PHYSICAL MOTIVATION ON DEFORMATION QUANTISATION

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ABSTRACT: *We study the Poisson structures associated with deformation quantisation and its non-degradable factor on the Casimir function. We also describe a filtered associative algebra in a quotient space as a Poisson algebra and the automorphism of the Poisson bracket is discussed.*

KEYWORDS: Poisson structures, Deformation quantisation, Casimir function, Quotient space, Automorphism.

INTRODUCTION

Deformation quantisation concerns the central physical concept of quantum theory; which comprises the algebra of observables and their dynamical evolution [7]. In other words, it is an alternative approach that prevents the issues of canonical quantisation (the replacement of classical observables with the corresponding operators) and its ordering ambiguity problem. The goal of deformation quantisation is to find a transformation that assigns quantum operators to classical observables in such a way that they respect the Poisson brackets and also, reproduce the appropriate quantum commutation relations. Whereas, the Poisson brackets are a foundational concept in classical mechanics, which is used to describe the dynamical evolution of classical systems. To this end, a general theory of formal deformation quantisation of associative algebras has been developed in the early works of [1] and [2], where quantisation is introduced as a deformation (star product) of the structure of the classical observables and they proffer solutions to differentiable deformation of the Lie algebra associated with the phase space, by defining the Poisson brackets on the deformation of the Lie algebra of smooth functions which generalizes the Moyal brackets. Also, in their works, ordinary multiplication to the Lie algebra is introduced, which gives rise to noncommutative associative algebras which are isomorphic to the operator algebras of quantum theory.

The works of Gerstenhaber [5] and [6], where the Hochschild 2- cocycles play the role of infinitesimal objects of such deformation and the definition of a skew bilinear form $\mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ as a Hochschild cocycle if and only if it defines a bi-derivation with respect to (Leibniz) rule and this bilinear form also satisfies the Jacobi identity and extends the deformation up to order two and the physical motivator for the deformation of this commutative algebra **A** is the Poisson bracket {. , .}.

Also, the works of De Wilde *et al.* [3], Fedosov [4] and Omori et al. [9] show that any non-degenerating Poisson bracket {. , .} on a smooth manifold of arbitrary dimension can be quantised. Since, the Poisson bracket is the first - order parameter as acknowledged by [7] in deformation quantisation. In this paper, we introduce Poisson Structures on deformation quantisation and its non-degradable factor on the Casimir function. Also, we show the relationship between the Poisson algebra with its morphism on a category. This paper has five sections. Section 1 is the introduction and Review of related Literature. In section 2, we discuss the Poisson brackets and its algebra. In section 3, we present Poisson brackets, its manifold and morphism. The Poisson bracket, deformation quantisation and the Casimir function is examined in section 4.

In section 5, we present the Poisson brackets and its automorphism.

Poisson Brackets and Poisson Algebra

There are some physical motivations that triggered the emergence of deformation quantisation. Some of them are: The Poisson brackets introduced by Joseph - Louis Lagrange and Simon - Denis de Poisson in the beginning of the 19th century, defined

as:

$$\{a,b\} = \sum_{i=1}^{n}\left(\frac{\partial a}{\partial q^i}\frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q^i}\right) \tag{2.1}$$

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where $\{a , b\} : R^{\infty} \times R^{\infty} - \rightarrow R$, are smooth functions and (q , p) are Lagrange's

canonical coordinates. Equation (2.1) is an algorithm used to solve the equation of motion in Hamiltonian systems.

In the thirties, Jacobi discovered a very simple proof of Poisson result, where he remarked that if a is a function, then the map; $g : \rightarrow \{a , b\}$ is a vector field because of the Leibniz identity for Poisson brackets and the Jacobi identity, given respectively as

$$\{a , bc\} = b\{c , a\} + c\{a , b\} \qquad (2.2)$$

and

$$\{a, \{b , c\}\} + \{b, \{c , a\}\} + \{c, \{a , b\}\} = 0 \qquad (2.3)$$

However, in the seventies, Marius Sophus Lie did a detailed research on the geometry of Partial Differential Equations. In his work, he brought to limelight a more systematic study of Poisson brackets, where he studied new examples of Poisson brackets, whose nature is different from those in Poisson's and Lagrange's works, which he named the Lie-Poisson brackets. According to [8], the unifying model for both Poisson and Lie brackets is the Poisson algebra.

