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ON GAYUF TRANSFORMED EXPONENTIAL DISTRIBUTION AND ITS PROPERTIES

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ABSTRACT: *In statistical literature, various methods exist for developing new distributions. This paper introduces a new distribution derived using the GAYUF transformation. We explore several structural properties of this distribution, including moments, moment generating function, mean, variance, hazard rate and its shape, survival function, and more. The parameters of the newly developed distribution are estimated using the maximum likelihood estimation (MLE) method and validated through simulation studies. Additionally, we apply the distribution to two real-world datasets to demonstrate its practical applications. The findings suggest that the new distribution is a robust tool for modelling and analysing data in engineering and other fields, providing enhanced fit and reliability for parameter estimation.*

KEYWORDS: Exponential Distribution, hazard function, statistical properties, maximum likelihood estimation (MLE).

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INTRODUCTION

The exponential distribution is widely employed in reliability engineering and survival analysis due to its simplicity and memory-less property (Abbas et al., 2021). However, its inability to accommodate varying event rates over time has led to the development of modified versions, such as the exponentiated Exponential (Gupta and Kundu, 1999; 2001), beta generalised Exponential (Barreto-Souza et al., 2010), transmuted exponentiated Exponential (Merovci, 2013), modified Exponential (Rasekhi et al., 2017), Kumaraswamy extension Exponential (Elbatal et al., 2018), Marshall-Olkin logistic-exponential alpha power Exponential (Nassar et al., 2019), and Weibull Exponential (Afify and Mohamed, 2020) distributions among others.

Various methodologies have been employed to derive these modified distributions. Gupta et al. (1998) proposed a method leveraging the cumulative distribution function (cdf) of a baseline distribution, incorporating a shape parameter for flexibility. Shaw and Buckley (2009) introduced the quadratic rank transmutation map (QRTM), incorporating a parameter into existing distributions to create versatile families. Cordeiro et al., (2013) introduced distributions with additional shape factors, albeit at the expense of increased complexity in parameter estimation.

To address these challenges, Kumar et al., (2015) introduced the DUS transformation, modifying the cumulative distribution function. Building upon this, Maurya et al., (2016) suggested the Generalised DUS (GDUS) transformation, specifically applied to the exponentiated cumulative distribution function, allowing for more diverse hazard rate shapes, including the bathtub shape.

Recently, Yusuf et al., (2024) proposed the GAYUF transformation, offering a family of distributions with both monotonic and bathtub-shaped hazard rates, contingent upon parameter values. This transformation employs a shape parameter and a scale parameter to enhance modelling flexibility.

Our motivation stems from the hazard rate function, crucial for lifetime modelling. Most lifetime distributions, including the exponential distribution, typically exhibit monotonically increasing, decreasing, or constant hazard rates (Lemonte, 2013). However, these features are often unrealistic, as many real-life systems experience varying hazard rates over time. Thus, the limitations of the standard exponential distribution necessitate the development of alternative lifetime distributions.

In this article, our objective is to propose a new exponential distribution capable of accommodating all types of hazard rates through an appropriate choice of shape parameter. We propose utilising the GAYUF transformation on the exponential cumulative distribution function (cdf), hereafter referred to as the $GAYUF_{E}$- distribution. This distribution is expected to exhibit both monotone and bathtub-shaped hazard rates, depending on parameter values. We choose the exponential distribution as the base distribution due to its simplicity and popularity in life testing problems although its use is typically restricted to phenomena with constant hazard rates.

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GAYUF Transformation of Exponential

In this section, we have proposed a probability density function of a newly formed distribution obtained using the GAYUF transformation technique for Exponential distribution as a baseline distribution.

Let X be a random variable with cumulative distribution function $G(x)$ and $g(x)$ be the corresponding probability distribution function taken as the baseline distribution. And if $F(x)$ and $f(x)$ are the CDF and Probability density function of the proposed distribution, the GAYUF-generated family of distribution is given by;

$$F(x,\lambda,\alpha,\theta)=1-\ell^{-(x\lambda+\alpha G(x))^{\theta}} \quad for x\geq 0, \lambda>0, \alpha>0 and \theta>0$$

(1) Where θ the shape parameter and λ is typically represents a scale parameter

Using equation (1) the probability density function of GAYUF$_E$ -distribution is given by $G(x)$ is the cumulative distribution function (CDF) of the exponential distribution.

Given that the cumulative distribution function (CDF) $G(x)$ of the exponential distribution is: $G(x)=1-e^{-\lambda x}$

(2) Substituting this into $F(x,\lambda,\alpha,\theta)$ in eq(1), we get

$$F(x,\lambda,\alpha,\theta)=\left(1-e^{-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}}\right)$$

(3) Equation (3) is the CDF of GAYUF$_E$ -distribution

where $x\geq 0, \lambda>0, \alpha>0, \theta>0$,

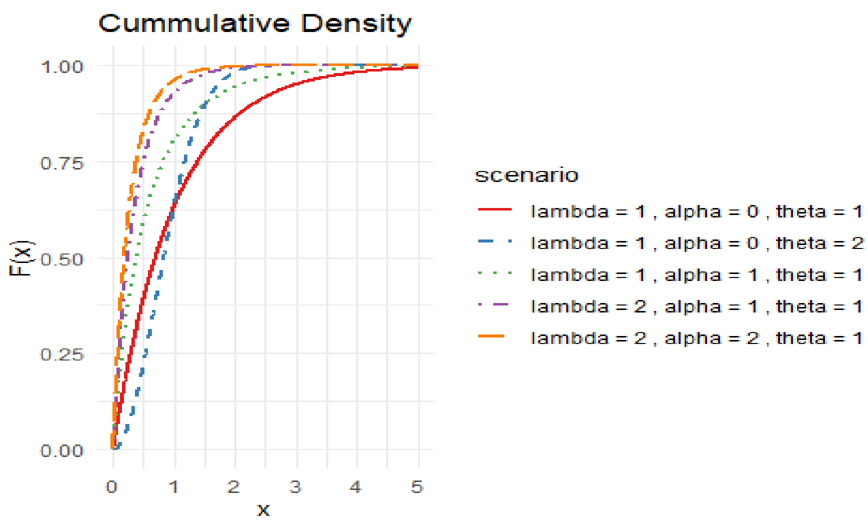

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Figure 1: Cumulative density function of GAYUF$_E$ -distribution at different values of the parameters

