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THEORETICAL STUDY OF FORCED VAN DER POL OSCILLATOR EQUATION USING MULTIPLE TWO-TIMING REGULAR PARAMETER PERTURBATION AND ASYMPTOTIC EXPANSION TECHNIQUES

Onuoha N.O.

Department of Mathematics, Imo State University, Owerri, Imo State, Nigeria.

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ABSTRACT: *This paper presents the theoretical study of forced Van der Pol oscillator equation. Oscillatory systems are studied to know measures that can reduce the amplitude of oscillation of the oscillatory system. Here, multiple two-timing regular parameter perturbation is applied since it is a kind of perturbation among other perturbation techniques that enables the study of the behaviour of a system under certain conditions. Asymptotic expansion technique was also applied. Excel Microsoft was used to analyse the uniformly valid asymptotic solution of the Van der Pol oscillator equation obtained. The uniformly valid asymptotic solution in the independent variable obtained, showed that damping alters the amplitude of the oscillatory system thereby affecting its motion. Increase in damping decreases the amplitude of oscillation of the system. With damping incorporated in the system though very small damping, the amplitude of oscillation reduces with time.*

KEYWORDS: Oscillatory system, Regular parameter perturbation, asymptotic expansion, damping, amplitude, oscillation.

INTRODUCTION

Investigations on Van der Pol oscillator equation have gained much patronage in recent research due to its vast applications in engineering, physical and biological sciences. Van der Pol oscillator equation is valuable for design investigation, simulation and presentation of real life oscillatory systems. The Van der Pol oscillator equation is a nonlinear model that has been solved and analysed using different techniques. Chen and Liu [4] applied Liao's homotopy analysis method to obtain a uniformly valid solution of limit cycle of the Duffing-Van der Pol oscillator equation. Sandile and Precious [17] solved forced-free Van der Pol and Duffing equations using successive linearisation method to obtain the limit cycle and bifurcation diagrams of the two equations, and they concluded that the method is accurate and effective in finding solutions of nonlinear equations with oscillatory solutions. Chen and Liu [3] studied the limit cycle of forced-free Van der Pol equation using homotopy analysis method to reduce the computational efforts. Li et al. [13], found series solutions of coupled Van der Pol oscillator equation by means of homotopy analysis method and showed that there exists either in-plane or out-phase periodic solutions. Kimiaeifer et al. [11], investigated the analysis of modified Van der Pol oscillator equation using He's parameter-expanding method in which they concluded that one term in the series expansions is sufficient to obtain a highly accurate solution which is valid for the whole solution. Cordshooli and Vahidi [5] solved Duffing-Van der Pol equation using Adomian's decomposition method. Their result showed that converting the differential equation to a system of equations in Adomian's transformation method gives more accurate answers in a short time of computations. Parameter expansion method was used by Darvish and Kheybari [6] to find the solution of the classical Van der Pol oscillator equation and its approximate frequencies. Lucero and Schoentegen [14], studied the Van der Pol equation as a model of right and left vocal fold oscillators. Jifeng et al. [8], investigated the stability of the periodic solution of Van der Pol's equation using homotopy perturbation method and they analysed the periodic solution they obtained using Floquet theory. Onuoha and Vincent [16], studied the free forced Van der equation to investigate theoretically the influence of nonlinear damping and geometric imperfections on oscillatory systems. Mohammadi et al. [15], solved Duffing Van der Pol equations numerically using the basis of hybrid functions. Their work showed that Duffing Van der Pol equation can be converted to a nonlinear volterra integral equation of the second kind. Khan [10], applied Homotopy perturbation method to obtain an analytic solution of forced-free Van der Pol differential equation. Serge [18] obtained various solutions in his research where he solved Van der Pol equation using numerical solution methods. The solution of Van der Pol equation can also be obtained in a complex domain. Victor and Alexander [19] found the analytic approximation solution of the Van der Pol equation in a complex domain. Comparison studies on different solution methods of solving Van der Pol equation were still carried out. Joel and Adedire [9] compared two solution methods; modified Adomian decomposition method and truncated Taylor series method of solving Van der Pol equation. Asma and Ahmed [2] applied active control approach with Laplace transform method to coupled generalized Van der Pol oscillator equation and presented different behaviours of the oscillator equation with distributed order. Alvaro et al [1] presented different analytical techniques and numerical method forced Van der Pol oscillator equation can be solved and analysed. Kuptsova [12], used Van der Pol oscillator equation to study dispersion resonance Fonkou et al [7], investigated the dynamic behaviour and real time control to a target trajectory using Van der Pol equation with sine nonlinearity.

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Forced Van der Pol Oscillator Equation

Van der Pol oscillator equation has been used to study many dynamical and oscillatory systems. Forced Van der Pol oscillator equation represents systems that external force is applied to. Here, regular parameter perturbation and asymptotic expansions are applied to forced Van der Pol oscillator equation to know the significance effect of the damping parameter, η

$$\ddot{x}+\eta\left(2-x^2\right)\dot{x}+x=\mu\cos 2t \tag{1a}$$

$$x(0)=\dot{x}(0)=0 \tag{1b}$$

For $0<\eta<<1,\ 0<\mu<<1$

Where η is the damping parameter and μ is the imperfection sensitivity parameter.

