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KERNEL CONSTRUCTION FOR EXPLORING TRENDS IN PROBABILITY DISTRIBUTION DEVELOPMENT

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ABSTRACT: *In this paper, we provided new methods that improve modeling flexibility of probability distributions. The methods focus on the construction of kernels for possible development of new probability models from (root) variable components or arbitrary functions. These approaches are further grouped into two different categories including construction of kernels from existing probability functions or directly using mathematical deterministic functions. The Direct substitution approach, homogeneous and inhomogeneous interaction methods are captured under kernel development from probabilistic functions. Two distributions namely, Lindley-Sine Distribution (LSD) and Alpha Lindley Distribution (ALD) were developed from the variable component of the Lindley distribution. More so, the combinations of normal and arcsine distribution, and Gumbel and exponential distributions birthed the Double Censored Normal-ArcSine Distribution (DCNAD) and Left Censored Gumbel-Exponential Distribution (LCGED) respectively. Interesting unconventional trends including decreasing sinusoidal, bathtub, triangular and circular trends realized from these developments validates the relevance of the approaches in probability forecasting. Finally, the asymptotic stability of the parameters of the derived distributions was established through simulation study.*

KEYWORDS: kernel development, censoring, trends, probability distribution.

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INTRODUCTION

Continuous probability distributions are realized majorly in two dimensions; either through density or distribution functions. Another approach could be from survival and hazard functions, which are technical component expressions of the probability density function (PDF) and or cumulative distribution function (CDF). Regardless of any approach adopted for methodological development, the common denominator is usually that the integral of the density function is one $\int_{-\infty}^{\infty} g(x,\omega)dx = 1$. This integral validation of PDFs gave rise to the concept of normalizing constant; which implies that deterministic functions $f(x,\omega)$ can be converted to probability functions.

The idea of normalization suggests that there are at most two components in a PDF: the variable component and the adjustable or unadjustable parameter component; where the parameter component is an inverted result of integrating the variable component. As reported in Feller (1998), Gaunt et al. (2019), Sun, Kong and Pal (2023), Mijoule et al. (2023), Wu et al. (2023), the concept of normalizing constant can be summarized thus:

$$g(x,\varphi) = Z_{\varphi}\, f(x,\varphi) \tag{1}$$

where $Z_{\varphi} = [\int_{-\infty}^{\infty} f(x,\varphi)dx\,]^{-1}$ is a parameter function, $x \; and \; \varphi$ are respectively the variable and the vector of constants or parameters. By extension, the multivariate representation is given by:

$$g(x_i,\varphi_i) = Z_{\varphi} f(x_i,\varphi_i)\,, \quad i = 1,2,3,\dots,n \tag{2}$$

and $$Z_{\varphi} = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_i,\varphi_i)dx_i\right]^{-1}$$

Lai (2013) studied different approaches to developing probability distributions including hazard and mean residual life function method, probability generating function induced G distribution, Laplace transform, Skewing, generalization, convolution, parameterization, mixture model and composite methods among others. Transformation, censoring and truncation of existing distributions are other methods too (Mandel 2007; Geskus 2011; Wang et al. 2023; Mohammed et al. 2023). Now, the offspring of various methodologies; however, can be treated in the light of equation (1); in fact all density functions follow the same principle when reaped apart. In other words, equations (1 and 2) are universal methodology for the development of PDFs.

So far, the trends recorded in literature under continuous distribution as a result of these methodologies include monotone increasing and decreasing trends, left and right skewed trends, (approximate) symmetric trend, U-shape, inverted and bathtub shapes, bimodal and multimodal trends, and triangular and circular trends to the best of my knowledge. But, comparatively, bimodal, multimodal, triangular and circular trends do not come by easily; especially that multimodal sine wave, increasing and decreasing sinusoidal trends have not been realized.

In this light, we seek to harness various uncommon methodological approaches by different construction of kernels for the development of PDFs, which will in turn yield some unprecedented trends. The sections that follow next include the methods of constructing kernels, vital developmental controls and the parameter estimation and simulation properties.

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METHODS OF CONSTRUCTING KERNELS

In this section, we investigate various means of constructing kernels or arbitrary functions for the development of PDFs. These various means of construction can be categorized under two different approaches:

- Kernel development from existing probability functions (PDFs and CDFs).
- Kernel development directly from deterministic functions.

We engage the study of the former followed by the latter.

Direct Substitution Method

This methodology deals with premeditated interjection or replacement of a given variable with a function in the variable component (root kernel) $f(x,\varphi)$ of an existing PDF. This function might be indicial, logarithmic, trigonometric or exponential, expressed as a parametric or variable factor, or even their combinations. This method leverages on the ease of adding extra parameters in contrast to many generalization methods. For example, we can realize new kernels $\nabla(x,\varphi)$ from the variable component $f(x,\varphi)$ of Lindley distribution (Lindley 1958) as:

$$(1+x)e^{-ax} \rightarrow \left(1+\left[\frac{1-sin[x]}{2}\right]\right)e^{-ax} \quad (3)$$

$$(1+x)e^{-ax} \rightarrow (1+x^b)e^{-ax^c} \quad (4)$$

where φ is the vector of parameters and $x>0$. These modified kernels when converted to PDF would definitely increase the modeling options of their root distributions. We give illustrations adopting equations (3 and 4);

$$\nabla(x,\varphi) = \left(1+\left[\frac{1-sin[x]}{2}\right]\right)e^{-\gamma x}$$

$$p(x,\gamma) = \frac{2\gamma(1+\gamma^2)}{3+\gamma(-1+3\gamma)}\left[1+\left(\frac{1-sin[x]}{2}\right)\right]e^{-\gamma x}; \; x>0, \; \gamma>0 \quad (5)$$