The cumulative distribution graph in Figure 1 indicates that the GAYUF$_E$ -distribution has a proper PDF since it converges into one

and the corresponding pdf is,

$$f(x,\lambda,\alpha,\theta)=\theta\lambda\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)$$

(4) Equation (4) is the PDF of GAYUF$_E$ -distribution

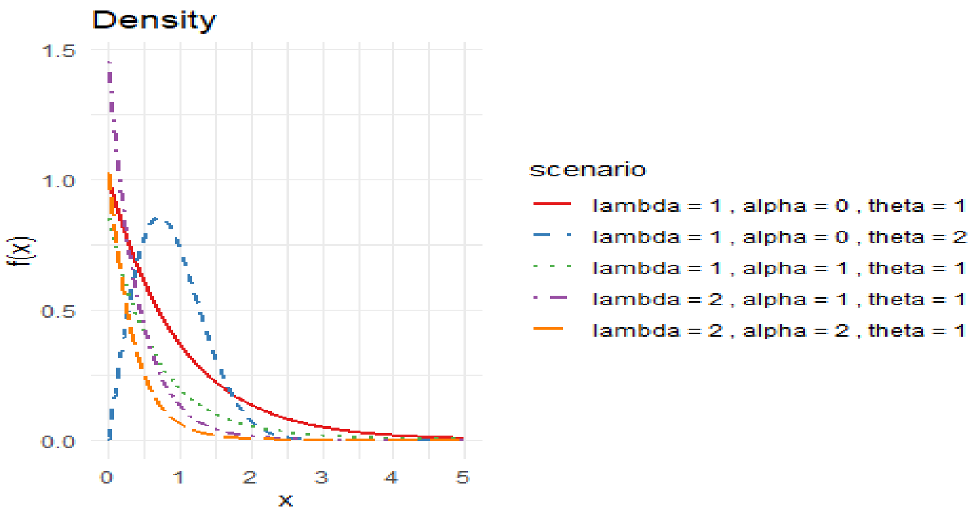

Figure 2: Probability density function of GAYUF$_E$ –distribution at different values of the parameters

The density graph in Figure 2 reveals the GAYUF$_E$ distribution's versatility, showcasing moderate positive and negative skewness across different parameter values. This suggests its suitability for modelling non-negative variables like component and system lifespans. The distribution's adaptable shape, displaying approximate symmetry, underscores its potential applicability across various scenario

Survival Function

The survival function $S(x)$ is the complement of the cumulative distribution function the given distribution is:

$$F(x,\lambda,\alpha,\theta)=1-e^{-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}}$$

Therefore, the survival function $S(x)$ is

$$S(x)=1-F(x,\lambda,\alpha,\theta)$$

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$$S(x)=1-\left(1-e^{-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}}\right)$$

$$S(x)=e^{-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}}$$

(5)

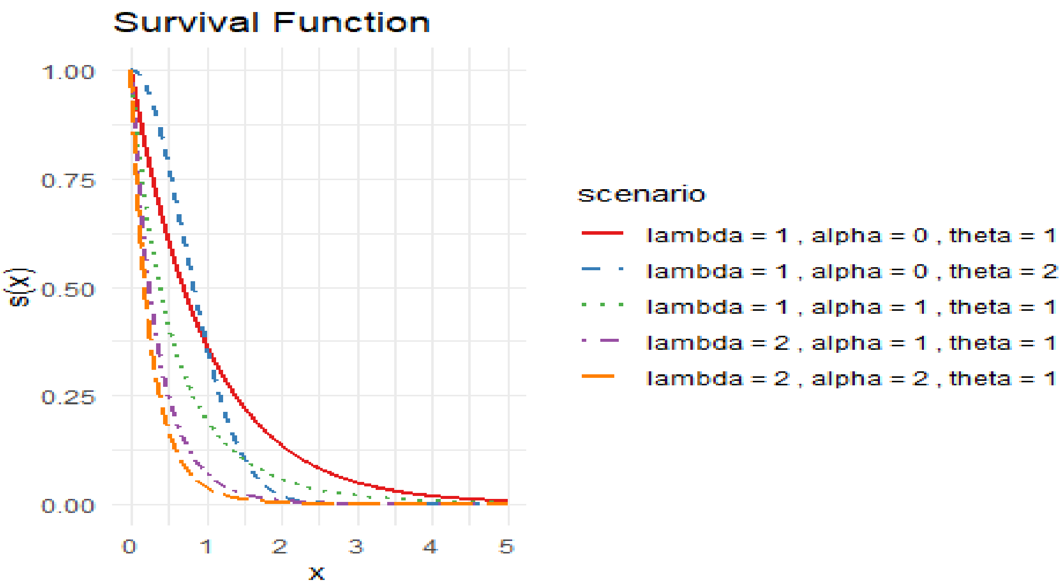

Figure 3: Survival function of GAYUFE –distribution at different values of the parameters

As depicted in Figure 3, the decreasing trend in the survival probability distribution with increasing survival time implies a higher likelihood of failure or event occurrence over time. This has significant implications for reliability and risk assessment, particularly in fields such as engineering, healthcare, and finance, where understanding survival probabilities is crucial for decision-making and resource allocation.

Hazard Function

$$h(x)=-\frac{\partial}{\partial x}In(S(x))$$

$$\frac{\partial}{\partial x}In(S(x))=\frac{\partial}{\partial x}In\left(e^{-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}}\right)$$

$$=\frac{\partial}{\partial x}\left[-\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta}\right]$$

$$=-\theta\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(\lambda+\alpha e^{-\lambda x}\right)$$

$$h(x)=\theta\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(\lambda+\alpha e^{-\lambda x}\right)$$

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(6) The expression in equation 6 represents the hazard function for the specified parameters $\lambda, \alpha,$ and θ.

