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UNIFIED FORMULA AND SYMMETRY OF PERFECT MAGIC SQUARE

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ABSTRACT: *Magic squares have long been used in divination and art, due to their magic and wonder. Among them, perfect magic squares are considered valuable as magic squares with special properties, and mathematicians have been interested in them and studied them. However, the achievements that are currently known are how to make a certain perfect magic square, and nothing is known about the number of all perfect magic squares, the unified formula, or the structure. This paper focused on symmetry and clarified the unified formula and structure.*

KEYWORDS: Magic square, Perfect magic square, Latin square, Point symmetry, Group theory.

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INTRODUCTION

Since ancient times, magic squares have been popular not only in Mathematics but also in fortune-telling and art. Of course, mathematicians have also done a lot of research. Among them, Euler's Latin square and the problem of 36 officers are too famous. Euler conjectured that it is impossible to find two orthogonal Latin squares of order n when n is half even. However, in practice only the sixth-order case proved impossible. Regarding this Latin square, unlike the magic square and perfect magic square, many studies and papers have been published. However, as a result, it is only showing the lower bound. The equation (1) is shown below (ref. [1]). Here, *l(n)* is the number of Latin squares of order *n.*

$$l(n) \geq \frac{(n!)^{2n-2}}{n^{n^2}} \qquad (1)$$

Also, as a real-life application, it was applied to agricultural planning to treat farmland with various conditions more fairly (ref. [2]). By the way, when it comes to the magic square of the sixth order, it is said that it will take about 30 million years to count and calculate the total number. In addition, although this paper deals with perfect magic squares, the total number is only known up to the fifth order. And it is proved that quadratic, cubic, and half-even numbers do not exist. Therefore, the total number of seven or more orders is not required. A perfect magic square is also called a magic square on a torus. Glue the upper and lower sides of a perfect magic square together to form a cylinder, stretch the cylinder further, and connect the two ends to form a doughnut. The surface of the torus is a perfect magic square in which all rows, columns and diagonals cannot be distinguished. Therefore, the pan diagonals of a perfect magic square can also be considered as rows and columns.

Description of a Perfect Magic Square

First of all, a perfect magic square means that in addition to the properties of a magic square, the sum of the numbers on the pandiagonal line is also the same. As already mentioned, it has been proved that a perfect magic square of half-even order (n=4m+2, where m is a natural number) does not exist. This proof is not only found in magic squares, but also in textbooks related to group theory, so I won't go out of my way to include it here.

And the other numbers are (1) n is an odd number that is a multiple of 3, (2) n is an odd number that is not a multiple of 3, (3) all even numbers (4m) and m is an odd number, (4) all even numbers (4m) and m is an even number. In the four cases, a method for making a perfect magic square is considered. As for (1), it is a method using Keima jump in Japanese shogi or Knight jump in chess. Since it can also jump sideways, it is called Knight jump. This creation method is described in a Japanese book called “Magic square world (ref [3])”.

In addition, the other (2) to (4) are created using perfect magic square of n-1 or less or auxiliary square* with such properties. However, it is known that there are overwhelmingly more irregular perfect magic squares that cannot be represented by these than regular perfect magic squares. Almost nothing is known about the latter irregular perfect magic square today. In this paper, I report that I found a certain rule about this irregular perfect magic square.

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**Auxiliary Square*

If it is an *n*-order square, it is expressed in *n*-adic numbers, and the second digit (upper digit) and the first digit (lower digit) are considered separately.

Relation between Perfect Magic Square and Symmetry

Regularity

First, I would like to clarify the definition of regularity in this paper. The regularity shown in other papers and related books mainly refers to those that can jump at Knight.

a) The arrangement of the same number represented by the auxiliary square is symmetrical about the number with the center. Please note that this is not limited to Knight jump.

Example: Knight jump case when not a multiple of 3 (see Fig.1). Arrows indicate Knight jumps about number 0.

34	3	22	41	10
21	40	14	33	2
13	32	1	20	44
0	24	43	12	31
42	11	30	4	23

4	3	2	1	0
1	0	4	3	2
3	2	1	0	4
0	4	3	2	1
2	1	0	4	3

3	0	2	4	1
2	4	1	3	0
1	3	0	2	4
0	2	4	1	3
4	1	3	0	2

first digit second digit

Figure 1: Fifth order perfect magic square

b) Each group distinguished by a modulus of *n* has point symmetry.