Multiple two-timing regular parameter perturbation and asymptotic expansion

Introducing the multiple two-time scale, we let

$$x(t)=\xi\left(t,\tau(t)\right) \tag{2}$$

where t and τ are time scales.

We also let

$$\tau=\eta t \tag{3}$$

Substituting for $x(t)$ in equation (1a), we have

$$\frac{d^2\xi}{dt^2}+\eta\left(2-\xi^2\right)\frac{d\xi}{dt}+\xi=\mu\cos 2t \tag{4a}$$

$$\xi(0,0)=\dot{\xi}(0,0)=0 \tag{4b}$$

We let

$$\xi(t,\tau)=\sum_{\substack{i=1\\ j=0}}^{\infty}\xi_{ij}(t,\tau)\mu^i\eta^j \tag{5}$$

Using equations (3) and (5), equation (4a) becomes

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$$\mu\left(\xi_{10,tt}+\eta\xi_{11,tt}+\eta^2\xi_{12,tt}+\ldots\right)+\ldots+2\eta\left\{\mu\left(\xi_{10,t\tau}+\eta\xi_{11,t\tau}+\eta^2\xi_{12,t\tau}+\ldots\right)+\ldots\right\}+$$
$$\eta^2\left\{\mu\left(\xi_{10,\tau\tau}+\eta\xi_{11,\tau\tau}+\eta^2\xi_{12,\tau\tau}+\ldots\right)+\ldots\right\}+2\eta\left\{\mu\left(\xi_{10,t}+\eta\xi_{11,t}+\eta^2\xi_{12,t}+\ldots\right)+\ldots\right\}+$$
$$2\eta^2\left\{\mu\left(\xi_{10,\tau}+\eta\xi_{11,\tau}+\eta^2\xi_{12,\tau}+\ldots\right)+\ldots\right\}-\eta\left[\mu\left(\xi_{10}+\eta\xi_{11}+\eta^2\xi_{12}+\ldots\right)+\ldots\right]^2$$
$$\left[\mu\left(\xi_{10,t}+\eta\xi_{11,t}+\eta^2\xi_{12,t}+\ldots\right)+\ldots+\eta\left\{\mu\left(\xi_{10,\tau}+\eta\xi_{11,\tau}+\eta^2\xi_{12,\tau}+\ldots\right)+\ldots\right\}\right]+$$
$$\mu\left(\xi_{10}+\eta\xi_{11}+\eta^2\xi_{12}+\ldots\right)+\mu^2\left(\xi_{20}+\eta\xi_{21}+\eta^2\xi_{22}+\ldots\right)=\mu\cos 2t$$
(6)

Solving equation (6), we equate the coefficients of powers of η^i and μ^j, $i=1,2,3\,;\,j=0,1,2$.

$$\left(\mu^1:\eta^0\right):\xi_{10,tt}+\xi_{10}=\cos 2t \tag{7}$$

$$\left(\mu^1:\eta^1\right):\xi_{11,tt}+2\xi_{10,t\tau}+2\xi_{10,t}+\xi_{11}=0 \tag{8}$$

$$\left(\mu^1:\eta^2\right):\xi_{12,tt}+2\xi_{11,t\tau}+\xi_{10,\tau\tau}+2\xi_{11,t}+2\xi_{10,\tau}+\xi_{12}=0 \tag{9}$$

$$\left(\mu^2:\eta^0\right):\xi_{20,tt}+\xi_{20}=0 \tag{10}$$

$$\left(\mu^2:\eta^1\right):\xi_{21,tt}+2\xi_{20,t\tau}+2\xi_{20,t}+\xi_{21}=0 \tag{11}$$

$$\left(\mu^2:\eta^2\right):\xi_{22,tt}+2\xi_{21,t\tau}+\xi_{20,\tau\tau}+2\xi_{21,t}+2\xi_{20,\tau}+\xi_{22}=0 \tag{12}$$

$$\left(\mu^3:\eta^0\right):\xi_{30,tt}+\xi_{30}=0 \tag{13}$$

$$\left(\mu^3:\eta^1\right):\xi_{31,tt}+2\xi_{30,t\tau}+2\xi_{30,t}-\left(\xi_{10}\right)^2\xi_{10,t}+\xi_{31}=0 \tag{14}$$

$$\left(\mu^3:\eta^2\right):\xi_{32,tt}+2\xi_{31,t\tau}+\xi_{30,\tau\tau}+2\xi_{31,t}+2\xi_{30,\tau}-\left(\xi_{10}\right)^2\xi_{11,t}-\left(\xi_{10}\right)^2\xi_{10,\tau}+\xi_{32}=0 \tag{15}$$

Next, we solve the resulting differential equations (7) to (15) using their respective initial conditions.