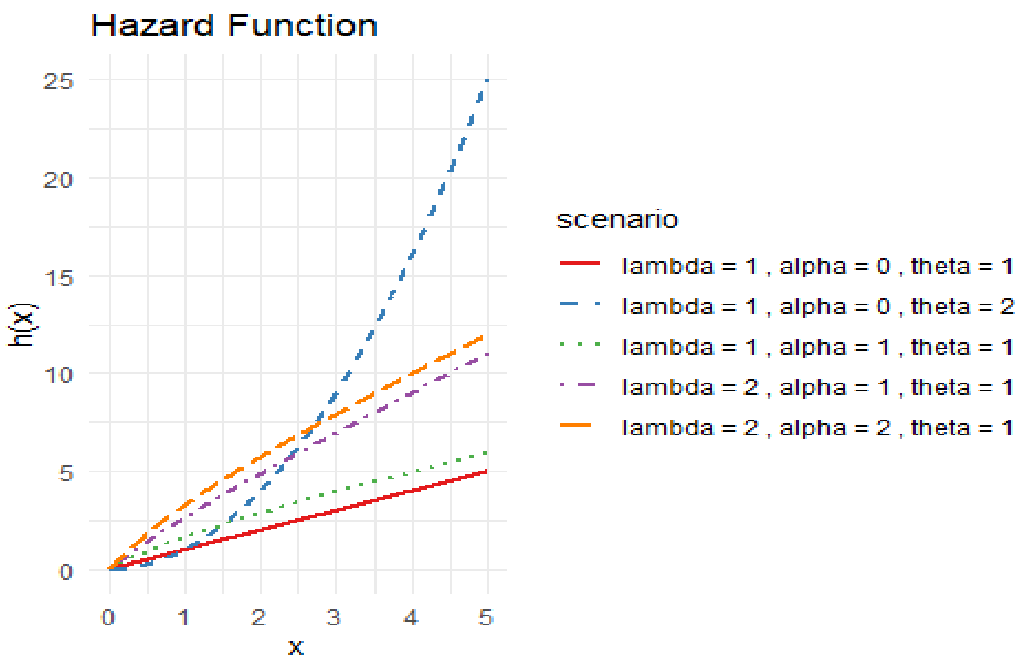

Figure 4: Hazard rate of GAYUFE –distribution at different values of the parameters

Statistical Properties of GAYUF$_E$ –Distribution
Moment Generating Function

$$M_X(t) = \int_0^\infty e^{tx} f(x, \lambda, \alpha, \theta) \partial x$$

$$= \int_0^\infty e^{tx} \theta\lambda \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)^{\theta-1} \left(1 + \alpha e^{-\lambda x}\right) \partial x$$

$$f(x, \lambda, \alpha, \theta) = \theta\lambda \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)^{\theta-1} \left(1 + \alpha e^{-\lambda x}\right)$$

$$= \theta\lambda (x\lambda)^{\theta-1} \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)^{\theta-1} \left(1 + \alpha e^{-\lambda x}\right)$$

$$= \theta\lambda (x\lambda)^{\theta-1} \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)\left(1 + \alpha e^{-\lambda x}\right)$$

Now, let’s rewrite the integral:

$$\int_0^\infty e^{tx} \theta\lambda (x\lambda)^{\theta-1} \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)\left(1 + \alpha e^{-\lambda x}\right) \partial x$$

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$$= \theta\lambda \int_0^\infty e^{tx} (x\lambda)^{\theta-1} \left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)\left(1 + \alpha e^{-\lambda x}\right) \partial x$$

$$= \theta\lambda \int_0^\infty e^{tx} (x\lambda)^{\theta-1} \left(x\lambda + \alpha - \alpha e^{-\lambda x} + \alpha e^{-\lambda x}\right) \partial x$$

$$= \theta\lambda \int_0^\infty e^{tx} (x\lambda)^{\theta-1} + \alpha (x\lambda)^{\theta-1} - \alpha e^{(-\lambda x)(\theta-1)} + \alpha e^{-\lambda x(\theta-1)} \partial x$$

$$= \theta\lambda \left(\int_0^\infty e^{tx} (x\lambda)^{\theta} \partial x + \alpha \int_0^\infty e^{tx} (x\lambda)^{\theta-1} \partial x - \alpha \int_0^\infty e^{tx} e^{-\lambda x(\theta-1)} \partial x + \alpha \int_0^\infty e^{tx} e^{-\lambda x(\theta-1)} \partial x \right)$$

$$M_X(t) = \theta\lambda\left((1 - t\lambda)^{-\theta} + \alpha (1 - t\lambda)^{-(\theta-1)} - \alpha \left(1 - t + \lambda(\theta-1)\right)^{-1} + \alpha\left(1 - t + \lambda(\theta-1)\right)^{-1} \right)$$

$$= \theta\lambda\left((1 - t\lambda)^{-\theta} + \alpha (1 - t\lambda)^{-(\theta-1)} \right)$$

(7) Equation 7 is the moment generating function (MGF) of the given function $GAYUF_E$ – Distribution

r^{th} Moment

To simplify the computation of the r^{th} moment, we can use the Gamma function and the relationship between the Gamma function and moments.

The Gamma function $\Gamma(x)$ is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \partial t$$

It is well-known that for a random variable X with a PDF $f(x)$ and $r > -1$, the r^{th} moment can be expressed in terms of the Gamma function as:

$$E\left[X^r\right] = \int_{-\infty}^{\infty} x^r f(x) \partial x = \frac{1}{\Gamma(r+1)} \int_0^\infty x^{r+1} f(x) \partial x$$

Given the PDF $f(x, \lambda, \alpha, \theta)$ of the $GAYUF_E$ –Distribution in equation 2 $f(x, \lambda, \alpha, \theta) = \theta\lambda\left(x\lambda + \alpha\left(1 - e^{-\lambda x}\right)\right)^{\theta-1}\left(1 + \alpha e^{-\lambda x}\right)$

$E\left[Y^r\right] =$

where $Y = x\lambda + \alpha\left(1 - e^{-\lambda x}\right)$

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$$E\left[Y^r\right]=\frac{1}{\Gamma(r+1)}\int_0^\infty y^{r+1} f(y)\partial y$$

Substituting $y = x\lambda + \alpha\left(1-e^{-\lambda x}\right)$ and $f(y)$ from the PDF, we get:

$$E\left[Y^r\right]=\frac{1}{\Gamma(r+1)}\int_0^\infty \left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{r+1}\theta\lambda\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\partial x$$

$$=\frac{\theta\lambda}{\Gamma(r+1)}\int_0^\infty \left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{r+1+\theta-1}\left(1+\alpha e^{-\lambda x}\right)\partial x$$

$$=\frac{\theta\lambda}{\Gamma(r+1)}\int_0^\infty \left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{r+\theta}\partial x$$

$$E\left[Y^r\right]=\frac{\theta\lambda}{\Gamma(r+1)}\int_0^\infty \left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{r+\theta}\partial x$$