Example: A case where even and odd numbers are point symmetric (see Fig.2). Even numbers are gray, odd numbers are white.

40	35	16	63	43	36	15	60
27	52	71	4	24	51	72	7
73	6	25	50	70	5	26	53
14	61	42	37	17	62	41	34
54	21	2	77	57	22	1	74
33	46	65	10	30	45	66	13
67	12	31	44	64	11	32	47
0	75	56	23	3	76	55	20

Figure 2: Perfect magic square of order 8

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Example: Multiples of 3, 3*m* and 3*m*+2 groups (*m* is a natural number) are point symmetric, and 3*m*+1 itself is point symmetric. (see Fig.3). Here, for the sake of clarity, it is written in decimal not *n*-ary. Dark gray is 3*m*+2, Light gray is 3*m*+1, White is 3*m*.

46	38	30	79	71	63	22	14	6
76	68	60	19	11	3	52	44	36
25	17	9	49	41	33	73	65	57
39	28	47	72	61	15	15	4	23
69	58	77	12	1	45	45	34	53
18	7	26	42	31	66	66	55	74
29	48	37	62	81	5	5	24	13
59	78	67	2	21	35	35	54	43
8	27	16	32	51	56	56	75	64

Figure 3: Perfect magic square of order 9

By the way, if regularity is defined as such, the above-mentioned introduction (1) to (4) will be arranged with regularity. For example, the Knight jump in (1) always has symmetry as shown in figure 1.

First, consider case a). Here we show a regular perfect magic square of order 5 (fig. 1). In addition, it is represented here by a quinary number -1. It is well known that it is convenient to think of an *n*-th order square as *n*-adic number -1 from the Latin square concept. Henceforth, all numbers are represented by *n*-adic numbers -1. The left figure is a normal perfect magic square in the middle, and the right figure is an auxiliary square in the first and second digits. For example, if you pay attention to all the 1s in the middle figure, you can see that they are points symmetrical about the central 1. Please make sure that the other numbers are similarly symmetrical about a certain point. In addition, I presume that the perfect square that satisfies the above is the meaning of the orthogonal Latin square plus the oblique direction. A Latin square means that *n* rows and *n* columns are arranged with *n* numbers from 1 to *n*, and there is one number from 1 to *n* in each row and column. After, add the diagonal and pandiagonal. Then, make two of them, and make two pieces with different pairs of cell numbers in the same position. It is a so-called orthogonal Latin square with an oblique direction added.

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Moreover, when the relational expression is expressed, it is as follows. Let *X* be the subgroup of the same numbers in each digit (see equation (2) and (3). Also, *i* and *j* are natural numbers less than or equal to *n*. If the image modeled by the subgroup represented by *X* is *G*, it can be expressed as follows (see (4). The letters *r, π* mean point symmetry. It is expressed in *n*-adic notation.

$$
\begin{aligned}
&X_{i0} = \{00_{(n)}, 10_{(n)}, 20_{(n)}, \dots, (n-2)0_{(n)}, (n-1)0_{(n)}\} \\
&X_{i1} = \{01_{(n)}, 11_{(n)}, 21_{(n)}, \dots, (n-2)1_{(n)}, (n-1)1_{(n)}\} \\
&\dots \\
&X_{i(n-2)} \\
&= \{0(n-2)_{(n)}, 1(n-2)_{(n)}, 2(n-2)_{(n)}, \dots, (n-2)(n-2)_{(n)}, (n-1)(n-2)_{(n)}\} \\
&X_{i(n-1)} \\
&= \{0(n-1)_{(n)}, 1(n-1)_{(n)}, 2(n-1)_{(n)}, \dots, (n-2)(n-1)_{(n)}, (n-1)(n-1)_{(n)}\}
\end{aligned}
\quad (2)
$$