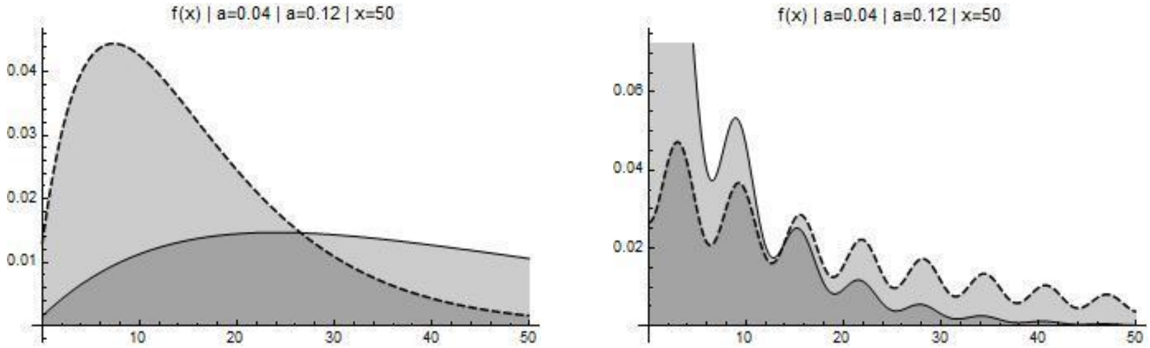

Figure 1: Graph plots of the PDFs as in equation (5)

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Remark

From Figure 1 we observe the outright consequence of the kernel development, transforming a unimodal function to multimodal model. This novel development assigns some credits to the root kernel. The development can be termed *Lindley-Sine Distribution (LSD)* and can serve as a forecast model in economies. Note that $a = \gamma$ as captured in the figure.

From the kernel in equation (4) we obtain a PDF.

$$\nabla(x, \varphi) = (1 + x^{\beta})e^{-\alpha x^{c}}$$

$$p(x, \varphi) = \frac{c \ \ \alpha^{\frac{1+\beta}{c}}}{R_{\circ}} \left(1 + x^{\beta}\right) e^{-\alpha x^{c}}, \ \ x > 0, \ \alpha > 0, \ \beta > 0, \ c > 0 \quad (6)$$

where
$$R_{\circ} = \alpha^{\frac{\beta}{c}} \ Gamma[1/c] + Gamma[(1 + \beta)/c]$$

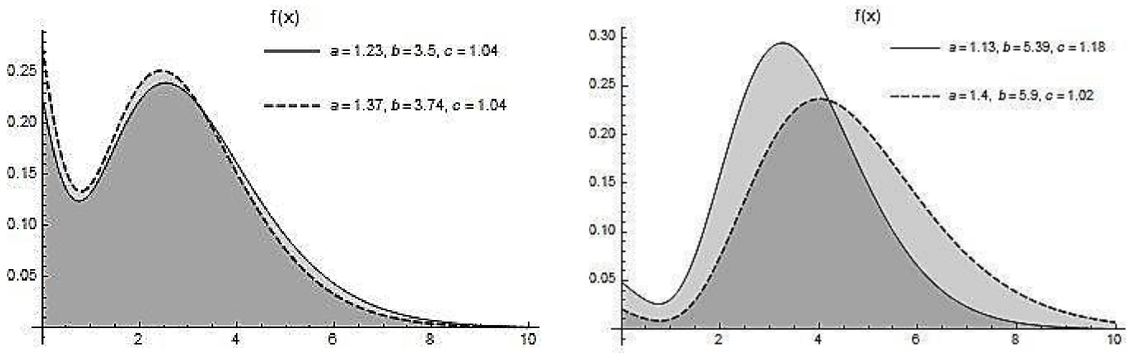

Figure 2: Graph plot of the PDF as in equation (6)

Lindley distribution is limited as it can only model left skewed and monotone increasing trend at $a < 1$ and monotone decreasing trend at $a \geq 1$. As seen in Figure 2, we have modified unimodal (approximately bimodal) trend and skewed trends in addition to the aforementioned shapes. By nomenclature, we term the PDF Alpha Lindley Distribution (ALD). Note: $\alpha = a \ and \ \beta = b$ as observed in the figure.

Homogeneous and Nonhomogeneous Interaction Method

Here, we treat various combinations of existing probability functions as kernels for new probability development. The combination across these functions might look alien, but provided there are matching variable supports, such that the kernels are integrable, we could use them as well. Let X be a random variable, we hence consider two PDFs $g(x, a) \ and \ t(x, b)$ with their corresponding CDFs $G(x, a) \ and \ T(x, b)$, then, we can both homogeneously and non-homogeneously combine them to realize new kernels, thus:

$$\nabla(x, \varphi) = g(x, \vartheta) \ G(x, \vartheta), \qquad \frac{g(x,\vartheta)}{G(x,\vartheta)}, \qquad g(x, \vartheta) \pm G(x, \vartheta)$$

$$\nabla(x, \varphi) = g(x, \vartheta) \ t(x, \beta), \qquad \frac{g(x,\vartheta)}{t(x,\beta)}, \qquad g(x, \vartheta) \pm t(x, \beta) \quad (7)$$