The initial conditions are obtained from equation (4b) using equation (5)

Solution to equation (7)

$$\xi_{10,tt}+\xi_{10}=\cos 2t \tag{16a}$$

$$\xi_{10}\left(0,0\right)=\xi_{10,t}\left(0,0\right)=0 \tag{16b}$$

Solving equation (16), we get

$$\xi_{10}\left(t,\tau\right)=A_{10}\left(\tau\right)\cos t+B_{10}\left(\tau\right)\sin t-\frac{1}{3}\cos 2t \tag{17}$$

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Applying the initial conditions, equation (16b) on equation (17), we obtain

$$A_{10}(0)=\frac{1}{3},\ B_{10}(0)=0 \tag{18}$$

Solution to equation (8)

$$\xi_{11,tt}+\xi_{11}=-2\left(-A'_{10}\sin t+B'_{10}\cos t\right)-2\left(-A_{10}\sin t+B_{10}\cos t+\frac{2}{3}\sin 2t\right) \tag{19a}$$

$$\xi_{11}(0)=\xi_{11,t}(0)=0 \tag{19b}$$

To ensure a uniformly valid asymptotic solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively

For $\sin t$

$$A_{10}(\tau)=k_1e^{-\tau} \tag{20a}$$

From equation (18)

$$k_1=\frac{1}{3}\Rightarrow A_{10}(\tau)=\frac{1}{3}e^{-\tau} \tag{20b}$$

For $\cos t$

$$\beta_{10}(\tau)=k_2e^{-\tau} \tag{21a}$$

From equation (21a)

$$k_2=0\Rightarrow B_{10}(\tau)=0 \tag{21b}$$

Solving the remaining part of equation (19a), we get

$$\xi_{11}(t,\tau)=A_{11}(\tau)\cos t+B_{11}(\tau)\sin t+\frac{2}{9}\sin 2t \tag{22}$$

Applying the initial conditions, equation (19b) on equation (22), we get

$$A_{11}(0)=0,\ B_{11}(0)=-\frac{4}{9} \tag{23}$$

Solution to equation (9)

$$\xi_{12,tt}+\xi_{12}=-2\left(A'_{11}\sin t+B'_{11}\cos\right)-\left(A''_{10}+2A'_{10}\right)\cos t-2\left(-A_{11}\sin t+B_{11}\cos t+\frac{4}{9}\cos 2t\right) \tag{24a}$$

$$\xi_{12}(0)=\xi_{12,t}=0 \tag{24b}$$

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To ensure a uniformly valid asymptotic solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively

For $\sin t$

$$A_{11}(\tau) = k_3 e^{-\tau} \tag{25a}$$

From equation (23),

$$k_3 = 0 \Rightarrow A_{11}(\tau) = 0 \tag{25b}$$

For $\cos t$

$$B_{11}(\tau) = e^{-\tau}\left\{\int_0^{\tau} H_1(\tau) e^{\tau} d\tau + B_{11}(0)\right\} \tag{26}$$

Solving the remaining part of equation (24a), we get

$$\xi_{12}(t,\tau) = A_{12}(\tau)\cos t + B_{12}(\tau)\sin t + \frac{8}{27}\cos 2t \tag{27}$$

Applying the initial conditions, equation (24b) on equation (27), we get

$$A_{12}(0) = -\frac{8}{27},\ B_{12}(0) = 0 \tag{28}$$

Solution to equation (10)

$$\xi_{20,tt} + \xi_{20} = 0 \tag{29a}$$

$$\xi_{20}(0) = \xi_{20,t}(0) = 0 \tag{29b}$$

Solving equation (29a), we get

$$\xi_{20}(t,\tau) = A_{12}(\tau)\cos t + B_{12}(\tau)\sin t \tag{30}$$

Applying the initial conditions, equation (29b) on equation (30), we get

$$A_{20}(0) = 0,\ B_{20}(0) = 0 \tag{31}$$

Solution to equation (11)

$$\xi_{21,tt} + \xi_{21} = -2\left(-A_{20}' \sin t + B_{20}' \cos t\right) - 2\left(-A_{20}\sin t + B_{20}\cos t\right) \tag{32a}$$

$$\xi_{21}(0) = \xi_{21,t}(0) = 0 \tag{32b}$$

To ensure a uniformly valid asymptotic solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively.

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For $\sin t$

$$A_{20}(\tau) = k_4 e^{-\tau} \tag{33a}$$

From equation (31)

$$k_4 = 0 \Rightarrow A_{20}(\tau) = 0 \tag{33b}$$

For $\cos t$

$$B_{20}(\tau) = k_5 e^{-\tau} \tag{34a}$$

From equation (31)

$$k_5 = 0 \Rightarrow B_{20}(\tau) = 0 \tag{34b}$$

Solving the remaining part of equation (32a), we get

$$\xi_{21}(t,\tau) = A_{21}(\tau)\cos t + B_{21}(\tau)\sin t \tag{35}$$

Applying the initial conditions, equation (32b) on equation (35), we get

$$A_{21}(0) = 0,\ \ B_{21}(0) = 0 \tag{36}$$

Solution to equation (12)

$$\xi_{22,tt} + \xi_{22} = -2\left(A'_{21}\sin t + B'_{21}\cos t\right) - 2\left(A_{21}\sin t + B_{21}\cos t\right) - \left(A''_{20}\cos t + B''_{20}\sin t\right) - 2\left(A'_{20}\cos t + B'_{20}\sin t\right) \tag{37a}$$

$$\xi_{20}(0) = \xi_{20,t} = 0 \tag{37b}$$

To ensure a uniformly valid solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively

For $\sin t$

$$A_{21}(\tau) = k_6 e^{-\tau} \tag{38a}$$

From equation (36)

$$k_6 = 0 \Rightarrow A_{21}(\tau) = 0 \tag{38b}$$

For $\cos t$

$$B_{21}(\tau) = k_7 e^{-\tau} \tag{39a}$$

From equation (36)

$$k_7 = 0 \Rightarrow B_{21}(\tau) = 0 \tag{39b}$$

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Solving the remaining part of equation (37a), we get

$$\xi_{22}(t,\tau) = A_{22}(\tau)\cos t + B_{22}(\tau)\sin t \tag{40}$$

Applying the initial conditions, equation (37b) on equation (40), we get

$$A_{22}(0) = 0,\ B_{22}(0) = 0 \tag{41}$$

Solution to equation (13)

$$\xi_{30,tt} + \xi_{30} = 0 \tag{42a}$$

$$\xi_{30}(0) = \xi_{30,t}(0) = 0 \tag{42b}$$

Solving equation (42a), we get

$$\xi_{30}(t,\tau) = A_{30}(\tau)\cos t + B_{30}(\tau)\sin t \tag{43}$$

Applying the initial conditions, equation (42b) on equation (43), we get

$$A_{30}(0) = 0,\ B_{30}(0) = 0 \tag{44}$$

Solution to equation (14)

$$\begin{aligned}\xi_{31,tt} + \xi_{31} = {} & -2\left(-A'_{30}\sin t + B'_{30}\cos t\right) - 2\left(-A_{30}\sin t + B_{30}\cos t\right) - \frac{1}{4}A_{10}^3\left(\sin t + \sin 3t\right) + \\ & \frac{2}{3}A_{10}^2\left(\frac{1}{2}\sin 2t + \frac{1}{4}\sin 4t\right) + \frac{1}{6}A_{10}^2\sin 4t - \frac{1}{9}A_{10}\left(\sin 5t + \sin 3t\right) + \\ & \frac{1}{54}\left(\sin 2t + \sin 6t\right) - \frac{1}{9}A_{10}\left(\frac{1}{2}\sin t + \frac{1}{4}\sin 5t + \frac{1}{4}\sin 3t\right)\end{aligned} \tag{45a}$$

$$\xi_{31}(0) = \xi_{31,t}(0) = 0 \tag{45b}$$

To ensure a uniformly valid asymptotic solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively

For $\sin t$

$$A_{30}(\tau) = e^{-\tau}\left\{\int_0^\tau H_2(\tau)e^{\tau}d\tau + A_{30}(0)\right\} \tag{46a}$$

where

$$H_2(\tau) = \frac{1}{8}A_{10}^3 = \frac{1}{36}A_{10} \tag{46b}$$

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For $\cos t$

$$B_{30}(\tau) = k_8 e^{-\tau} \tag{47}$$

From equation (44)

$$k_8 = 0 \Rightarrow B_{30}(\tau) = 0 \tag{48}$$

Solving the remaining part of equation (45a), we get

$$\xi_{31}(t,\tau) = A_{31}(\tau)\cos t + B_{32}(\tau)\sin t - \frac{1}{3}G_1(\tau)\sin 2t + \frac{1}{8}G_2(\tau)\sin 3t - \frac{1}{45}G_3(\tau)\sin 4t + \frac{5}{864}G_4(\tau)\sin 5t - \frac{1}{3510}\sin 6t \tag{49}$$

where

$$G_1(\tau) = \frac{1}{3}A_{10}^2 + \frac{1}{54} \tag{50a}$$

$$G_2(\tau) = \frac{1}{4}A_{10}^3 + \frac{1}{9}A_{10} \tag{50b}$$

$$G_3(\tau) = \frac{1}{3}A_{10}^2 \tag{50c}$$

$$G_4(\tau) = \frac{5}{36}A_{10} \tag{50d}$$

Applying the initial conditions, equation (45b) on equation (49), we get

$$A_{31}(0) = 0,\ B_{31}(0) = \frac{2}{3}G_1(0) - \frac{3}{8}G_2(0) + \frac{4}{45}G_3(0) - \frac{25}{864}G_4(0) \tag{51}$$

Solution to equation (15)