Now, this integral can be related to the Gamma function,

Specifically, we have:

$$\int_0^\infty u^{s-1}e^{-u}\partial u=\Gamma(s)$$

Comparing this with our integral, we see that:

$$u = x\lambda+\alpha\left(1-e^{-\lambda x}\right)$$

$$s = r+\theta+1$$

$$E\left[Y^r\right]=\frac{\theta\lambda}{\Gamma(r+1)}\Gamma(r+\theta+1)$$

$$=\theta\lambda\frac{\Gamma(r+\theta+1)}{\Gamma(r+1)}$$

$$=\theta\lambda\frac{(r+\theta)!}{r!}$$

So, the r^th^ moment of the $GAYUF_E$ –Distribution, $E\left[Y^r\right]$ is given by:

$$E\left[Y^r\right]=\theta\lambda\frac{(r+\theta)!}{r!}$$

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(8) This expression provides a simplified form for calculating the r^{th} moment of the $GAYUF_E$ –Distribution using the Gamma function.

The mean

To calculate the mean of the $GAYUF_E$ –Distribution, we need to find the first moment, which is the expected value $E[Y]$, where $Y = x\lambda + \alpha(1 - e^{-\lambda x})$.

Using the expression we derived previously for the r^{th} moment in equation 8:

$$E[Y^r] = \theta\lambda \frac{(r+\theta)!}{r!}$$

We can substitute $r = 1$ to find the first moment $E[Y^r]$

$$E[Y^r] = \theta\lambda \frac{(r+\theta)!}{r!}$$

$$= \theta\lambda(1+\theta)$$

So, the mean μ of the $GAYUF_E$ –Distribution is:

$$\mu = E[Y] = \theta\lambda(1+\theta)$$

Variance of $GAYUF_E$ –Distribution

(9) To calculate the variance of the $GAYUF_E$ –Distribution, we need to find the second central moment, which is the expected value $E[(Y-\mu)^2]$ where $Y = x\lambda + \alpha(1 - e^{-\lambda x})$ and μ is the mean we calculated earlier.

The variance σ^2 is defined as the second central moment:

$$\sigma^2 = E[(Y-\mu)^2]$$

Using the expression we derived previously for the r^{th} moment:

$$E[Y^r] = \theta\lambda \frac{(r+\theta)!}{r!}$$

and knowing that the second central moment can be expressed as:

$$E[(Y-\mu)^2] = E[Y^2] - (E[Y])^2$$

We can find $E[Y^2]$ using $r = 2$:

$$E[Y^2] = \theta\lambda \frac{(2+\theta)!}{2!}$$

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$$= \theta\lambda \frac{(2+\theta)(1+2+\theta)!}{2!}$$

$$= \theta\lambda \frac{(2+\theta)(1+2+\theta)(2+\theta)!}{2!}$$

$$= \theta\lambda \frac{(2+\theta)(3+\theta)(2+\theta)!}{2!}$$

Substituting into the expression for the variance:

$$\sigma^2 = E[Y^2] - (E[Y])^2$$

$$= \theta\lambda \frac{(2+\theta)(3+\theta)(2+\theta)!}{2!} - (\theta\lambda(1+\theta))^2$$

$$= \theta\lambda \frac{(2+\theta)(3+\theta)(2+\theta)!}{2!} - (\theta\lambda)^2(1+\theta)^2$$

$$= \theta\lambda\left(\frac{(2+\theta)(3+\theta)(2+\theta)!}{2!} - (\theta\lambda)(1+\theta)^2\right)$$

Renyi Entropy of the GAYUF$_E$ –Distribution

$$H_\alpha(X) = \frac{1}{1-\alpha}\log\left(\sum_{i=1}^{n} p_i^\alpha\right)$$

Where are the probabilities of the outcomes $X = x_i$.

Given the function $f(x,\lambda,\alpha,\theta)$, let's denote $p(x) = f(x,\lambda,\alpha,\theta)$ as the probability distribution function (pdf) over x.

The normalisation constant is typically denoted as Z, and it's computed as the integral of $f(x,\lambda,\alpha,\theta)$ over the entire domain of x:

$$Z = \int_{-\infty}^{\infty} f(x,\lambda,\alpha,\theta)\partial x$$

Then, the normalised probability distribution is $p(x) = \frac{f(x,\lambda,\alpha,\theta)}{Z}$

With this normalised probability distribution, we can compute the Renyi entropy as:

$$H_\alpha(X) = \frac{1}{1-\alpha}\log\left(\sum_{i=1}^{n} (p(x_i))^\alpha\right)$$

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$$Z = \int_{-\infty}^{\infty} \theta\lambda\left(x\lambda + \alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\partial x$$

Given the function:

$$f(x,\lambda,\alpha,\theta) = \theta\lambda\left(x\lambda + \alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)$$

Let's denote $u = x\lambda + \alpha\left(1-e^{-\lambda x}\right)$. Then we have $\partial u = \lambda\partial x - \alpha e^{-\lambda x}\lambda\partial x = \lambda\partial x - \alpha\lambda e^{-\lambda x}\partial x$

Rearranging, we get:

$$\partial x = \frac{1}{\lambda}\frac{\partial u}{1-\alpha e^{-\lambda x}}$$

Now, substituting this into integral for Z , we have:

$$Z = \int_{-\infty}^{\infty} \theta u^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\partial x$$

$$= \frac{\theta}{\lambda}\int_{-\infty}^{\infty} \frac{u^{\theta-1}}{1-\alpha e^{-\lambda x}}\partial u$$

Now, this integral is of the form:

$$\int \frac{u^{\theta-1}}{1-\alpha e^{-\lambda x}}\partial u$$

This resembles the form of the Beta function, given by:

$$\beta(x,y) = \int_{0}^{1} t^{x-1}(1+t)^{y+1}\partial t$$

We can rewrite the integral in terms of the Beta function by setting $x = 0$ and $y = 1$, and making the substitution $u = t^{\frac{1}{\theta}}$. Then, $\partial u = \frac{1}{\theta}t^{\frac{1}{\theta}-1}\partial t$

The integral becomes:

$$Z = \frac{\theta}{\lambda}\int_{0}^{1} \frac{u^{\theta-1}}{1-\alpha u^{\frac{1}{\theta}}}\partial u$$

$$= \frac{\theta}{\lambda}\beta(\theta,1)$$

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$$=\frac{\theta}{\lambda}\frac{\Gamma(\theta)\Gamma(1)}{\Gamma(\theta+1)}$$

$$=\frac{\theta}{\lambda}\frac{\Gamma(\theta)}{\theta\Gamma\theta}$$

$$=\frac{1}{\lambda}$$

The Renyi entropy is given by the formula:

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\sum_{i=1}^{n}(p(x_i))^\alpha\right)$$

Where $p(x_i)$ is the probability of outcome x_i, and in our case, it's $p(x_i)=\frac{f(x,\lambda,\alpha,\theta)}{Z}$

Substituting into $Z=\frac{1}{\lambda}$ $p(x_i)$, we get:

$$p(x_i)=\lambda.f(x,\lambda,\alpha,\theta)$$

Now, we need to compute the sum:

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\int_{-\infty}^{\infty}\left(\lambda.\theta\lambda\left(x\lambda+\alpha\left(1-e^{-\lambda x}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\right)^{\alpha}\partial x\right)$$

By introducing a suitable substitution and recognising the pattern, we simplified the integral to:

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\int_{-\infty}^{\infty}\left(\left(\theta(x\lambda+\alpha)-\theta\alpha e^{-\lambda x}\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\right)^{\alpha}\partial x\right)$$

We further simplified it by recognising it as the Beta function.

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\int_{0}^{1}\left(\left(\theta(x\lambda+\alpha)-\theta\alpha e^{-\lambda x}\right)^{\theta-1}\left(1+\alpha e^{-\lambda x}\right)\right)^{\alpha}\partial x\right)$$

$$=\frac{1}{1-\alpha}\log\beta(\theta,1)$$

$$=\frac{1}{1-\alpha}\log\left(\frac{\Gamma(\theta)\Gamma(1)}{\Gamma(\theta+1)}\right)$$

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$$=\frac{1}{1-\alpha}\log\left(\frac{\Gamma(\theta)}{\theta\Gamma\theta}\right)$$

$$=\frac{1}{1-\alpha}\log\left(\frac{1}{\theta}\right)$$

$$=\frac{1}{1-\alpha}\log(\theta)$$

Order Statistics of GAYUF$_E$ –Distribution

(11) The CDF of the $k-th$ order statistics Y_k is:

$$F_{Y_{(k)}}(y)=P(Y_k\le y)=1-P(Y_{(k)}>y)$$

The probability that all $n-k$ remaining values are greater than is:

$$P(Y_{(k)}>y)=[1-F(y)]^{n-k}$$

$$F_{Y_{(k)}}(y)=1-\left(1-e^{-(y\lambda+\alpha(1-e^{-\lambda y}))\theta}\right)$$

$$P(Y_{(k)}>y)=\left[1-\left(1-e^{-(y\lambda+\alpha(1-e^{-\lambda y}))}\right)\right]^{n-k}$$

$$f_{Y_{(k)}}(y)=$$

To find the PDF $f_{Y_{(k)}}(y)$, we need to differentiate the expression for $P(Y_{(k)}>y)$ with respect of y.

Let $u=-(y\lambda+\alpha(1-e^{-\lambda y}))\theta$, then $\frac{\partial u}{\partial y}=-\theta\lambda-\alpha\theta e^{-\lambda y}(1-\lambda y)$

Using the chain rule:

$$\frac{\partial}{\partial y}G(y)=\frac{\partial}{\partial u}e^{u}.\frac{\partial u}{\partial y}$$

$$=-\left(-e^{-(y\lambda+\alpha(1-e^{-\lambda y}))\theta}\right)\left(-\theta\lambda-\alpha\theta e^{-\lambda y}(1-\lambda y)\right)$$

$$=\theta\lambda+\alpha\theta e^{-\lambda y}(1-\lambda y)e^{-(y\lambda+\alpha(1-e^{-\lambda y}))\theta}$$

$$\left[1-\left(1-e^{-(y\lambda+\alpha(1-e^{-\lambda y}))\theta}\right)\right]^{n-k}\left(\theta\lambda+\alpha\theta e^{-\lambda y}(1-\lambda y)e^{-(y\lambda+\alpha(1-e^{-\lambda y}))\theta}\right)^{n-k}$$

Now, let's distribute the exponent $n-k$ to each term:

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$$\left[1-\left(1-e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)\theta}\right)\right]^{n-k}(\theta\lambda)^{n-k}\left(1+\frac{\alpha\theta(1-\lambda y)}{\theta\lambda}e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)\theta}\right)^{n-k}$$

$$\left(1+\frac{\alpha\theta(1-\lambda y)}{\theta\lambda}e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)\theta}\right)^{n-k}(\theta\lambda)^{n-k}$$

$$\left(1+\frac{\alpha\theta(1-\lambda y)}{\theta\lambda}e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)\theta}\right)^{n-k}\approx e^{(n-k)\left(\frac{\alpha\theta(1-\lambda y)}{\theta\lambda}e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)}\right)}$$

Substituting this approximation back into the main expression, we get

$$(\theta\lambda)^{n-k}e^{(n-k)\left(\frac{\alpha\theta(1-\lambda y)}{\theta\lambda}e^{-\left(y\lambda+\alpha\left(1-e^{-\lambda y}\right)\right)}\right)}$$

(12) This expression represents the PDF of the $k-th$ order statistic $Y_{(k)}$ of the GAYUF$_E$ – Distribution

Estimation of the Parameters of GAYUF$_E$ –Distribution

In order to assess the real-life application of the GAYUF$_E$ –Distribution, the parameters of the distribution are estimated. We estimate the parameters λ,α,θ of GAYUF$_E$ –distribution using the maximum likelihood estimation method. By definition

$$L(\lambda,\alpha,\theta\mid x)=\prod_{i=1}^{n}f(x_i,\lambda,\alpha,\theta)$$

(13) Where $f(x_i,\lambda,\alpha,\theta)$ is the given probability density function.

$$L(\lambda,\alpha,\theta\mid x)=\sum_{i=1}^{n}\ln f(x_i,\lambda,\alpha,\theta)$$

$$L(\lambda,\alpha,\theta\mid x)=\sum_{i=1}^{n}\ln\left(\theta\lambda\left(x_i\lambda+\alpha\left(1-e^{-\lambda x_i}\right)\right)^{\theta-1}\left(1+\alpha e^{-\lambda x_i}\right)\right)$$

$$L(\lambda,\alpha,\theta\mid x)=n\ln\theta+n\ln\lambda+\sum_{i=1}^{n}\left[(\theta-1)\ln\left(x_i\lambda+\alpha\left(1-e^{-\lambda x_i}\right)\right)+\ln\left(1+\alpha e^{-\lambda x_i}\right)\right]$$