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$$\nabla(x,\varphi) = G(x,\vartheta)\, T(x,\beta), \quad \frac{G(x,\vartheta)}{T(x,\beta)}, \quad G(x,\vartheta) \pm T(x,\beta)$$

$$\nabla(x,\varphi) = g(x,\vartheta)\, T(x,\beta), \quad \frac{g(x,\vartheta)}{T(x,\beta)}, \quad g(x,\vartheta) \pm T(x,\beta) \qquad (8)$$

where $\vartheta\ and\ \beta$ are the vectors of parameters for their respective distributions. It is worthy of mention that this combination $2g(x,a)\ G(x,a)$ would yield a new PDF for distributions with the support range $x > 0\ and -\infty < x < \infty$; since ½ is the result of the integration of $g(x,a)\ G(x,a)$. Now, the various arrangements are realizable when we further introduce the concept of censoring in cases where the original support(s) are not converging. By convergence, we mean the integrability of a function over a range of support. We note again that in combining $g(x,\vartheta)\ T(x,\beta)$, where $g(x,\vartheta)$ has its range in the positive real line, $x > 0$ and $T(x,\beta)$ takes support such that $x \in R$; the kernel converges by left censoring in view of the support of $T(x,\beta)$.

Censored Distribution in this context implies the adjustment of either the lower or upper bound of a distribution. More so, in necessary cases, the upper and lower limits can be modified as well. These are referred to as left, right and double censoring $(R_{cs},\ l_{cs}\ and\ D_{cs})$ respectively; where for $\int_{-\infty}^{\infty} \nabla(x,\varphi)dx$:

- $\int_{-\infty}^{x_{max}} \nabla(x,\varphi)dx \to R_{cs}$
- $\int_{x_{min}}^{\infty} \nabla(x,\varphi)dx \to l_{cs}$
- $\int_{x_{min}}^{x_{max}} \nabla(x,\varphi)dx \to D_{cs}$

We therefore give some illustrative examples. Let $g(x,\vartheta) \sim N(0,\sigma^2)$ and $t(x,\beta) \sim ArcSine[x,\ \pi]$. It is well known that normal and arcsine distribution are symmetric; although the former is bell shaped and the latter has a U-shape. By combining them according to equation (7), given as $g(x,\vartheta)/t(x,\beta)$, we apply integration method over the support range $0 < x < 1$; and then normalizing the outcome we obtain a one parameter distribution thus:

$$\nabla(x,\varphi) = \frac{g(x,\ \sigma)}{t(x,\ \pi)} = \frac{\{\pi\sqrt{(1-x)x}\}\ e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

Note: Although π is a number; however, in distribution theory, it is recognized as *unadjustable parameter.*

$$p(x,\sigma) = Z_{\varphi}\left(\frac{g(x,\ \sigma)}{t(x,\ \pi)}\right) = \frac{8\, e^{-\frac{x^2}{2\sigma^2}}\sqrt{(1-x)x}}{\pi\ \aleph_{PFQ}},\ \ 0 < x < 1,\ \sigma > 0$$

(9)

where $\aleph_{PFQ} = hypergeometricPFQ\left[\left\{\frac{3}{4},\ \frac{5}{4}\right\},\ \left\{\frac{3}{2}, 2\right\},\ -\frac{1}{2\sigma^2}\right]$ is a generalized hyper-geometric function. It is to be noted that the choice of the support range between the two distributions was ultimately made due to convergence. Howbeit, we might view the example as double censoring D_{cs} with respect to the support range of normal distribution $x \in R$.

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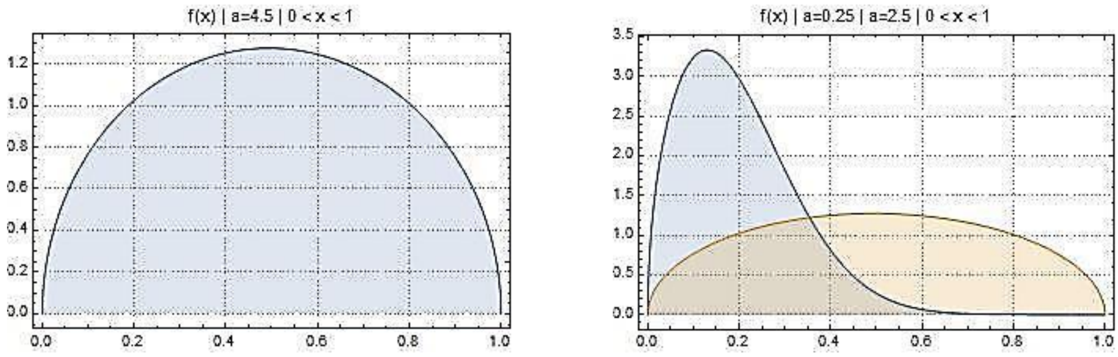

Figure 3: Graph plots of the PDF as in equation (9)

Remark

The shapes realized from this combination as clearly shown in Figure 3 are alien to the original shapes of the two distributions. These have increased the modeling options of the combo; especially that perfect semi-circle symmetric distribution seems to be rare in literature. We hence call the new development *Double Censored Normal-Arcsine Distribution (DCNAD).* Note: $\sigma = a$ as observed in the figure.

In the next illustration, we exemplify the case of left censoring l_{cs} using the product kernel in equation (8) $\nabla(x,\varphi) = g(x,\vartheta)\, T(x,\beta)$, which represents a combination of a PDF and CDF from different distributions. If $g(x,\vartheta) \sim Exponential[\theta]$ and $T(x,\beta) \sim Gumbel[0,1]$; then we realize a new PDF as:

$$\nabla(x,\theta) = g(x,\theta)\, T(x) = \left(\theta e^{-\theta x}\right)\left(1 - e^{-e^{x}}\right)$$

$$\rightarrow \qquad p(x,\theta) = \frac{\theta e^{-\theta x}\left(1-e^{-e^{x}}\right)}{1-\theta\,\Gamma[-\theta,\ 1]}; \quad x > 0,\ \theta > 0 \qquad (10)$$

where $\Gamma[-\theta,\ 1]$ is an incomplete gamma function.