Definition 2.1 (Poisson Algebra)**.** A Poisson Algebra is an associative algebra A

(over a field K) with a linear bracket $\{. , .\}: A \times A \rightarrow A$, such that

(i) $\{a , b\} = - \{b , a\}$ (Anti-symmetry)

(ii) $\{ab , c\} = a\{b , c\} + \{a , c\}b$ (Leibniz)

(iii) $\{a, \{b , c\}\} + \{c, \{a , b\}\} + \{b, \{c , a\}\} = 0$ (Jacobi Identity)

For all $a, b, c, \in A$.

1.1 Examples of Poisson Algebras

Example 2.2. 1. An associative algebra A is a Poisson algebra, if we say;

$$\{a , b\} = ab - ba$$

This implies that

$$\{ab , c\} = (ab)c - c(ab)$$

$$= a(bc) - a(cb) + (ac)b - (ca)b$$

$$= a\{(bc) - (cb)\} + \{(ac) - (ca)\}b$$

$$= a\{b , c\} + \{a , c\}b$$

The example above shows that the second property (Leibniz) of a Poisson

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algebra holds.

2. Every Lie algebra is a Poisson algebra with respect to its null associative product $a \cdot b = 0$ and every associative algebra is a Poisson algebra with respect to null Poisson brackets $\{a, b\} = 0$. Such an algebra is a null Poisson algebra.

There are many more examples of Poisson algebras. (See [10])

Proposition 2.3. *If G is a filtered associative algebra in a quotient space*

$G^0 \subseteq G^1 \subseteq \ldots, G^i \cdot G^j \subseteq G^{i+j}$ *such that* $G = \bigoplus_{i=0}^{\infty} G^i/G^j$ *is commutative, and*

$[x] \in G$ *be the class of* $x \in G^i$ *and we define* $\{[x], [y]\} = [xy - yx] \in G$. *Then G is*

a Poisson algebra.

Proof. The filtered associative algebra must satisfy the properties of the Poisson algebra. By linearity of the Poisson brackets, we have, $\forall a, b \in R$ and $x, y \in G$.

$\{[x], [y]\} = [xy - yx]$

$$= a[xy - yx] - b[xy - yx]$$

$= a[x][y] - b[x][y]$

$$= a\{[x], [y]\} - b\{[x], [y]\}$$

Next, we test for anti-symmetry

$\{[x], [y]\} = [xy - yx]$

$$= [-yx + xy]$$

$$= -\{[y], [x]\}$$

For the Leibniz rule and Jacobi identity, let $x, y, z \in G$ such that

$\{[x], \{[y], [z]\}\} + \{[y], \{[z][x]\}\} + \{[z], \{[x][y]\}\} = [x\{[y][z]\} - \{[y][z]\}x] + [y\{[z,[x]\} - [\{z[x]\}y]$

$$+ [z\{[x],[y]\} - \{[x][y]\}z]$$

$= [x[yz - zy] - [yz - zy]x] + [y[zx - xz]$

$- [zx - xz]y] + [z[xy - yx] - [xy - yx]z]$

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$= xyz - xzy - yzx + zyx + yzx - yxz - zxy$

$+ xzy + zxy - zyx - xyz + yxz$

$= 0.$

G having satisfied the properties of a Poisson algebra, *G* is a Poisson algebra.

Poisson Brackets and Poisson Manifolds

Definition 3.1. Poisson Manifolds

Let *M* be a manifold. Then *M* is a Poisson manifold if *M* is a smooth manifold equipped with a Poisson bracket, which is a skew symmetric bilinear operation

$\{. , .\}$ on $C^\infty(M)$, such that

$\{. , .\}: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M),$

with the Leibniz property: $\{ab , c\} = a\{b , c\} + \{a , c\}b$ and the Jacobi identity: $\{a, \{b, c\}\} + \{c, \{a , b\}\} + \{b, \{c , a\}\} = 0.$ $\forall a, b, c \in C^\infty(M).$

Poisson Morphism

Definition 3.2. Poisson Morphism

Let $\tau : (A, \{. , .\}) \rightarrow (B, \{. , .\})$ be a morphism of algebras such that $\tau\{\{a , b\}_A\} = \{\tau(a) , \tau(b)\}_B,$ $\forall a, b, \in A.$

Definition 3.3. Category

Category is the collections of objects and morphisms between these objects which are subject to;