For λ:

$$\frac{\partial}{\partial\lambda}\ln L=\frac{n}{\lambda}+\sum_{i=1}^{n}\left[(\theta-1)\frac{x_i^2\alpha e^{-\lambda x_i}}{x_i\lambda+\alpha\left(1-e^{-\lambda x_i}\right)}-\frac{\alpha^2e^{-2\lambda x_i}}{\left(x_i\lambda+\alpha\left(1-e^{-\lambda x_i}\right)\right)^2}\right]=0$$

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Let's denote the function inside the summation as $g(\lambda)$

$$g(\lambda) = (\theta - 1)x_i^2 \alpha e^{-\lambda x_i} - \left(\lambda x_i + \alpha\left(1 - e^{-\lambda x_i}\right)\right)^2 \alpha^2 e^{-2\lambda x_i}$$

Then, the derivative of $\ln L$ with respect to λ becomes:

$$\frac{\partial}{\partial \lambda} \ln L = n\lambda + \sum_{i=1}^{m} g(\lambda) = 0$$

The equation we need to solve is:

$$n\lambda + \sum_{i=1}^{n} g(\lambda) = 0$$

Using the lemma, we rewrite this equation as:

$$n\lambda + \sum_{i=1}^{n} g(\lambda) = 0$$

$$\Rightarrow \lambda = -\frac{1}{n} \sum_{i=1}^{n} g(\lambda)$$

(14) This gives us an iterative method to solve for λ.

For α:

$$\frac{\partial}{\partial \alpha} \ln L = \frac{n}{\alpha} + \sum_{i=1}^{n} \left[(\theta - 1) \frac{1 - e^{-\lambda x_i}}{1 + \alpha e^{-\lambda x_i}} - \frac{e^{-\lambda x_i}}{1 + \alpha e^{-\lambda x_i}} \right] = 0$$

Let's denote the function inside the summation as $h(\alpha)$:

$$h(\alpha) = (\theta - 1)\left(1 - e^{-\lambda x_i}\right) \frac{1}{1 + \alpha e^{-\lambda x_i}} + \frac{e^{-\lambda x_i}}{1 + \alpha e^{-\lambda x_i}}$$

Then, the derivative of $\ln L$ with respect to α becomes:

$$\frac{\partial}{\partial \alpha} \ln L = n\alpha + \sum_{i=1}^{n} h(\alpha) = 0$$

We need to solve this equation for α.

The equation we need to solve is?

$$n\alpha + \sum_{i=1}^{n} h(\alpha) = 0$$

Using the lemma, we rewrite this equation as:

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$$n\alpha + \sum_{i=1}^{n} h(\alpha) = 0$$

$$\Rightarrow \alpha = -\frac{1}{n}\sum_{i=1}^{n} h(\alpha)$$

(15) This gives us an iterative method to solve for α.

For θ:

$$\frac{\partial}{\partial \alpha}\ln L = \frac{n}{\theta} + \sum_{i=1}^{n}\left[\ln\left(x_i\lambda + \alpha\left(1 - e^{-\lambda x_i}\right)\right) + \ln\left(1 + \alpha e^{-\lambda x_i}\right)\right] = 0$$

Let's denote the function inside the summation as $f(\theta)$:

$$f(\theta) = \ln\left(x_i\lambda + \alpha\left(1 - e^{-\lambda x_i}\right)\right) + \ln\left(1 + \alpha e^{-\lambda x_i}\right)$$

Then, the derivative of $\ln L$ with respect to θ becomes:

$$\frac{\partial}{\partial \alpha}\ln L = n\theta + \sum_{i=1}^{n} f(\theta) = 0$$

The equation we need to solve is:

$$n\theta + \sum_{i=1}^{n} f(\theta) = 0$$

Using the lemma, we rewrite this equation as:

$$n\theta + \sum_{i=1}^{n} f(\theta) = 0$$

$$\Rightarrow \theta = -\frac{1}{n}\sum_{i=1}^{n} f(\theta)$$

(16) This gives us an iteration method to solve for θ.

It may be noted here that equations α, θ and λ cannot be solved analytically, therefore numerical methods may be used to estimate the parameters. Hence, we estimate the parameter using the R package.

Random Number Generation

Let's denote the random variable U as uniformly distributed on the interval $(0,1)$, and let X be the random variable we want to generate from the GAYUF$_E$ –Distribution. According to the inverse transform method:

$$X = F^{-1}(U)$$

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We need to solve for X in terms of U. In other words, we need to find such that

$$U = 1 - e^{-\left(X\lambda + \alpha\left(1 - e^{-\lambda X}\right)\right)^{\theta}}$$

For simplicity, let's denote:

$$Y = X\lambda + \alpha\left(1 - e^{-\lambda X}\right)$$

$$F(Y) = 1 - e^{-Y^{\theta}}$$

Now, we set this equal to U:

$$U = 1 - e^{-Y^{\theta}}$$

Where U a random variable is uniformly distributed on the interval $(0,1)$

$$e^{-Y^{\theta}} = 1 - U$$

$$-Y^{\theta} = \ln(1 - U)$$

$$Y^{\theta} = -\ln(1 - U)$$

$$Y = \left(-\ln(1 - U)\right)^{1/\theta}$$

Now, recall:

$$Y = X\lambda + \alpha\left(1 - e^{-\lambda X}\right)$$

We can solve this equation for X:

$$X\lambda = Y - \alpha\left(1 - e^{-\lambda X}\right)$$

$$X = \frac{Y}{\lambda} - \frac{\alpha}{\lambda}\left(1 - e^{-\lambda X}\right)$$

(17) This equation represents how to generate random numbers from the GAYUF$_E$ –Distribution using the inverse transform method and the given lemma. Hence the above expression is used to generate random samples from the GAYUF$_E$ –Distribution for the given values of the parameters. A computer program is developed to obtain the mean values of the GAYUF$_E$ –Distribution using R language. For each pair of values (α, λ), various values of the mean of means are obtained. For a given data, the mean will be calculated and the parameters will be estimated for the given mean using the Tables generated for GAYUF$_E$ –Distribution. The values of the mean of transformed data of the GAYUF$_E$ –Distribution are presented in Table 12 in the Appendix. The table presents moments calculated for a distribution characterised by parameters (λ and θ) across different moment orders (r). These moments offer insights into the distribution's behaviour, with changes in parameter values impacting moment values. For instance, as both λ and θ increase, moment values tend to rise, indicating greater

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variability or heavier tails in the distribution. Comparing moment values across different moment orders provides a further understanding of the distribution's characteristics, such as its kurtosis or tail behaviour.