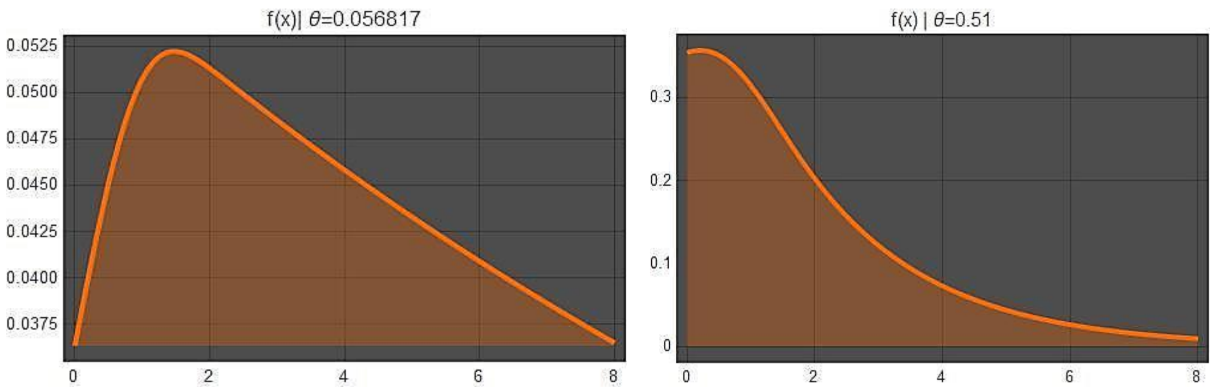

Figure 4: Graph plots of the PDF as in equation (10)

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Remark

The shapes obtained in Figure 4 are products of the combination of Gumbel and exponential distributions. The derived distribution is left censored, in view of the Gumbel distribution with the support range $x \in R$. These distributions are characterized by right skewed and monotone decreasing trend respectively. However, their combo resulted to triangular and half-bell-curve trend distribution, which is majorly not a kind in their individual trend. The half-bell-curve trend can co-model scenarios where truncated normal distribution fits. We would refer to this development as Left Censored Gumbel-Exponential Distribution (LCGED).

Kernel Development Directly From Deterministic Functions

The first section detailed the construction of kernels from existing probability functions. However, we briefly will give a more direct illustration using potential mathematical functions.

Table 1: ***Mathematical arbitrary functions for PDF developments***

S/N	Kernels $f(x,\varphi)$	Function Type	Variable Support	$\int_{-\infty}^{\infty} f(x,\varphi)dx$	Parameter Support	$Z_\varphi f(x,\varphi)$
1	$a^x e^{-bx}$	Indicial / Exponential	$\{0,\infty\}$	$\frac{1}{b-log[a]}$	$log[a]<0$ $0<a<1$ $b>0$	$\frac{(b-log[a])\,a^x}{e^{bx}}$
2	$\frac{1-sin\,(ax)}{2}$	Trigonometric	$\{0,b\}$	$\frac{-1+ab+Cos[ab]}{2a}$	{ … }	$\frac{a(1-sin[ax])}{ab-1+cos[ab]}$
3	$x^a + cos\,(bx)$	Trigonometric	$\{0,c\}$	$\frac{c^{1+a}}{1+a}+\frac{Sin[bc]}{b}$	$a>-1$ $b>0$ $c>0$	$\frac{x^a+cos[bx]}{\frac{c^{1+a}}{1+a}+\frac{sin[bc]}{b}}$
4	$log\left[\frac{ax}{b}\right]$	Logarithmic	$\{0,1\}$	$-1+log\left[\frac{a}{b}\right]$	{ … }	$\frac{log[\frac{ax}{b}]}{-1+log[\frac{a}{b}]}$

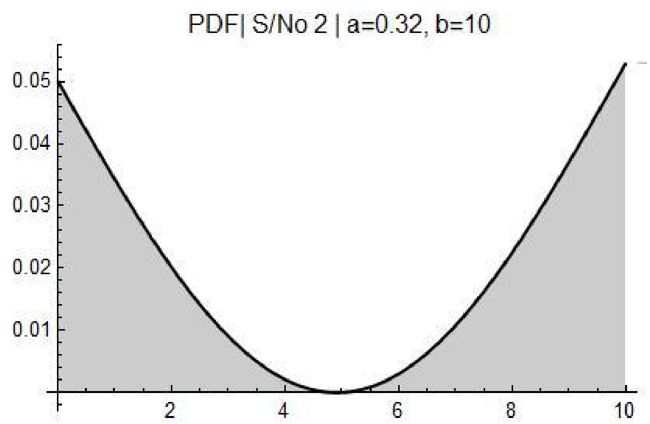

Figure 5: Graph plots of the PDFs from the kernels in Table 1, S/Ns 2.

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Figure 5 presents a trend for the PDF of the kernel in S/N 2; which features bathtub trend. In addition, S/N 1 and 4 is characterized by conventional monotone decreasing trend.

Vital Developmental Controls

There are three major inevitable challenges one is very much likely to encounter in various attempts, to develop distributions through these mechanisms, and they include:

- Zero realization of parametric support(s) over a given support range $x \in R$.
- Having special functions as component(s) of the new distribution.
- Combining distributions with many parameters.

