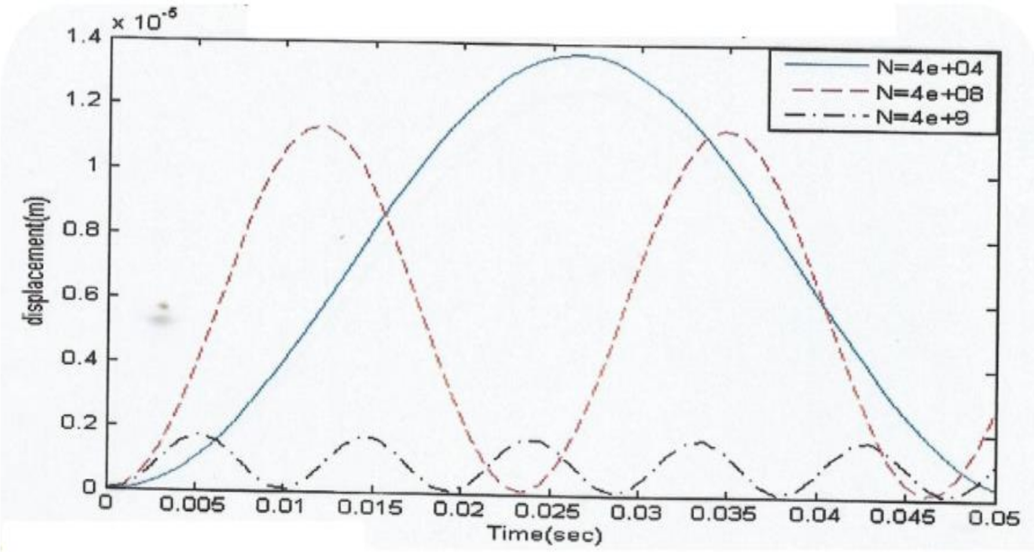


**Figure 3: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of axial force N and fixed values of K(4000) and G(4000) that traversed by moving distributed mass.**


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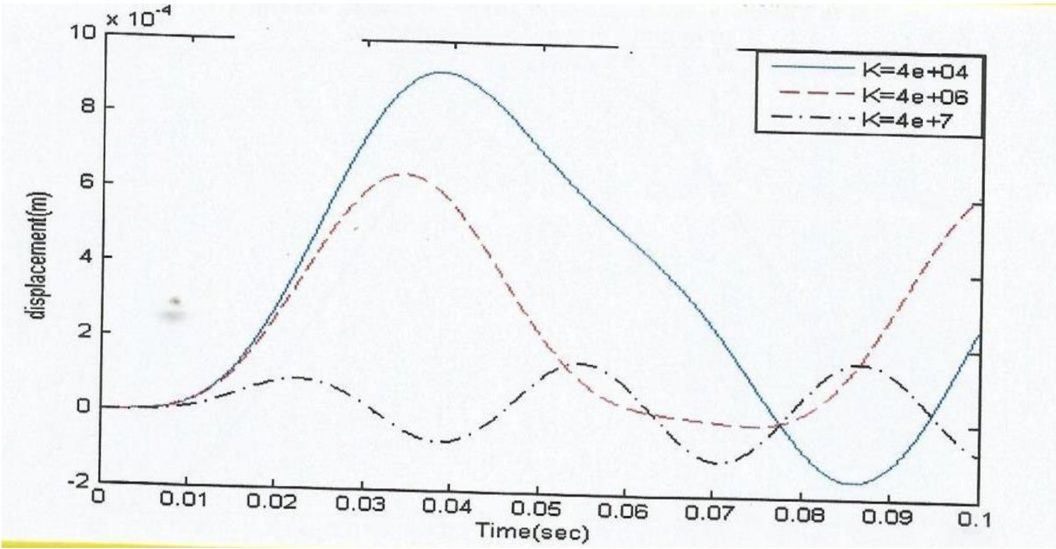