<Odd number not multiple of 3>

$$
\begin{aligned}
&X_{0j} = \{00_{(n)}, 01_{(n)}, 02_{(n)}, \dots, 0(n-2)_{(n)}, 0(n-1)_{(n)}\} \\
&X_{1j} = \{10_{(n)}, 11_{(n)}, 12_{(n)}, \dots, 1(n-2)_{(n)}, 1(n-1)_{(n)}\} \\
&\dots \\
&X_{(n-2)j} \\
&= \{(n-2)0_{(n)}, (n-2)1_{(n)}, (n-2)2_{(n)}, \dots, (n-2)(n-2)_{(n)}, (n-2)(n-1)_{(n)}\} \\
&X_{(n-1)j} \\
&= \{(n-1)0_{(n)}, (n-1)1_{(n)}, (n-1)2_{(n)}, \dots, (n-1)(n-2)_{(n)}, (n-1)(n-1)_{(n)}\}
\end{aligned}
\quad (3)
$$

$$
\begin{aligned}
&G(X_{i0}): r, \pi && G(X_{0j}): r, \pi \\
&G(X_{i1}): r, \pi && G(X_{1j}): r, \pi \\
&\dots && \dots \\
&G(X_{i(n-2)}): r, \pi && G(X_{(n-2)j}): r, \pi \\
&G(X_{i(n-1)}): r, \pi && G(X_{(n-1)j}): r, \pi
\end{aligned}
\quad (4)
$$

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Next, consider case b). In the case of multiples of 3 or even numbers, this Knight jump is not possible. (See Fig. 2, Fig. 3). Instead, it is represented by the product of *n*-1 or less, and can be composed with them. For example, in the case of even numbers, if we consider the relationship between even and odd numbers to have a definite difference of 1, then the even and odd groups have a symmetrical relationship. Also, if you think that the average of 3*m* and 3*m*+2 is 3*m*+1 when it is a multiple of 3, you can see that 3*m* and 3*m*+2 are symmetrical, and 3*m*+1 itself is symmetrical, which makes it even. The relational expression is as follows. Let *H* be the image modeled by the subgroup also denoted by *Y*. And it can be expressed as follows (see (5) ~ (8)). (5) and (6) are even cases, and (7) and (8) are non-prime odd cases. For the sake of clarity, the formulas here are expressed in decimal numbers instead of *n*-ary numbers. This will naturally become like this when you make a pair and make a definite difference or take an average value.

<Even number>

$$\begin{aligned} Y_0 &= \{2,4,6,\dots 2k-2, 2k\} \\ &= \{0(mod2)\} \\ Y_1 &= \{1,3,5,\dots 2k-3, 2k-1\} \\ &= \{1(mod2)\} \\ n &= 2k \\ Y_0, Y_1 &\in N \end{aligned} \tag{5}$$

$$G(X_0) \overset{r,\pi}{\leftrightarrow} G(X_1) \tag{6}$$

<Non-prime odd> ex. If the divisor of *n* is l

$$\begin{aligned} Y_0 &= \{0(mod\ l)\} \\ Y_1 &= \{1(mod\ l)\} \\ Y_2 &= \{2(mod\ l)\} \\ &\dots \\ Y_{(l-2)} &= \{(l-2)(mod\ l)\} \\ Y_{(l-1)} &= \{(l-1)(mod\ l)\} \\ Y &\in N \end{aligned} \tag{7}$$

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$$
\begin{aligned}
&G(Y_0) \overset{r,\pi}{\leftrightarrow} G(Y_{l-1}) \\
&G(Y_1) \overset{r,\pi}{\leftrightarrow} G(Y_{l-2}) \\
&G(Y_2) \overset{r,\pi}{\leftrightarrow} G(Y_{l-3}) \\
&\ldots \\
&G\left(Y_{\frac{l-1}{2}-2}\right) \overset{r,\pi}{\leftrightarrow} G\left(Y_{\frac{l-1}{2}+2}\right) \\
&G\left(Y_{\frac{l-1}{2}-1}\right) \overset{r,\pi}{\leftrightarrow} G\left(Y_{\frac{l-1}{2}+1}\right) \\
&G\left(Y_{\frac{l-1}{2}}\right) : r, \pi
\end{aligned}
\tag{8}
$$

In the case of a) as well, if we think in terms of the definite difference, we distinguish the group with the definite difference of 0 as an auxiliary square. Each has point symmetry. Including this definite difference of 0 would cover all the definite differences and would represent the necessary condition for a regular perfect magic square.Then, with this, if either a) or b) holds true, the perfect magic square is defined as having regularity.