As there has been technological advancement, parameter supports for $\int_{-\infty}^{\infty} \nabla(x,\varphi)dx$ are realized by default. However, in cases where the parameter(s) are not conditioned, we "manually" constrain them graphically. But, it is advisable to work with components that ensure-these supports are realized by default. This is because; it covers all the probability space for such distributions. Now, if we engage manual restrictions in a problem, we would realize some reasonable coverage, but may not outrightly cover the probability space for such distributions. Software like Wolfram Mathematica (WM) or Math Lab could be of great assistance here:

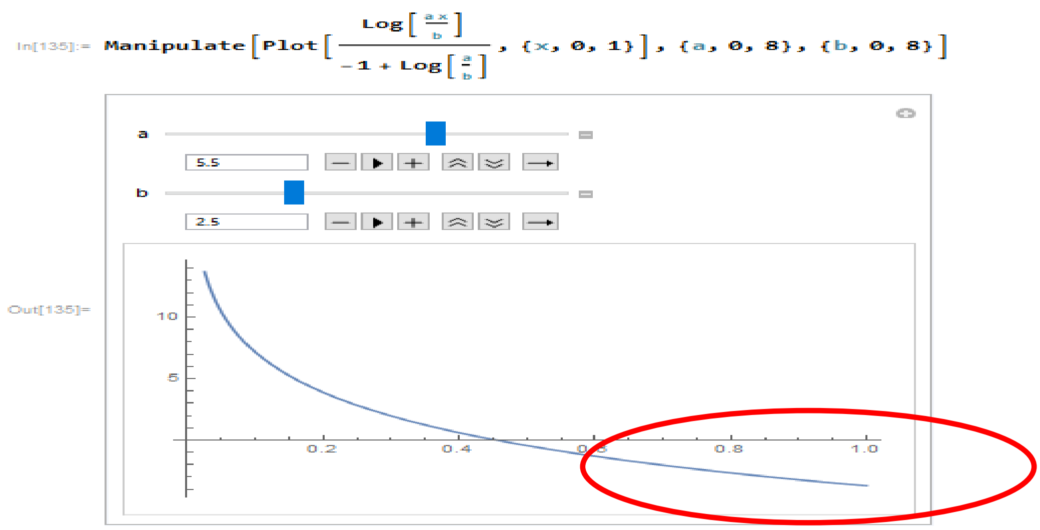

Figure 6: Graph showing the plot likelihood in a case of zero parametric support as in S/N 4

As seen in Figure 6, the adjustments can be made using the blue knots until the plot aligns within the probability space. Of course a probability space is the first or second quadrant in a graph $\{x,y\}$ *and* $\{-x,y\}$.

In handling functions, it might be discouraging sometimes to work with special kinds like Bessel, Lommel, Erf, Hyper-geometric, Poly-gamma, Incomplete gamma (and in their various regularized or generalized forms), and many other functions alike (Andrew 1998). These functions notwithstanding, as observed in the study, can provide some relevant trends in probability modeling. So, to process the complexities that come with their computations,

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simulations and inferential analysis, the use of advanced mathematical software is recommended.

More so, combinatory preference is given to distributions with one or no parameters each, such that the total number of parameters is at most two. In cases where the supports are in the positive real line, as in equation (6), the combined model can house up to three parameters. These in essence, will reduce many complexities one is likely to encounter.

Parameter Estimation and Simulation Study

Here, we study the behavior of the parameters of the derived distributions, to ascertain whether they are consistent and asymptotically stable. The major goal of the maximum likelihood estimator is to obtain the parameter values of a probability function that maximizes the likelihood function in the parametric spatial dimension. This can be studied under different data conditions, which include uncensored data and or censoring (Fang et al. 2015; Kinaci et al. 2014). Censoring is a concept that describes the timing, in which data are recorded during a procedural observation; in the sense that truncated events are not treated like exhaustive investigations. In general, the likelihood functions for both data conditions are respectively given as:

$$L(x,\theta) = \prod_{i=1}^{n} \quad [g(x_i)]$$

$$L(x,\theta) = \frac{n!}{(N-n)!} \{\prod_{i=1}^{n} \quad g(x_i)\}\{1 - G(x_T)\}^{N-n} \tag{11}$$

where $g(x_i)$ and $G(x_i)$ are the PDF and CDF of a distribution with an independent random observations $x_i, i = 1, 2, \dots, n$; N is the number of specimens being investigated or number of trials. Now, if the fixed time or cycle or count to an event (say failure time) is x_0, then for Type 1 censoring according to equation (11), the time of termination $x_T = x_0$; and $x_T = x_n$ for Type 2 case. However, for this study, we have our emphasis only on the complete or uncensored investigations.

Now, let $x_i, i = 1,2, \dots, n$ be a vector of observations from LSD, and then the log-likelihood for the complete data is defined by:

$$l(x,\gamma) = logL(x,\gamma) = \sum_{i=1}^{n} \quad log\, log\, \{f(x,\gamma)\}$$

$$l(x,\gamma) = \sum_{i=1}^{n} \quad log\, log\, \left\{\frac{2\gamma(1+\gamma^2)}{(3+2\gamma^2)} \left[1 + Sin^2\left(\frac{x}{2}\right)\right] e^{-\gamma x}\right\}$$

$$= log\left[\frac{2\gamma(1+\gamma^2)}{(3+2\gamma^2)}\right]^n + \sum_{i=1}^{n} \quad log\, log\, \left[1 + Sin^2\left(\frac{x_i}{2}\right)\right] - \gamma \sum_{i=1}^{n} \quad x_i \quad = nlog\,(2\gamma) + nlog\,(1+\gamma^2) - nlog\,(3+2\gamma^2) + \sum_{i=1}^{n} \quad log\, log\, \left[1 + Sin^2\left(\frac{x}{2}\right)\right] - \gamma \sum_{i=1}^{m} \quad x_i \tag{12}$$