**Figure 4: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of foundation stiffness K and fixed values of G(4000) and N(4000) that traversed by moving distributed force.**

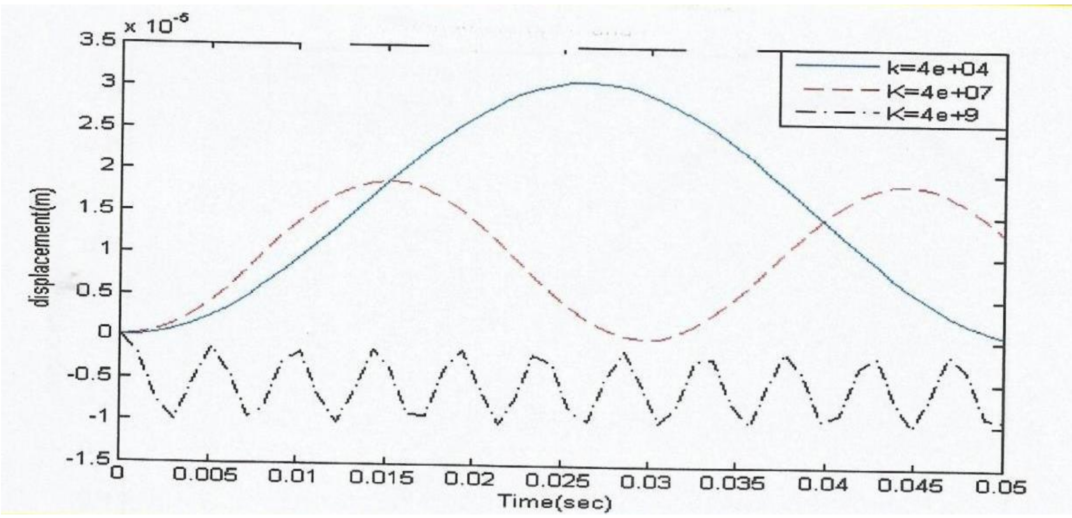


**Figure 5: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of foundation stiffness K and fixed values of G(4000) and N(4000) that traversed by moving distributed mass**.


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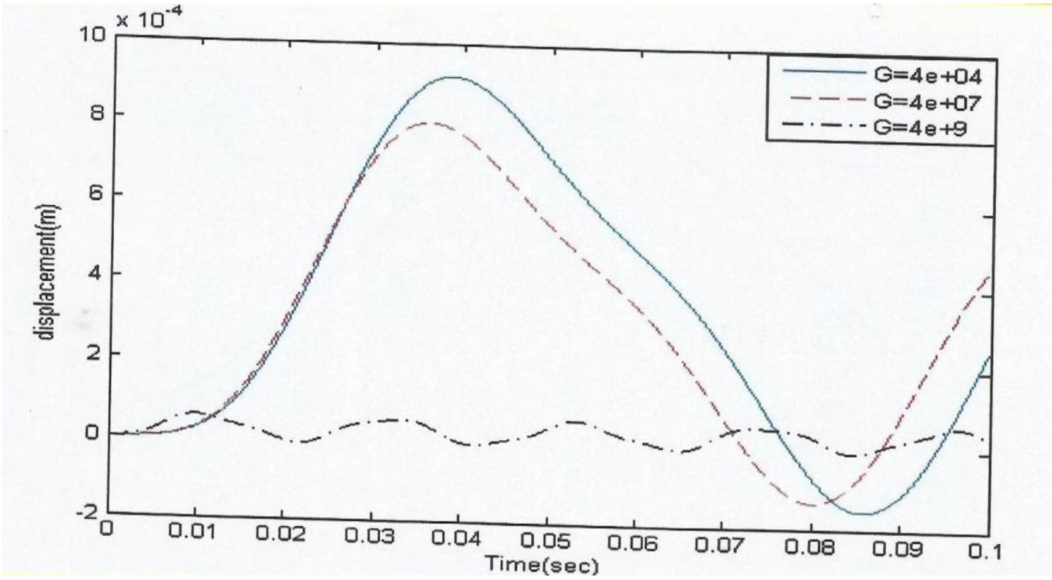