Irregularities and Symmetric Permutation

Then, I would like to think about how to create an irregular perfect magic square that does not belong to a regular perfect magic square. I will give the answer first, but it can be done by exchanging two groups with point symmetry in a perfect magic square with regularity. In this paper, it is named partial symmetric permutation because there is no suitable one in current mathematical terminology. I will call it that from now on. As shown in figure 4, when distinguishing between the positive and negative (large and small) of the difference between the exchanged numbers, the so-called two groups become symmetrically transformed. And they are always point-symmetrical, symbolically expressed as *r, π*. In addition, since the perfect magic square as a whole has the same condition even if it is vertically and horizontally shifted, the above regularity and partial symmetric permutation are considered including it. An example of an irregular perfect magic square of degree 7 is shown in the figure 4.

0⇔1、4⇔5、22⇔23、34⇔35、44⇔45、52⇔53

As above, there are six pairs of transformations with a definite difference of $1_{(7)}$, and if we exchange these, we get a regular Knight jump (see Fig. 4,5). If we distinguish between the positive and negative (large and small) of the exchanged numbers, we can see that they are point symmetrical. Here, positive values are colored in light gray and negative values in dark gray.

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31	40	54	63	6	12	25
2	15	21	30	45	52	66
43	56	62	4	11	20	35
10	24	33	46	53	65	0
55	61	1	14	22	36	42
26	32	44	51	60	5	13
64	3	16	23	34	41	50

Before transformation

31	40	54	63	6	12	25
2	15	21	30	44	53	66
43	56	62	5	11	20	34
10	24	33	46	52	65	1
55	61	0	14	23	36	42
26	32	45	51	60	4	13
64	3	16	22	35	41	50

After transformation

Figure 4: Irregular perfect magic square of degree 7

1	0	4	3	6	2	5
2	5	1	0	4	3	6
3	6	2	5	1	0	4
0	4	3	6	2	5	1
5	1	0	4	3	6	2
6	2	5	1	0	4	3
4	3	6	2	5	1	0

First digit

3	4	5	6	0	1	2
0	1	2	3	4	5	6
4	5	6	0	1	2	3
1	2	3	4	5	6	0
5	6	0	1	2	3	4
2	3	4	5	6	0	1
6	0	1	2	3	4	5

Second digit

Figure 5: Auxiliary square after transformation of Fig 4

Now let me explain why this happens. First, at a certain point, the number of transforms in the vertical, horizontal, and diagonal directions is always an even number. This corresponds to even permutation in group theory. Next, if the number of transformations is seven or less, all points do not satisfy the even permutation in the vertical, horizontal, and diagonal directions, so it is clear that eight points are always necessary as shown in the figure 6. For example, if one point is replaced and the difference is positive, there are four points in the vertical, horizontal, and diagonal directions with respect to that point, and the difference is negative. So, correspondingly, at least three positive points are required. Therefore, six points cannot be replaced. And 8 points is the minimum.

It is a simple conversion diagram of a perfect magic square assuming that numbers are placed in the crossing parts and the circle parts (Fig.6). Again, it would be nice if you could understand that the shading of the black circles is the positive or negative difference. If one point is a dark black circle, the vertical, horizontal, and diagonal lines connected to it naturally become light black circles. Then you can easily see that it will be placed below. If each distance of the circle is the same, there is no problem.

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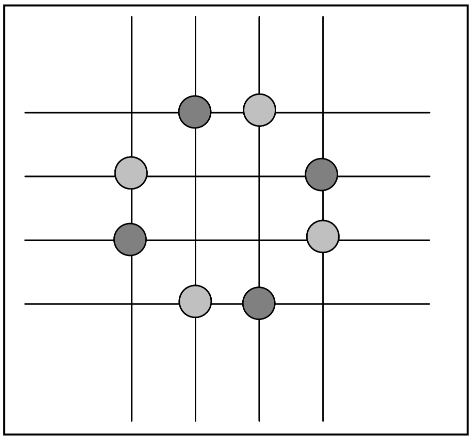

Figure 6: Octagonal transformation diagram of perfect magic square

If even one of the above symmetrical states is asymmetrical, it cannot be evenly replaced in the so-called vertical, horizontal, and diagonal directions, and new permutations are required. And it will require more than 8 transformations (4-tuple octagonal transformations).

This is illustrated by an example of a perfect magic square of order 7. Perfect magic square on the left side of Fig. 7 is a transformation diagram with a definite difference of $10_{(7)}$, which is a 6-tuple transformation. And the central figure is a transformation diagram with a definite difference of $3_{(7)}$, which is a 6-tuple transformation. It can be seen that each is point-symmetrically transformed around the black circle. The figure after transformation is shown on the right. You can see that the second and first digits are Knight jumps. In Fig. 7, for the sake of clarity, I chose the case of Knight jump as a regular perfect magic square, but of course there are other cases as well. Figure 5 is the same.