The score function for equation (12) is defined by

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} + \frac{2\gamma n}{1+\gamma^2} - \frac{4\gamma n}{3+2\gamma^2} - \sum_{i=1}^{n} \quad x_i$$

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$$\frac{n}{\gamma}+\frac{2\gamma n}{1+\gamma^2}-\frac{4\gamma n}{3+2\gamma^2}-\sum_{i=1}^{n} x_i=0$$

$$\frac{n}{\gamma}+\frac{2\gamma n}{1+\gamma^2}-\frac{4\gamma n}{3+2\gamma^2}-\sum_{i=1}^{n} x_i=0$$

$$\frac{3+7\gamma^2+2\gamma^4}{3\gamma+5\gamma^3+2\gamma^5}=\frac{\sum_{i=1}^{n} x_i}{n}$$

$$(3+7\gamma^2+2\gamma^4)-\underline{x}(3\gamma+5\gamma^3+2\gamma^5)=0$$

Let $x_i, i=1,2,\dots,n$ be a vector of observations from ALD; then the log-likelihood estimate is defined by

$$l(x,\alpha,\beta,c)=\sum_{i=1}^{n} log\, log\left[\frac{c\ \alpha^{\frac{1+\beta}{c}}(1+x^{\beta})\,e^{-\alpha x^{c}}}{\alpha^{\frac{\beta}{c}}\,\Gamma\left[\frac{1}{c}\right]+\Gamma\left[\frac{1+\beta}{c}\right]}\right]$$

$$=log\left[\frac{c\ \alpha^{\frac{1+\beta}{c}}}{\alpha^{\frac{\beta}{c}}\,\Gamma\left[\frac{1}{c}\right]+\Gamma\left[\frac{1+\beta}{c}\right]}\right]^{n}+\sum_{i=1}^{n} log(1+x^{\beta})-\alpha\sum_{i=1}^{n} x_i^{c}$$

$$=nlog\,c+n\left(1+\frac{\beta}{c}\right)log\,log\ \alpha-n\left(\frac{\beta}{c}\right)log\,log\,\alpha-log\,log\,\Gamma\left(\frac{1}{c}\right)$$

$$-log\,log\,\Gamma\left(\frac{1+\beta}{c}\right)+\sum_{i=1}^{n} log(1+x^{\beta})-\alpha\sum_{i=1}^{n} x_i^{c} \quad (13)$$

The score function for (13) is defined by

$$\frac{\partial l}{\partial\alpha}=n\left(1+\frac{\beta}{c}\right)\left(\frac{1}{\alpha}\right)-n\left(\frac{\beta}{c}\right)\left(\frac{1}{\alpha}\right)-\sum_{i=1}^{n} x_i^{c}$$

$$\frac{n}{\alpha}-\sum_{i=1}^{n} x_i^{c}=0$$

$$\frac{\partial l}{\partial\beta}=n\left(\frac{1}{c}\right)log\,log\,\alpha-n\left(\frac{1}{c}\right)log\,log\,\alpha-\frac{P_g\left(0,\ \frac{1+\beta}{c}\right)}{c}+\sum_{i=1}^{n}\frac{x^{\beta}logx}{(1+x^{\beta})}$$

$$\sum_{i=1}^{n}\frac{x_i^{\beta}logx}{(1+x_i^{b})}-\frac{P_g\left(0,\frac{1+\beta}{c}\right)}{c}=0$$

$$\frac{\partial l}{\partial c}=\frac{n}{c}-n\left(\frac{\beta}{c^2}\right)log\,log\,\alpha+n\left(\frac{\beta}{c^2}\right)log\,log\,\alpha+\frac{P_g\left(0,\ \frac{1}{c}\right)}{c^2}+\frac{(1+\beta)\,P_g\left(0,\ \frac{1+\beta}{c}\right)}{c^2}-\alpha\sum_{i=1}^{n} x_i^{c}\,log\,log\,(x_i)$$

$$\frac{n}{c}+\frac{P_g\left(0,\ \frac{1}{c}\right)}{c^2}+\frac{(1+\beta)\,P_g\left(0,\ \frac{1+\beta}{c}\right)}{c^2}-\alpha\sum_{i=1}^{n} x_i^{c}\,log\,log\,(x_i)=0$$

where $P_g(*,*)$ is poly-gamma.

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Let $x_i, i = 1,2, \ldots, n$ be a vector of observations from DCNAD, and then the log-likelihood for the complete data is defined by:

$$l(x,\sigma) = logL(x,\sigma) = \sum_{i=1}^{n} \quad log\, log\, \{f(x,\sigma)\}$$

$$l(x,\sigma) = \sum_{i=1}^{n} \quad log\, log \left\{ \frac{8\, e^{-\frac{y^2}{2\sigma^2}} \sqrt{(1-x)x}}{\pi\, ℵ_{PFQ}} \right\}$$

$$= log \left[\frac{8}{\pi\, ℵ_{PFQ}}\right]^n + \sum_{i=1}^{n} \quad log\, log \left[\left((1-x)x\right)^{\frac{1}{2}}\right] - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \quad x_i^2$$

$$= nlog\, 8 - nlog(\pi) - nlog\left(ℵ_{PFQ)}\right) + \sum_{i=1}^{n} \quad log\, log \left[\left((1-x)x\right)^{\frac{1}{2}}\right] - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \quad x_i^2 \qquad (14)$$