1. Identity of morphism: $id_A : A \rightarrow A.$
2. Composition of morphism: $\rho = \sigma \circ \tau$
3. Associative of morphism: $(\rho \circ \sigma) \circ \tau = \rho \circ (\sigma \circ \tau)$

Proposition 3.4. *Let* $(A, \{. , .\})$ *be a Poisson algebra with its Morphism. Then, the Poisson algebras with its morphism form a category.*

Proof. Here, we test for identity, composition and associativity of morphisms. Let A be the Poisson algebra. Then, the identity map; $id_A : A \rightarrow A$, is such that it

preserves the Poisson brackets;

$\{[id_A(a) , id_A (b)]\} = \{[a,b]\}.$ $\forall a , b \in A.$

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For the composition. Let A_1, A_2 and A_3 be any Poisson algebra. Then,

$\tau : A_1 \rightarrow A_2$

and

$\sigma : A_2 \rightarrow A_3$

be Poisson morphisms such that σ o τ is a Poisson morphism, we have;

$\{[\tau(a) , \tau(b)]\} = \tau\{[a , b]\}.\ \forall\ a , b \in A_1$

and

$\{[\sigma(b) , \sigma(c)]\} = \sigma\{[b , c]\}.\ \forall\ b , c \in A_2.$

Let $\rho = \sigma$ o τ. Then, we consider ρ (a) and ρ (b) for any a , $b \in A_1$ such that;

$\{[\rho(a) , \rho(b)]\} = \{[\sigma(\tau(a)) , \sigma(\tau(b))]\}$

$= \sigma\{\tau(a) , \tau(b)]\}$

$= \sigma\{\tau\{[a , b]\}\}$

$= \{\sigma(\tau\{[a,b]\})\}$

$= \{\rho\{[a , b]\}\}.$

The Poisson brackets are preserved. Also, these compositions of the morphisms must be associative. Let A_1, A_2, A_3 and A_4 be Poisson algebras and $\tau : A_1 \rightarrow A_2$, $\sigma : A_2 \rightarrow A_3$ and $\rho : A_3 \rightarrow A_4$ be the Poisson morphisms. Then,

$(\rho \text{ o } \sigma) \text{ o } \tau = \rho \text{ o } (\sigma \text{ o } \tau)$

$\rho(a)\ \rho(b) = [\rho(\sigma(a) , \rho(\sigma(b)] \text{ o } \tau$

$(\rho \text{ o } \sigma).\ \tau(a) = \rho\ (\sigma(\tau(a)))$

$= \rho(\sigma(b))$

$= \rho(c)$

$= d$

and

$\rho \text{ o } (\sigma \text{ o } \tau) = \rho\ (\sigma(\tau(a)))$

$= \rho(c)$

$= d$

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$\forall\ a, b, c, d \in A_1,\ A_2, A_3, A_4 \subset A.$

$\Rightarrow (\rho \circ \sigma) \circ \tau(a) = \rho \circ (\sigma \circ \tau)\, a = d.$

Thus, the Poisson algebra with its morphisms form a category. ☐

Poisson Brackets and Deformation Quantisation

Let $C^{\infty}(M)$ be a Poisson manifold. Then we denote:

$A\{[\epsilon]\}$ to be the set of all formal power series with coefficients in A.

$R\{[\epsilon]\}$ as the set of all formal power series on R.

We define the differential operators that is globally bounded as,

Definition 4.1. Let T_k: A × A → A, be differential operators with respect to each argument of globally bounded order, then

$T_k(fg) \leq A \parallel f \parallel\parallel g \parallel,\ \forall f, g \in A$ satisfying the following properties, such that

(1) $\sum j+k=n\ T_j\,(T_k\,(f, g)\,h) = \sum j+k=n\ T_j(f, T_k(g, h))$ (Associativity)

(2) $T_0(f, h) = f\,g$ (Classical limit)

(3) $T_1(f, g) - T_1(g, h) = \{f, g\}$ (Semi-classical limit)

The idea of star product depends on the basic definitions of formal deformation of an algebra.

Definition 4.2. Let A be an associative and unital algebra over a commutative ring *K*. Then a formal deformation of the algebra A is a formal power series

$$f * g = f.g + \sum_{k=1}^{\infty} \epsilon^k\, T_k\ (f, g)$$

where, $f, g \in A \subset A\{[\epsilon]\}$ and T_k : A × A → A are bilinear maps such that the product $(*)$ is associative.