**Figure 6: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of shear modulus G and fixed values of N(4000) and K(4000) that traversed by moving distributed force.**

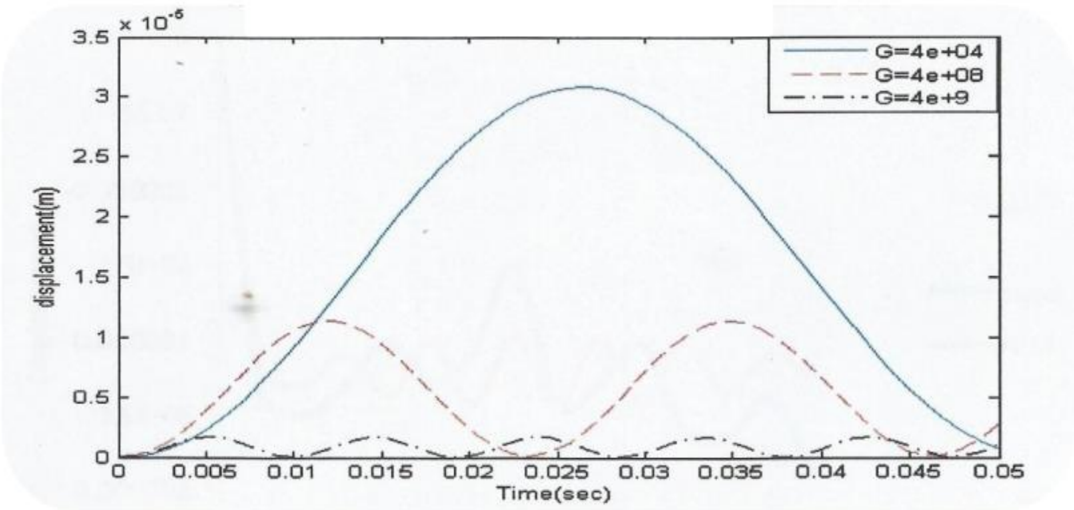


**Figure 7: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of shear modulus G and fixed values of N(4000) and K(4000) that traversed by moving distributed mass.**


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Figures 2-4 depict the transverse displacement responses of a non-uniform clamped-clamped Bernoulli-Euler beam subjected to distributed moving load traveling at constant velocity under the influence of moving distributed force. The figures display various values of (i) axial force *N* while other parameters remain fixed, (ii) foundation stiffness *K* while other parameters remain fixed, and (iii) shear modulus *G* while other parameters remain fixed. It is observed that as *N*, *K*, and *G* increase, there is a decrease in the deflection of the beam. Similar outcomes are achieved when the beam encounters moving mass, as shown in Figures 5-7.