The score function for equation (14) is defined by

$$\frac{\partial l}{\partial \sigma} = \frac{n\{ℶ_{PFQ}\}}{ℵ_{PFQ)}} + \frac{1}{\sigma^3} \sum_{i=1}^{n} \quad x_i^2$$

$$\frac{n\{ℶ_{PFQ}\}}{ℵ_{PFQ)}} + \frac{1}{\sigma^3} \sum_{i=1}^{n} \quad y_i^2 = 0$$

Where $ℶ_{PFQ} = \frac{5}{16e^3} HypergeometricPFQ\left[\left\{\frac{7}{4}, \frac{9}{4}\right\}, \left\{\frac{5}{2}, 3\right\}, -\frac{1}{2e^2}\right]$

More so, if $x_i, i = 1,2, \ldots, n$ is a vector of observations from LCGED, then the log-likelihood for the complete data is defined by:

$$l(x,\theta) = logL(x,\theta) = \sum_{i=1}^{n} \quad log\, log\, \{f(x,\theta)\}$$

$$l(x,\theta) = \sum_{i=1}^{n} \quad log\, log \left\{ \frac{\theta e^{-\theta x}\left(1-e^{-e^x}\right)}{1-\theta\, \Gamma[-\theta,\ 1]} \right\}$$

$$= log \left[\frac{\theta}{1-\theta\, \Gamma[-\theta,1]}\right]^n + \sum_{i=1}^{n} \quad log\, log \left[1 - e^{-e^{x_i}}\right] - \theta \sum_{i=1}^{n} \quad x_i$$

$$= nlog\theta - nlog(1-\theta\, \Gamma[-\theta,\ 1]) + \sum_{i=1}^{n} \quad log\, log \left[1 - e^{-e^{x_i}}\right] - \theta \sum_{i=1}^{n} \quad x_i \qquad (15)$$

The score function for equation (15) is defined by

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{n\left(\theta M_G - \Gamma(-\theta,1)\right)}{1-\theta\, \Gamma[-\theta,1]} - \sum_{i=1}^{n} \quad x_i$$

$$(1-\theta\, \Gamma[-\theta,\ 1]) - \theta\left(\theta M_G - \Gamma(-\theta,1)\right) - \theta(1-\theta\, \Gamma[-\theta,\ 1])\hat{x} = 0$$

Where $M_G = MeijerG[\{\{\},\{1,1\}\},\{\{0,0,-\theta\},\{\}\},1]$

A numerical analysis like Newton-Raphson iterative method, which is a root finding algorithm, can be used to obtain the MLEs of $\hat{\gamma},\ \hat{a}, \hat{b}, \hat{c}, \hat{\sigma}, \hat{\theta}\ and$. This scheme is given by

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$$\hat{\varphi} = \varphi - H^{-1}(\varphi)\, S(\varphi)$$

where $S(\omega)$ is the score function and $H^{-1}(\omega)$ is the second derivatives of the log-likelihood function termed the Hessian matrix (Akram and Ann 2015; Bayal et al. 2022). Finally, it is expected that the different methods show efficiency with respect to the selected size of data samples under consideration.

Furthermore, the asymptotic character of the maximum likelihood estimates of the parameters of Lindley-Sine, Alpha Lindley, Double Censored Normal-Arcsine, and Left Censored Gumbel-Exponential distributions is investigated, through Monte Carlo simulation study. For different sample sizes $n = 20,\ 50,\ 75,\ 100\ \&\ 250$, a 10000 times trials is carried out; and the steps are given by the algorithm:

i) Choose a value M (which represents the number of Monte Carlo trials).

ii) Select the values $\varphi_0 = (\gamma_0, \alpha_0, \beta,\ c_0, \sigma_0, \theta_0)$ within the domain of their parameter supports.

iii) Simulate a sample of size *n* from the derived distributions.

iv) Compute the maximum likelihood estimates $\hat{\varphi}_k\ of\ \varphi_k$

v) Redo steps 3-4 for *N* number of times

vi) The computations of the following measures are obtained:

$$Average\ Bias = \left[\ \frac{1}{M}\sum_{i=1}^{M} \quad (\hat{\varphi}_i - \varphi)\right]\ and$$

$$MSE = \left[\ \frac{1}{M}\sum_{i=1}^{M} \quad (\hat{\varphi}_i - \varphi)^2\ \right]$$

See Table 2-7 for the *Average Bias and MSE of the LSD, ALD, DCNAD and LCGED*

Table 2: *Average Bias and MSE of the (LSD) Estimator $\hat{\gamma}$*

Parameter	*n*	Average Bias (γ)	MSE (γ)
$\gamma = 0.1$	20	0.31626	5.27540
	50	0.19005	3.57610
	75	0.10085	2.24623
	100	0.09485	2.05691
	250	0.05852	1.35571
$\gamma = 0.5$	20	0.06401	0.25472
	50	0.03385	0.12843
	75	0.01876	0.07529
	100	0.01232	0.04211
	250	0.01029	0.03573
$\gamma = 1.5$	20	-0.00161	0.03749
	50	-0.00182	0.01141
	75	-0.00250	0.00905

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	100	-0.00414	0.00884
	250	-0.00560	0.00546
$\gamma = 2.5$	20	-0.00279	0.15275
	50	-0.00635	0.08667
	75	-0.01125	0.03772
	100	-0.01708	0.02354
	250	-0.02033	0.02418