However, the deformation quantisation of $C^{\infty}(M)$ is the formal deformation of $A = C^{\infty}(M)$ such that the unit of the algebra is preserved.

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Definition 4.3. Star Product

A star product of M is an R{[ϵ]} bilinear map, on $C^{\infty}(M)\{[\epsilon]\} \times C^{\infty}(M)\{[\epsilon]\} \rightarrow C^{\infty}(M)\{[\epsilon]\}$ denoted by, $(a\ ,\ b) \rightarrow a * b$ such that the following conditions are satisfied:

1) $a * b = a.b + \sum_{k+1}^{\infty} \epsilon^k T_k\ (a\ ,b)$ Point-wise multiplication

2) $(a * b) * c = a * (b * c)$, $\forall\ a, b, c \in C^{\infty}(M)$. Associativity.

3) $a * 1 = 1 * a = a$, $\forall\ a \in C^{\infty}(M)$. Identity

In other words, a deformation quantisation of a Poisson manifold $(C^{\infty}(M)\{.\ ,.\})$ is given by a star product on $C^{\infty}(M)\{[\epsilon]\}$ with the following the following properties;

1. $(C^{\infty}(M)\{[\epsilon]\},*)$ is a deformation of the algebra structure on $C^{\infty}(M)$
2. The terms $T_1\,(a\ ,b)$ are given by the bi-differential operators in a and b.
3. $T_1(a\ ,b) - T_2(a\ ,b) = \{a\ ,b\}$.
4. $a * 1 = 1 * a = a$, $\forall a \in C^{\infty}(M)$

Remark 4.4. Deformation quantisation uses the Poisson brackets and star product to map classical observables to quantum operators while preserving important physical properties and principles, such as the Heisenberg uncertainty principle.

Poisson Brackets and Casimir Function

Given a Poisson bracket on a Poisson algebra A. Then, an element $a \in$ A is a

Casimir function, if $\{a\ ,b\} = 0$, $\forall\ b \in$ A. That is, it is a Casimir function, if its Poisson structure commutes with all other functions, where the Poisson brackets of a Casimir function with any other function is zero.

Proposition 4.5. *If b is a function whose differentials span the cotangent bundles everywhere, then,* $a \in C^{\infty}(M)$ *is a Poisson manifold, such that* $\{a\ ,b\} = o$ *is a*

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Casimir function.

Proof. Let $\{a\ , b\} = 0$. This implies that Poisson brackets between the manifold and the Casimir function is zero. i.e $\{a\ , b\} = 0$, for any other function b

$$\{a\ , b\} = ab - ba.$$

Since, b is a Casimir function, we have;

$$ab = ba = 0.$$

Since b commutes with a. Then, $\{a\ , b\} = 0$. □

Remark 4.6. A Casimir function is a function on the phase space that commutes with all other functions under the Poisson bracket. The existence and properties of Casimir functions are closely tied to the symmetries and conserved quantities of the system described by the Poisson bracket. Casimir functions often arise when dealing with systems with certain Lie group symmetries, such as mechanical systems with rotational or translational symmetries. Also, in the background of Hamiltonian mechanics, they correspond to constants of motion or conserved quantities. For example, the energy of a system is often represented by a Casimir function.

Poisson Brackets and Poisson Automorphism

A Poisson automorphism of a Poisson manifold is a diffeomorphism;

$$\phi : C^{\infty}(M) \rightarrow C^{\infty}(M)$$

that preserves the Poisson brackets. That is, for any $a\ , b \in C^{\infty}(M)$, we have;

$$\{a\ , b\} = \{\phi^{*}a\ , \phi^{*}b\}.$$

Where, ϕ^{*} is the pullback operation that transform functions on $C^{\infty}(M)$ into the function on the diffeomorphism image of $C^{\infty}(M)$ under ϕ.

Remark 5.1. Intuitively, a Poisson automorphism is a smooth and invertible transformation that preserves the Poisson structure. This means that when we apply this transformation to the functions on the manifold M, the Poisson brackets, $\{.\ ,.\}$ relations between the functions remain unchanged. The significance of this preservation is that Poisson automorphisms represent symmetries or transformations that maintain the fundamental Poisson algebraic structure of the manifold. In other words, the manner the functions interact with each other through the Poisson brackets remain consistent under such transformations.

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