$$\begin{aligned}\xi_{32,tt} + \xi_{32} &= -2\left(-A_{31}'\sin t + B_{31}'\cos t - \frac{2}{3}G_1'\cos 2t + \frac{3}{8}G_2'\cos 3t - \frac{4}{45}G_3'\cos 4t + \frac{25}{864}G_4'\cos 5t\right) - A_{30}''\cos t - \\ &2A_{30}'\cos t - 2\left(-A_{31}\sin t + B_{31}\cos t - \frac{2}{3}G_1\cos 2t + \frac{3}{8}G_2\cos 3t - \frac{4}{45}G_3\cos 4t + \frac{25}{864}G_4\cos 5t - \frac{6}{3510}\cos 6t\right) + \\ &\frac{1}{4}A_{10}^2 B_{11}(\cos 3t + 3\cos t) + \frac{4}{9}A_{10}^2\left(\frac{1}{2}\cos 2t + \frac{1}{4}\cos 4t + \frac{1}{4}\right) - \frac{2}{3}A_{10}B_{11}\left(\frac{1}{2}\cos 2t + \frac{1}{4}\cos 4t + \frac{1}{4}\right) + \\ &\frac{1}{9}B_{11}\left(\frac{1}{2}\cos t + \frac{1}{4}\cos 5t\frac{1}{4}\cos 3t\right) + \frac{4}{81}\left(\frac{3}{4}\cos 2t + \frac{1}{4}\cos 6t\right) - \frac{8}{27}A_{10}\left(\frac{1}{2}\cos t + \frac{1}{4}\cos 5t + \frac{1}{4}\cos 3t\right) + \\ &\frac{1}{4}A_{10}^2 A_{10}'(\cos 3t + 3\cos t) - \frac{2}{3}A_{10}'\left(\frac{1}{2}\cos 2t + \frac{1}{4}\cos 4t + \frac{1}{4}\right) + \frac{1}{9}A_{10}'\left(\frac{1}{2}\cos t + \frac{1}{4}\cos 5t + \frac{1}{4}\cos 3t\right)\end{aligned} \tag{52a}$$

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$$\xi_{32}(0)=\xi_{32,t}(0)=0 \tag{52b}$$

To ensure a uniformly valid solution in t, we equate to zero the coefficients of $\sin t$ and $\cos t$ respectively

For $\sin t$

$$A_{31}(\tau)=k_9 e^{-\tau} \tag{53}$$

From equation (51)

$$k_9=0 \Rightarrow A_{31}(\tau)=0 \tag{54}$$

For $\cos t$

$$B_{31}(\tau)=e^{-\tau}\left\{\int_0^{\tau} H_3(\tau)e^{\tau}d\tau+B_{31}(0)\right\} \tag{55}$$

Solving the remaining part of equation (52a), we get

$$\xi_{32}(t,\tau)=A_{32}(\tau)\cos t+B_{32}(\tau)\sin t+G_5(\tau)\cos 2t+G_6(\tau)\cos 3t+G_7(\tau)\cos 4t+G_8(\tau)\cos 5t+F_1\cos 6t+\frac{1}{4} \tag{56}$$

where

$$G_5(\tau)=-\frac{1}{3}\left(\frac{4}{3}G_1'+\frac{4}{3}G_1+\frac{2}{9}A_{10}^2-\frac{1}{3}A_{10}B_{11}+\frac{1}{3}A_{10}'+\frac{3}{81}\right) \tag{57a}$$

$$G_6(\tau)=-\frac{1}{8}\left(-\frac{3}{4}G_2'-\frac{3}{4}G_2+\frac{1}{4}A_{10}^2B_{11}+\frac{1}{36}B_{11}+\frac{2}{27}A_{10}+\frac{1}{4}A_{10}^2A_{10}'+\frac{1}{36}A_{10}'\right) \tag{57b}$$

$$G_7(\tau)=-\frac{1}{15}\left(\frac{8}{45}G_3'+\frac{4}{45}G_3+\frac{1}{3}A_{10}'\right) \tag{57c}$$

$$G_8(\tau)=-\frac{1}{24}\left(-\frac{25}{432}G_4'-\frac{25}{432}G_4+\frac{1}{36}B_{11}+\frac{2}{27}A_{10}+\frac{1}{36}A_{10}'\right) \tag{57d}$$

$$F_1=-\frac{1}{35}\left(\frac{6}{3510}+\frac{1}{81}\right) \tag{57e}$$

Applying the initial conditions, equation (52b) on equation (56), we get

$$A_{32}(0)=-G_5(0)-G_6(0)-G_7(0)-G_8(0)-F_1-\frac{1}{4},\ B_{32}(0)=0 \tag{58}$$

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The Amplitude (The maximum displacement)

The approximate solution of equations (4a) and (4b) serves as solution to equation (1a) and (1b) respectively. From equation (5), the displacement taking to be the maximum displacement is

$$\xi(t,\tau)=\mu\left(\xi_{10}+\eta\xi_{11}+\eta^2\xi_{12}\right)+\mu^3\left(\xi_{30}+\eta\xi_{31}+\eta^2\xi_{32}\right) \tag{59}$$

For simplicity, we take the first two terms of $\xi(t,\tau)$. Hence equation (59) becomes

$$\xi(t,\tau)=\mu\left[\left\{A_{10}(\tau)\cos t-\frac{1}{3}\cos 2t\right\}+\eta\left\{B_{11}(\tau)\sin t+\frac{2}{9}\sin 2t\right\}\right] \tag{60}$$

Substituting for $A_{10}(\tau)$ and $B_{11}(\tau)$ in equation (60), we get

$$\xi(t,\tau)=\frac{1}{3}\mu\left(e^{-\eta t}\cos t-\cos 2t\right)+\frac{1}{3}\mu\eta\left\{e^{-\eta t}\left(\eta t-\frac{4}{3}\right)\sin t+\frac{2}{3}\sin 2t\right\} \tag{61}$$

Analysis

Equation (61) is the solution to equation (6) and by equation (4a), it is an approximate solution to equation (1). Graphs of $\xi(t,\tau)$ against t at various values η and fixed value of the imperfection sensitive parameter μ are plotted to know the effect of damping on the amplitude of a forced oscillatory system.