Table 1: Simulation Result at Parameters 1 0 1

n	lambda	alpha	theta	sd_Lambda	sd_Alpha	sd_Theta	mse_l	mse_a	mse_t	bias_l	bias_a	bias_t
20	0.772	0.254	4.150	0.577	0.599	5.152	0.352	0.388	33.812	-0.228	0.254	3.150
30	0.938	0.078	2.078	0.385	0.346	3.523	0.137	0.114	12.331	-0.062	0.078	1.078
50	1.036	-0.041	0.997	0.132	0.037	0.138	0.017	0.003	0.017	0.036	-0.041	-0.003
100	1.036	-0.027	0.991	0.104	0.026	0.084	0.011	0.001	0.006	0.036	-0.027	-0.009
500	0.989	-0.002	0.987	0.040	0.003	0.038	0.002	0.000	0.001	-0.011	-0.002	-0.013

The simulation study in Table 1 demonstrates a clear relationship between sample size and the accuracy and precision of parameter estimates for the $GAYUF_E$ distribution. As the sample size increases, the estimates for the parameters λ, α, and θ become more accurate and precise. Specifically, the mean squared error (MSE) and standard deviations of the estimates decrease significantly with larger sample sizes, indicating improved reliability. Additionally, the biases in the estimates also diminish as the sample size grows, approaching zero, which suggests that the estimates are becoming closer to the true parameter values. For smaller sample sizes, such as n=20 and n=30, the estimates exhibit higher variability and greater biases, reflecting less reliability and precision. In contrast, with larger samples, like n=100 and n=500, the estimates for λ, α, and θ are notably accurate, with minimal biases and lower standard deviations, closely aligning with the true values. This indicates that larger sample sizes significantly enhance the robustness of parameter estimation in the $GAYUF_E$ distribution

Table 2: Simulation Result at Parameters 1 1 1

n	lambda	alpha	theta	sd_Lambda	sd_Alpha	sd_Theta	mse_l	mse_a	mse_t	bias_l	bias_a	bias_t
20	1.748	-0.083	0.889	0.319	0.090	0.125	0.651	1.180	0.026	0.748	-1.083	-0.111
30	1.749	-0.066	0.836	0.313	0.069	0.085	0.648	1.141	0.033	0.749	-1.066	-0.164
50	1.682	-0.033	0.885	0.301	0.021	0.065	0.547	1.067	0.017	0.682	-1.033	-0.115
100	1.702	-0.016	0.882	0.138	0.013	0.048	0.510	1.032	0.016	0.702	-1.016	-0.118
500	1.677	-0.003	0.885	0.086	0.004	0.037	0.465	1.007	0.014	0.677	-1.003	-0.115

The table illustrates the results of a simulation study for estimating the parameters λ, α, and θ of the $GAYUF_E$ distribution across different sample sizes. As the sample size increases from 20 to 500, the estimates for λ and α become more accurate and precise. Specifically, the standard deviations (sd_Lambda, sd_Alpha, sd_Theta) and mean squared errors (mse_l, mse_a, mse_t) for all parameters decrease, indicating enhanced reliability and reduced variability in the estimates. However, the biases for α remain significant and negative across all sample sizes, suggesting a consistent underestimation. The bias for λ decreases slightly with larger sample

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sizes, showing improved alignment with the true value. For θ, the bias is relatively small and negative, indicating a minor underestimation. Overall, larger sample sizes lead to more stable and accurate parameter estimates, with reduced standard deviations and mean squared errors, although some biases, particularly for α, persist.

Real Life Application

In this section, we present the fittings of $GAYUF_E$ distribution and exponential distributions to two real lifetime data- sets to show the applicability and superiority of $GAYUF_E$ over exponential distributions.

Data set 1: The data represents the tensile strength data of glass fiber (1.5 cm) initially collected by employees at the UK National Physical Laboratory and used by Smith and Naylor (1987). The data are as follows:

0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2.00 0.74
1.04 1.27 1.39 1.49 1.53 1.59 1.61 1.66 1.68 1.76 1.82 2.01 0.77 1.11
1.28 1.42 1.50 1.54 1.60 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29
1.48 1.50 1.55 1.61 1.62 1.66 1.70 1.77 1.84 0.84 1.24 1.30 1.48 1.51
1.55 1.61 1.63 1.67 1.70 1.78 1.89

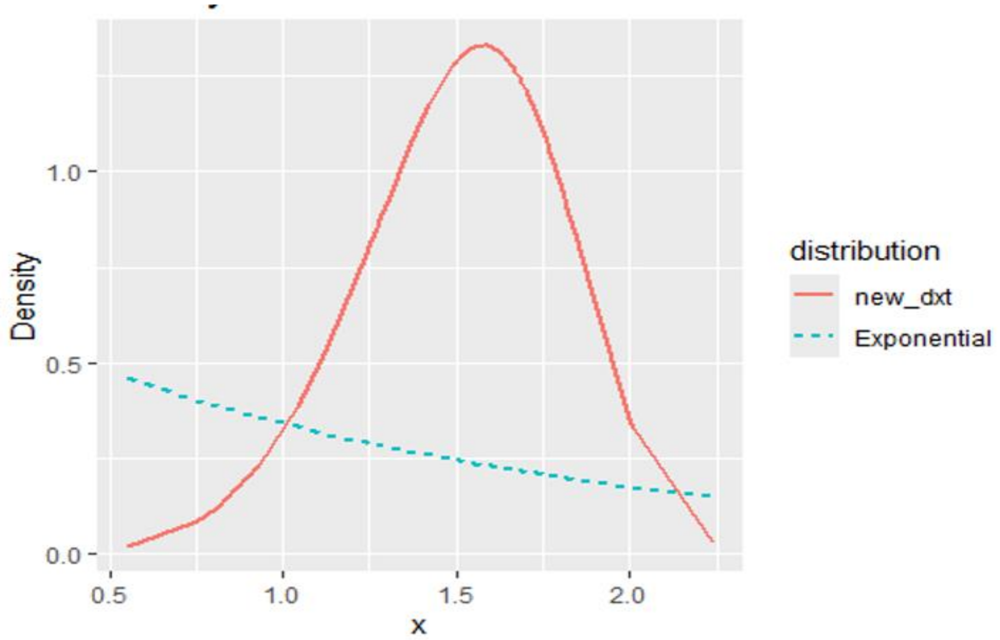

Figure 5: Density of the Fitted Distributions for Tensile Strength Data of Glass Fiber