40	54	6	61	33	12	25
13	22	35	60	4	66	31
64	46	14	20	32	55	0
52	2	50	34	26	24	43
16	44	63	5	62	30	11
45	10	21	36	51	3	65
1	53	42	15	23	41	56

Before transformation
(Permutation 1)

40	54	6	61	33	12	25
13	22	35	60	4	66	31
64	46	14	20	32	55	0
52	2	50	34	26	24	43
16	44	63	5	62	30	11
45	10	21	36	51	3	65
1	53	42	15	23	41	56

Before transformation
(Permutation 2)

40	54	6	61	33	12	25
13	22	45	50	4	66	31
64	36	11	23	42	55	0
52	5	60	34	16	21	43
26	41	53	2	65	30	14
35	10	24	46	51	3	62
1	63	32	15	20	44	56

After transformation

Figure 7: Irregular perfect magic square of order 7 (permutation twice)

These are represented by the following formulas (see (9) ~ (12). The letter G indicates the image before partial symmetry permutation, and the letter H indicates the image after partial symmetry permutation. And, α indicates a partially symmetric permutation operation. The characters + and - indicate whether the definite difference has increased (+) or decreased (-) before and after partial symmetric permutation. (9) and (10) are the equations of a single partial symmetric permutation. Then, (11) and (12) become general expressions. Characteristics of α

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and G are also shown in (13), and a simple proof for them is given in (14). G' represents the transposed matrix.

$$\alpha_{+} \overset{r,\pi}{\leftrightarrow} \alpha_{-} \tag{9}$$

$$H = \alpha_{+,-} G \tag{10}$$

$$\begin{aligned} &to\ be\ \alpha_x \neq \alpha_y, \\ &\alpha_{1+} \overset{r,\pi}{\leftrightarrow} \alpha_{1-} \\ &\alpha_{2+} \overset{r,\pi}{\leftrightarrow} \alpha_{2-} \\ &\alpha_{3+} \overset{r,\pi}{\leftrightarrow} \alpha_{3-} \\ &\ldots \end{aligned} \tag{11}$$

$$H = \alpha_{1+,-} \cdot \alpha_{2+,-} \cdot \alpha_{3+,-} \cdot \ldots \cdot G \tag{12}$$

<Characteristics of α and G>

$$\alpha_x G \neq \alpha_y G \tag{13}$$

Because

$$\begin{aligned} &if\ \alpha_x G = \alpha_y G, \\ &\alpha_x G G' = \alpha_y G G' \\ &Therefore, \\ &\alpha_x = \alpha_y \end{aligned} \tag{14}$$

This contradicts the first setting. Therefore, the expression (13) holds.

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And this idea also applies to the regularity of the archetypes mentioned above. For example, in the case of pair permutation such as 2-1(b), if there is even one asymmetrical number, it requires at least 8 even numbers to be permuted vertically, horizontally, and diagonally. And finally, having symmetry in everything is the condition for becoming a perfect magic square.

Let $F(n)$ be the total number of perfect magic squares, $A(n)$ be the total number of original regular perfect magic squares, and $B(n)$ be an irregular perfect magic square after partial symmetric permutation. As I mentioned earlier, the perfect magic square has the same conditions even if it is vertically or horizontally shifted as a whole, so it will be the product of the squares of n. Then the following formula will be derived.

$$F(n) = n^2 \times A(n) + n^2 \times B(n) \tag{15}$$

CONCLUSION AND FUTURE OUTLOOK

In this paper, I clarified that an irregular perfect magic square can always be formed by applying partial symmetric permutation from a regular perfect magic square. It was discovered that all perfect magic squares consist of irregular perfect magic squares and regular perfect magic squares.

And this means that perfect magic squares are always associated with symmetry. And this hidden symmetry is probably the reason why people have been fascinated by perfect magic squares for a long time.

In the natural world, only things with high symmetry and regular movements have attracted attention and been elucidated. However, there are overwhelmingly more asymmetrical and irregular ones like this time. Perhaps, like this time, there may be many things that are asymmetric and irregular at first glance, but are constructed by symmetrical permutation. In particular, there may be applications in the world of chemistry and physics, where the rule of conservation of energy holds true in all directions.

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