Table1: Computed values of $\xi(t,\tau)$ at various values of t and η at fixed values of $\mu, (\mu=0.01)$

t	$\xi(t,\tau)$					
	$\eta=0$	$\eta=0.01$	$\eta=0.02$	$\eta=0.03$	$\eta=0.04$	$\eta=0.05$
0.5	0.575028	0.574672	0.574319	0.573972	0.573628	0.573289
1.0	0.641111	0.640565	0.640034	0.639517	0.639014	0.638525
1.5	0.129908	0.129442	0.328999	0.128579	0.128180	0.127802
2.0	-0.499140	-0.499260	-0.499350	-0.499420	-0.499470	-0.499510
2.5	-0.678640	-0.678230	-0.677820	-0.677410	-0.676990	-0.676560
3.0	-0.252210	-.0251300	-0.250400	-0.249530	-0.248680	-0.247860
3.5	0.383836	0.385056	0.386221	0.387334	0.388398	0.389414
4.0	0.645932	0.647099	0.648189	0.649208	0.650159	0.651048
4.5	0.299464	0.300173	0.300807	0.301379	0.301873	0.302317
5.0	-0.327070	-0.327140	-0.327260	-0.327410	-0.327600	-0.327820
5.5	-0.646520	-0.647500	-0.648470	-0.649410	-0.650350	-0.651260
6.0	-0.355630	-0.357380	-0.359050	-0.36063	-0.362140	-0.363570
6.5	0.283817	0.281683	0.279697	0.277848	0.276127	0.274524
7.0	0.684279	0.682323	0.680543	0.678923	0.677447	0.676102
7.5	0.472570	0.471371	0.470326	0.469415	0.468621	0.467931

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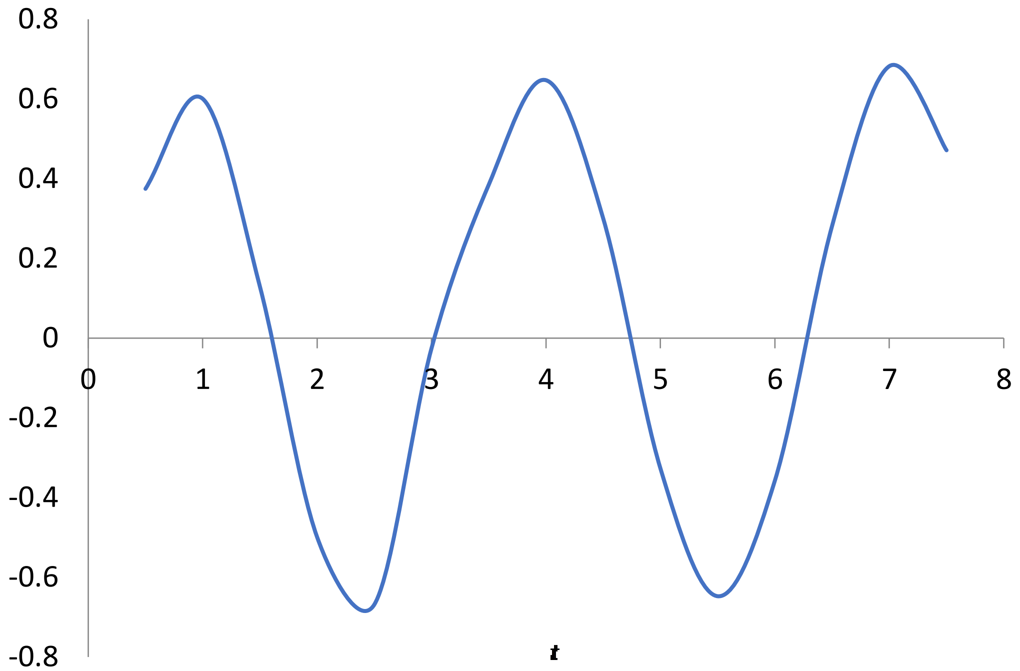

Fig 1. Variation of the displacement $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0.01$