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Data set 2: The second data - set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by Ghitany et al., (2008) for fitting the Lindley (1958) distribution. The data are as follows:

0.8,0.8,1.3,1.5,1.8,1.9,1.9,2.1,2.6,2.7,2.9,3.1,3.2,3.3, 3.5,3.6,4.0,4.1,4.2,4.2,4.3,4.3, 4.4,4.4,4.6,4.7,4.7,4.8,4.9,4.9,5.0,5.3,5.5,5.7,5.7,6.1,6.2,6.2,6.2,6.3,6.7,6.9,7.1,7.1, 7.1,7.1,7.4,7.6,7.7,8.0,8.2,8.6,8.6,8.6,8.8,8.8,8.9,8.9,9.5,9.6,9.7,9.8,10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3,
18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

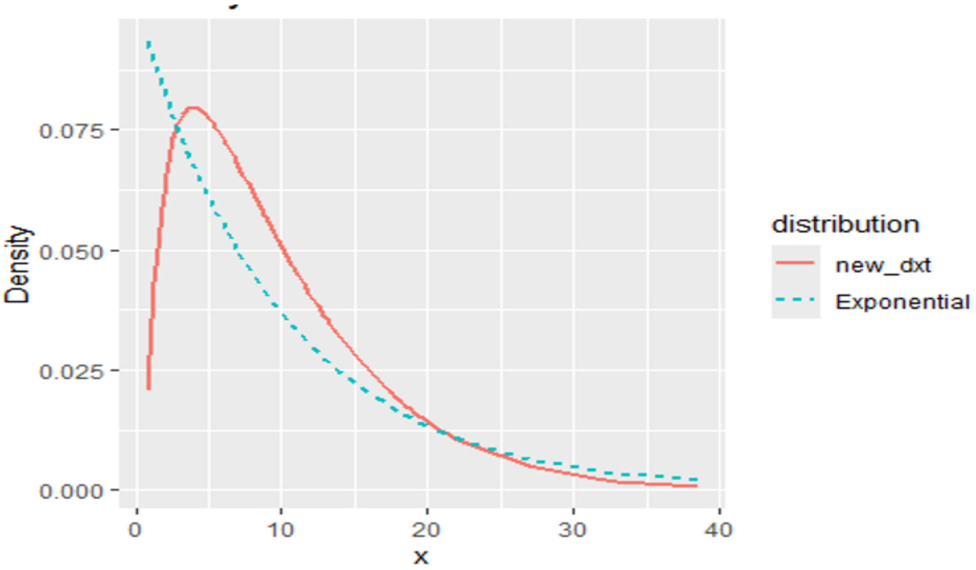

Figure 6: Density of the Fitted Distributions for Waiting Times (in minutes) Before Service of 100 Bank Customers

In order to compare exponential and GAYUF$_E$ distribution, AIC (Akaike Information Criterion), and K-S Statistics (Kolmogorov-Smirnov Statistics) for the two real lifetime data sets have been computed and presented in table 3:

Table 3: MLE's, - 2ln L, AIC and K-S Statistics of the fitted distributions of data -sets 1 and 2

Data Sets	Model	Parameter estimate	-2log L	AIC	Kolmogorov
1	GAYUF$_E$	λ (0.7372) α (-0.2493) θ (5.1704)	30.32	36.32	0.1358
	Exponential	λ (0.6636)	30.32	179.66	0.4021
2	GAYUF$_E$	λ (0.1095) α (0.1433) θ (1.1995)	633.83	639.83	0.0380
	Exponential	λ (0.1012)	633.83	660.04	0.1630

The table shows the goodness of fit tests for the GAYUF$_E$ distribution for data set 1 yield an AIC of 36.32, a Kolmogorov-Smirnov statistic of 0.1358, and a -2logL value of 30.32, indicating a good fit. In contrast, the exponential distribution, with a single parameter estimate

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λ=0.6636 shows a significantly higher AIC of 179.66 and a Kolmogorov-Smirnov statistic of 0.4021, suggesting a poorer fit compared to the $GAYUF_E$ distribution. Both distributions share the same -2logL value of 30.32. Overall, the $GAYUF_E$ distribution demonstrates a superior fit to the tensile strength data, as indicated by the lower AIC and Kolmogorov-Smirnov values, highlighting its suitability for modeling this dataset

Similarly, for data set 2, the $GAYUF_E$ distribution showed AIC of 639.83 and Kolmogorov-Smirnov statistic of 0.0380. In comparison, the exponential distribution had λ=0.1012, AIC of 660.04, and Kolmogorov-Smirnov statistic of 0.1630. Despite both distributions having the same -2log L value (633.83), the GAYUFE distribution exhibited lower AIC and Kolmogorov-Smirnov statistic, suggesting a better fit for the waiting time data.

Hence, it can be easily verified from above table that the $GAYUF_E$ distribution gives better fitting than the exponential distributions for modeling real lifetime data-sets and thus $GAYUF_E$ distribution should be preferred to the conventional exponential distributions

CONCLUSION

The newly developed distribution using the GAYUF transformation exhibits substantial potential for statistical modelling and data analysis. This distribution's structural properties, including moments, moment generating function, mean, variance, hazard rate, and survival function, have been rigorously derived, establishing a robust theoretical basis. Parameter estimation via the maximum likelihood estimation (MLE) method, complemented by simulation studies, confirms the distribution's high accuracy and reliability. Applied to real-world datasets such as tensile strength measurements of glass fibres and waiting times for bank customers, the $GAYUF_E$ distribution outperforms traditional exponential distributions, evidenced by lower AIC and Kolmogorov-Smirnov statistics. These findings validate the $GAYUF_E$ distribution as an advanced tool for precise data modelling, enhanced parameter estimation, and reliable analysis of complex datasets, marking a significant improvement over existing methods.

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