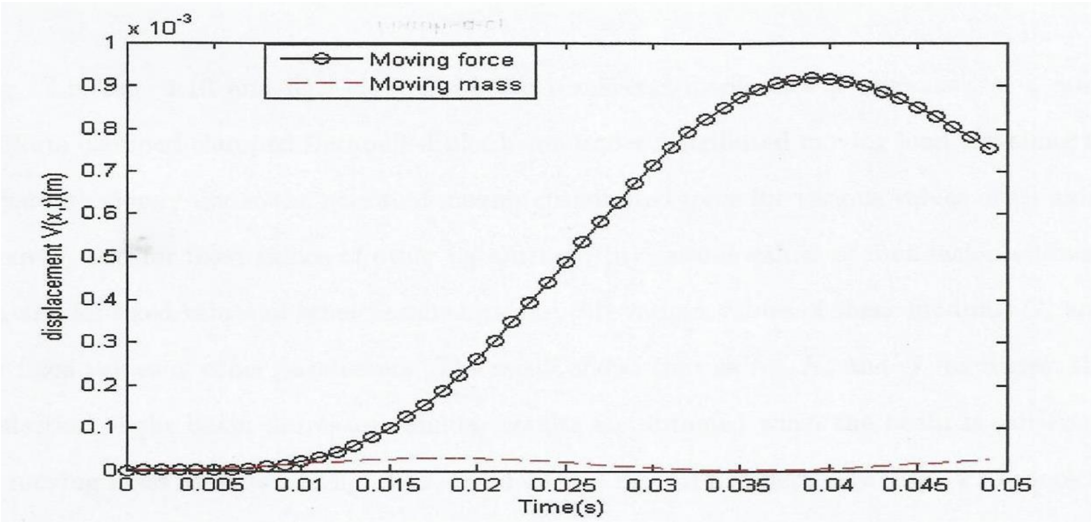


**Figure 8: Comparison of the transverse displacement of the moving distributed mass and moving distributed force for the non-uniform clamped-clamped Bernoulli-Euler beam.**

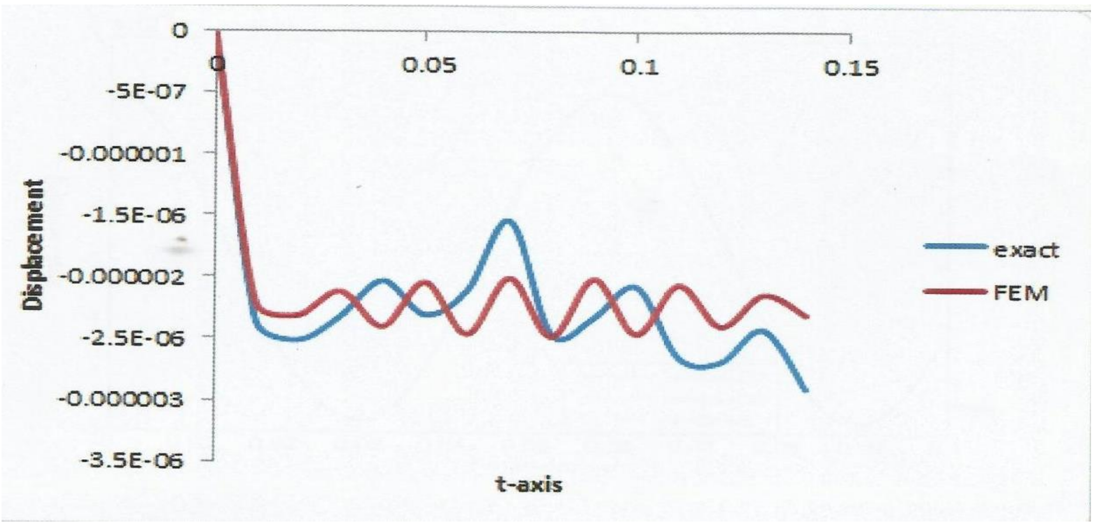


**Figure 9: Comparison of the transverse displacement of the exact and numerical solutions for the non-uniform clamped-clamped Bernoulli-Euler beam.**


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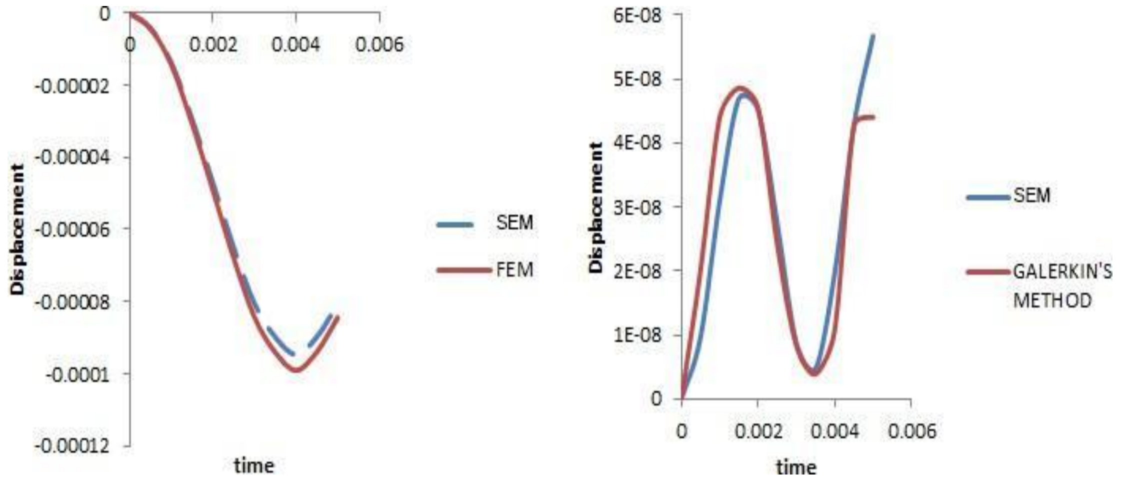