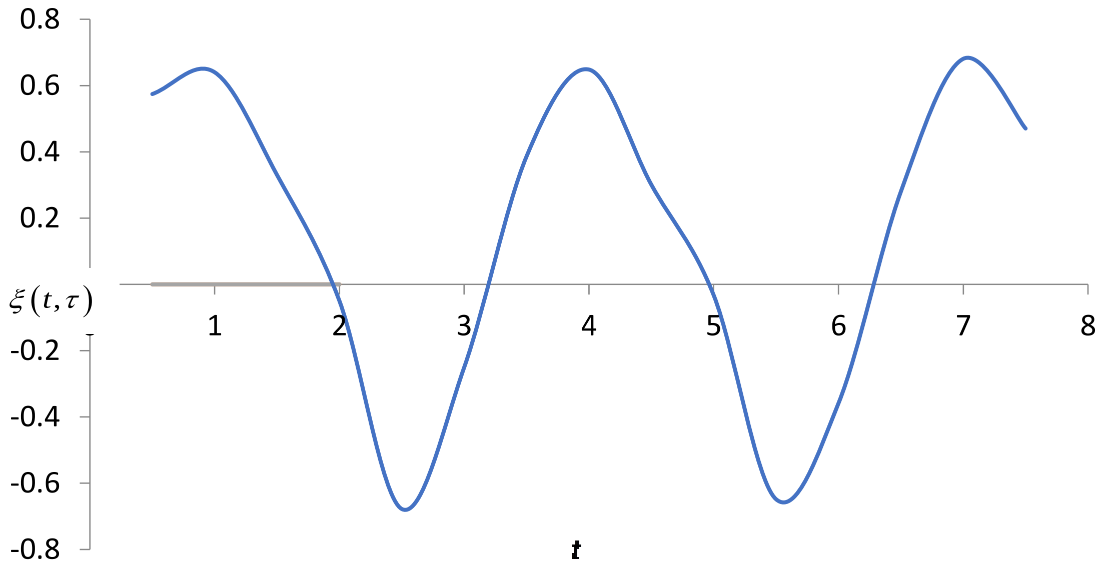

Fig 2. Variation of the displacement, $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0.02$

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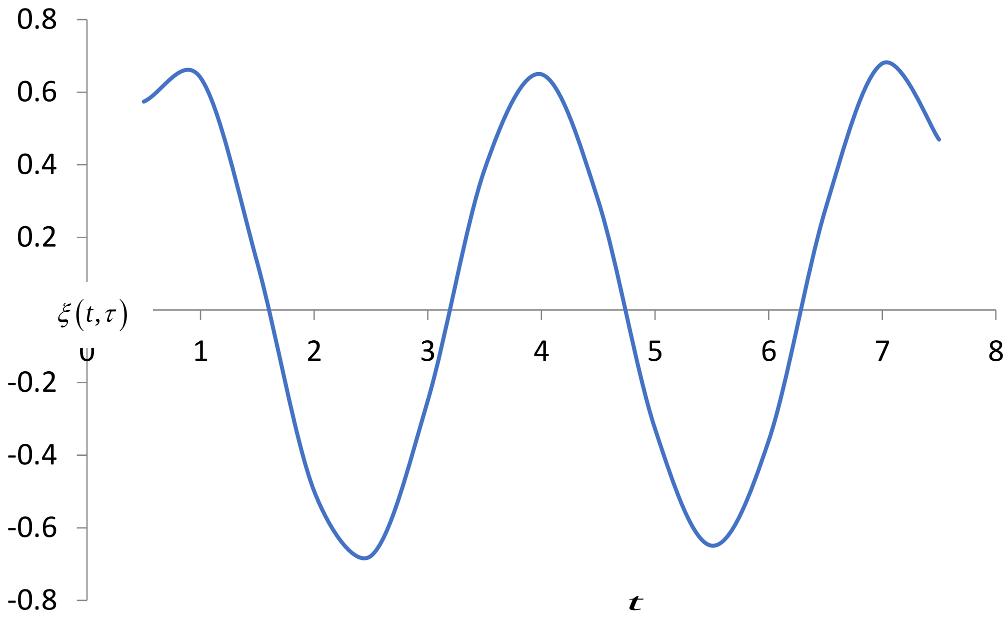

Fig 3. Variation of the displacement, $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0.03$

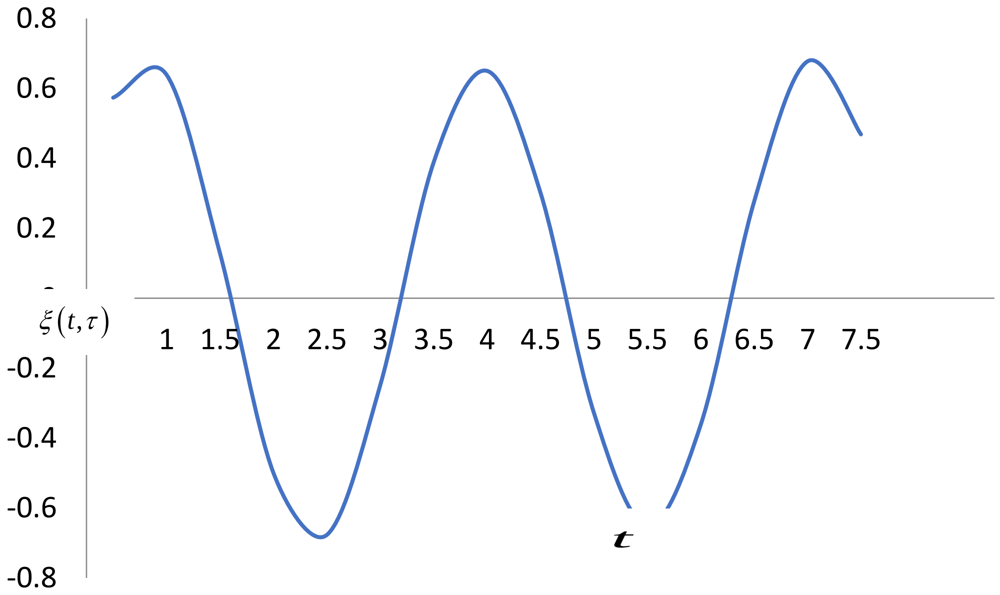

Fig 4. Variation of the displacement, $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0.04$