Table 3: ***Average Bias and MSE of the (ALD) Estimator $\hat{\alpha}$***

Parameter	N	Average Bias (α)	MSE (α)
$\alpha = 1.5$ $\beta = 2.0$ $c = 1.0$	20	0.1207	0.02012
	50	0.1162	0.00635
	75	0.1038	0.00479
	100	0.1001	0.00306
	250	0.0967	0.00209
$\alpha = 1.5$ $\beta = 2.0$ $c = 1.5$	20	0.4851	0.03705
	50	0.3300	0.02553
	75	0.1941	0.01800
	100	0.1792	0.01766
	250	0.0620	0.01669
$\alpha = 2.5$ $\beta = 2.5$ $c = 4.0$	20	0.6190	0.00256
	50	0.5606	0.00183
	75	0.4970	0.00170
	100	0.3792	0.00164
	250	0.2781	0.00158

Table 4: ***Average Bias and MSE of the (ALD) Estimator $\hat{\beta}$***

Parameter	n	Average Bias (β)	MSE (β)
$\alpha = 1.5$ $\beta = 2.5$ $c = 1.0$	20	0.6945	0.09015
	50	0.4514	0.08102
	75	0.4003	0.07087
	100	0.3908	0.05835
	250	0.3100	0.03254
$\alpha = 1.5$ $\beta = 2.0$ $c = 1.5$	20	0.5321	0.12350
	50	0.3143	0.05421
	75	0.3018	0.03254
	100	0.1011	0.01831
	250	0.0258	0.00214
$\alpha = 0.5$ $\beta = 2.5$ $c = 4.0$	20	0.8354	0.21470
	50	0.6358	0.12014
	75	0.6014	0.08211
	100	0.4532	0.05360
	250	0.2145	0.00389

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Table 5: ***Average Bias and MSE of the (ALD) Estimator $\hat{c}$***

Parameter	*N*	Average Bias (c)	MSE (c)
$\alpha = 1.5$ $\beta = 2.5$ $c = 1.0$	20	0.8674	0.23512
	50	0.7586	0.05386
	75	0.5381	0.05123
	100	0.3254	0.02519
	250	0.3012	0.01534
$\alpha = 1.5$ $\beta = 2.0$ $c = 1.5$	20	0.5247	0.09241
	50	0.5213	0.09008
	75	0.3254	0.07235
	100	0.1253	0.06247
	250	0.1103	0.05321
$\alpha = 0.5$ $\beta = 2.5$ $c = 4.0$	20	0.7586	0.03125
	50	0.4251	0.02141
	75	0.4011	0.02012
	100	0.2345	0.01785
	250	0.1568	0.01239

Table 6: ***Average Bias and MSE of the (DCNAD) Estimator $\hat{\sigma}$***

Parameter	*n*	Average Bias (σ)	MSE (σ)
$\sigma = 0.1$	20	-1.6e-16	0.00499
	50	-0.3e-16	0.00576
	75	-7.8e-16	0.00387
	100	-8.0e-16	0.00312
	250	-10.6e-16	0.00125
$\sigma = 0.5$	20	-2.6e-15	0.05451
	50	-2.8e-15	0.04729
	75	-3.2e-15	0.04154
	100	-6.1e-15	0.03215
	250	-8.2e-15	0.03012
$\sigma = 1.5$	20	-3.1e-15	0.06999
	50	-3.7e-15	0.06865
	75	-8.3e-15	0.04824
	100	-7.9e-15	0.05234
	250	-9.7e-15	0.01425
$\sigma = 2.5$	20	6.2e-15	0.06017
	50	-2.2e-16	0.05934
	75	-3.9e-15	0.05239
	100	-4.5e-15	0.04236
	250	-1.2e-15	0.06546

Table 7: ***Average Bias and MSE of the (LCGED) Estimator $\hat{\theta}$***

Parameter	*n*	Average Bias (θ)	MSE (θ)
	20	3.5e-15	4.7921
	50	3.1e-15	2.1045

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$\theta = 0.1$	75	4.6e-15	5.2365
	100	1.9e-14	0.2004
	250	1.4e-14	0.0158
	20	-1.1e-14	2.0365
	50	-2.8e-14	1.5867
$\theta = 0.5$	75	-4.8e-14	0.5876
	100	-8.5e-14	0.2365
	250	-3.1e-14	0.4986
	20	3.8e-15	0.3437
	50	3.3e-15	0.3014
$\theta = 1.5$	75	2.5e-15	0.2358
	100	1.2e-15	0.1356
	250	-4.4e-16	0.0258
	20	5.1e-15	0.2145
	50	6.2e-15	0.3254
$\theta = 2.5$	75	-9.7e-15	0.0214
	100	-7.7e-14	0.0145
	250	-5.5e-13	0.0015

The estimates for the average bias and mean square error are presented in Tables 2-7, and at different selected values of parameters. Apparently, from the Tables, we deduce that the estimates of the average bias and mean square error decrease as the sample size n increases, apart from DCNAD and LCGED that showed slight relativity. These simply indicate that the estimators of the derived distributions are consistent and asymptotically stable.

CONCLUSION

In the course of study, we developed different probability distributions through various means; by engaging the construction of kernels from either already existing probability functions or directly using mathematical deterministic functions. These approaches entail directly substituting parametric functions appropriately in to a variable component of existing functions, suitable combinations of PDFs and CDFs given that there is convergence over its censored or uncensored variable support range. The remarkable trends obtained are horizontal, increasing and decreasing sinusoidal trends, bathtub, triangular and circular trends. The modeling possibilities as revealed by the trends validate the adoption of the approaches of constructing kernels for the development of probability distributions. More so, a simulation study was carried out to ascertain the stability of the different parameters in the variety of the derived distributions. The investigation showed that they are consistent and asymptotically normal. It is uniquely recommended that extreme events including economic data, sine-wave data, flood data and many others can be modelled by adopting the sinusoidal distributions as captured in Figures 1 and 5. More so, other forms of kernel development that are not treated here can be explored; as they might sustain the propensities for more novel developments.

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