**Figure 10: Comparison of the transverse displacement of the moving distributed mass and moving distributed with SEM force for the non-uniform clamped-clamped Bernoulli–Euler beam.**

Various comparisons of the lateral displacements are depicted in Figures 8-10. To authenticate the precision of the current approach, we compare the vibration caused by moving distributed masses with varying magnitudes on a non-uniform Bernoulli-Euler beam that is clamped-clamped and rests on a Pasternak elastic foundation, as obtained through our method and frequency-domain spectral element method (SEM) at two different velocities illustrated in Figure 10. The findings indicate that dynamic responses generated through our procedure are nearly identical to those acquired via SEM.

## CONCLUSION

The investigation concerns the vibration of distributed masses that vary in magnitude and move under a clamped-clamped non-uniform Bernoulli-Euler beam resting on an elastic foundation governed by fourth-order partial differential equations with variable and singular coefficients. The primary objective is to obtain a closed-form solution for this type of dynamical problem, specifically when dealing with the non-uniform Bernoulli-Euler beam that varies along its span. Finite integral transform cannot be used to solve the governing equation due to its complexity, hence Galerkin's method - commonly employed in solving such problems - is utilized instead to transform the governing equation with singular and variable coefficients. The resulting equations of Galerkin are subsequently solved through (i) the utilization of Laplace transformation and convolution theory to obtain analytical solutions for the one-dimensional dynamic problem caused by a moving force, and (ii) finite element analysis in conjunction with Newmark method for instances involving a moving mass, which is analytically unsolvable due to its harmonic nature. To validate the accuracy of the aforementioned methodology used in (i), dynamic responses obtained via finite element method (FEM) for a clamped-clamped non-uniform Bernoulli-Euler beam are compared in Fig. 9, while those from frequency-domain spectral element method (SEM) are presented in Fig. 10. The acquired analytical solutions undergo an analysis where resonance conditions related to the problems at hand are identified. Numerical analysis is conducted whereby this study exhibits several interesting features:


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1. As the axial force values escalate, the displacement amplitude of a non-uniform Bernoulli-Euler beam that is clamped-clamped and subjected to uniformly distributed force decreases. This result holds true for fixed shear modulus *G* and foundation stiffness *K*. The same outcomes and analyses are obtained in the case of moving mass.

2. In the dynamic scenario, as the stiffness of the Pasternak foundation increases, the displacement of a clamped-clamped non-uniform Bernoulli-Euler beam subjected to a moving distributed force and resting on said foundation decreases. This holds true when both axial force *N* and shear modulus *G* remain constant. Similar findings and analyses are observed for cases involving moving mass.

3. For a constant axial force *N* and foundation stiffness *K*, the response amplitude of the clamped-clamped non-uniform Bernoulli-Euler beam subjected to a moving force decreases as the shear modulus *G* is increased. Similar findings and analyses are applicable for cases involving moving masses. This research has suggested valuable techniques for solving dynamic problems concerning clamped-clamped non-uniform Bernoulli-Euler beams under variable magnitude distributed masses.

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