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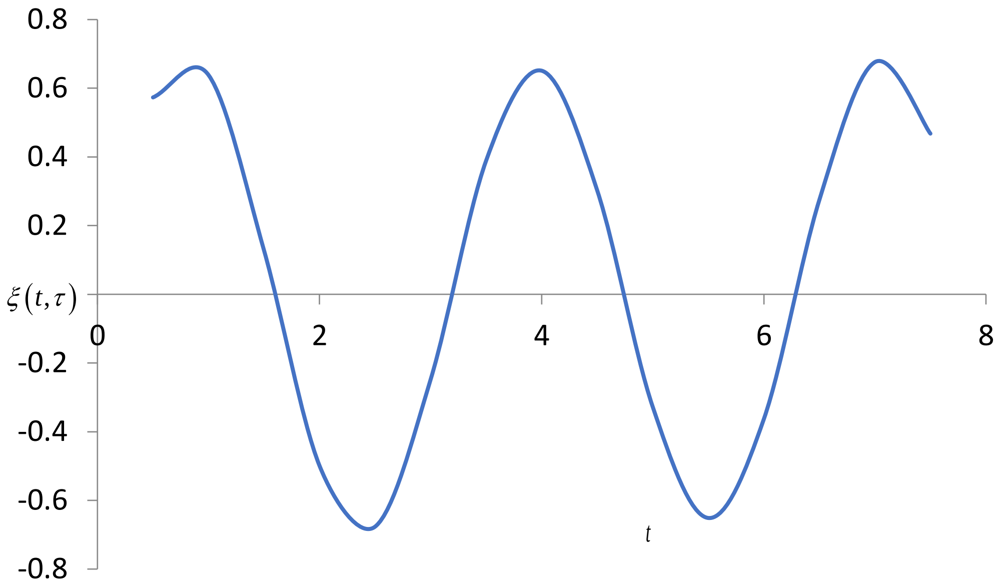

Fig 5. Variation of the displacement, $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0.05$

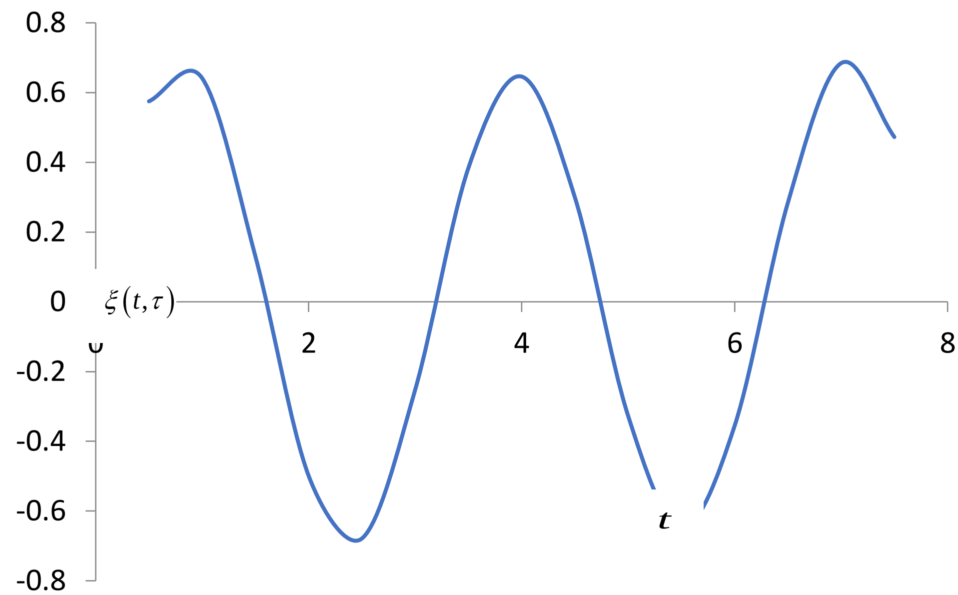

Fig 6. Variation of the displacement, $\xi(t,\tau)$ with time t at fixed value of damping $\eta = 0$

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CONCLUSION

This theoretical study of a forced Van der Pol oscillator equation can be a generalization of forced oscillatory systems. Multiple two-timing regular parameter perturbation method and asymptotic expansion were applied to analyse the forced Van der Pol oscillator equation to obtain a uniformly valid asymptotic approximate solution. The asymptotic expansion was used to obtain good estimate for the amplitude, $\xi(t,\tau)$ as t tends to infinity. The oscillator equation is damped though the damping is sufficiently small. The result obtained is a uniformly valid asymptotic approximate solution in t. We observed that the amplitude of oscillation of the forced Van der Pol oscillator decreases in the absence of damping in a particular time. From Table 1, it is seen that damping alters the amplitude of oscillation of the Van der Pol oscillator. Increase in damping reduces the amplitude of oscillation of any oscillatory system.

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