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ON THE TOTAL SCORE FOR A NEGATIVE-BINOMIALLY ROLL OF A TRUNCATED TURN-UP SIDE OF $(v-u+1)$-SIDED DIE

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ABSTRACT: *Let $u, v \in N(u < v)$ and $\{T(x,y) = p^x q^y : x, y = u, u+1, u+2, u+3, \ldots, v\}$ be a string of sequence of success-failure events constituting the Bernoulli trials, with success p and failure q. Several probability distributions have derived their roots from the sequences of this form. However, it is our purpose to introduce new probability distribution functions that unify some of the existing ones generated by sets of this form mentioned in the literature and then give some of the statistics associated with it.*

KEYWORDS AND PHRASES: Probability mass function, geometric random variable, independent identically distributed (iid) random variable, moments, binomial expansion.

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INTRODUCTION AND PRELIMINARIES

In 1994, Balasubramanian [3] considered the set $\{E_x \in [0,1]: x \in \{0,1,2,3,\ldots,m-1\}\}$ of turn-up side probability of sequence of events, by taking

$$E_x = p^x q^{m-1-x}\,; x = 0,1,2,3,\ldots,m-1 \quad (1.0)$$

They used this set to describe a die that has m faces marked $\{0,1,2,3,\ldots,m-1\}$ with E_x representing the turn-up side probabilities which are in geometric progression.

They considered consecutive rolls of this m-faced die with side probabilities E_x for a given positive integer k and the random variable $Z_k^{(m)}$ defined as

$$Z_k^{(m)} = \textit{total score until face marked } 0 \textit{ appears } k \textit{ times.} \qquad (1.1)$$

It was shown that $Z_k^{(m)}$ follows the standard negative binomial distribution when $m = 2$. In order to derive the PGF of $Z_k^{(m)}$, they viewed the above experiment as a two-stage process first by generating a value n of

$$T_{m,k} = \textit{number of rolls until face marked } 0 \textit{ appears } k \textit{ times} \qquad (1.2)$$

Then, secondly, rolling n times a "reduced" die with faces marked $\{1,2,3,\ldots,m-1\}$ with the corresponding side probabilities $F_x = \frac{p^x q^{m-1-x}}{(1-q^{m-1})}$. They computed the total score among the n rolls to obtain $Z_k^{(m)}$ with the convention that $Z_n^{(m)} = 0$ whenever $n = 0$, where the pmf of $T_{m,k}$ has the negative binomial distribution given by

$$P(T_{m,k} = x; p) = \binom{x+k-1}{k-1} q^{(m-1)k}(1-q^{m-1})^x\,; 0 \le x < \infty \qquad (1.3)$$

which is a standard negative binomial distribution if $m = 2$. It was observed that $Z_k^{(m)}$ can be seen as the total score from a negative binomial random number of rolls of the reduced die and from the basic theory on compounding of distributions [see 13, pp. 44-45]. The PGF of $Z_k^{(m)}$ can be written as

$$H(t) = E\left(t^{Z_k^{(m)}}\right) = G_{T_{m,k}}\big(G_R(t)\big) \qquad (1.4)$$

where $G_{T_{m,k}}(t)$ is the PGF of $T_{m,k}$ and $\big(G_R(t)\big)$ is the PGF of the score in one roll of the "reduced" die. With

$$G_{T_{m,k}}(t) = \left(\frac{q^{m-1}}{1-(1-q^{m-1})t}\right)^k, G_R(t) = \frac{pt}{1-q^{m-1}}\frac{q^{m-1}-p^{m-1}t^{m-1}}{q-pt}$$

which implies that

$$H(t) = q^{(m-1)k}\left(1 - pt\,\frac{q^{m-1}-p^{m-1}t^{m-1}}{q-pt}\right)^{-k} \qquad (1.5)$$

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Using the familiar negative binomial expansion, it follows that the probability distribution of the random variable $Z_k^{(m)}$ is given by

$$P\left(Z_k^{(m)} = r; p\right) = \sum_{x=0}^{r}\sum_{s=0}^{\beta}(-1)^s\binom{x}{s}\binom{x+k-1}{x}\binom{r-(m-1)s-1}{x-1}p^r q^{(m-1)k-r};$$

$$0 \le r < \infty \qquad (1.6)$$

with the condition that $q^{(m-1)} = 1 - \frac{p(q^{m-1}-p^{m-1})}{q-p}$, where $\beta = \left\{x, \left[\frac{r-x}{m-1}\right]\right\}$ and the mean of $Z_n^{(m)}$ is given as

$$E\left(Z_k^{(m)}\right) = H'(1) = \frac{kp}{q^{m-1}}\frac{1-mp^{m-1}}{q-p}$$

When $m = 2$, we have that $E\left(Z_k^{(2)}\right) = \frac{kp}{q}$, which is the expected value of a standard negative binomial random variable.

Let $X_n^{(m)}$ be a random variable that counts the total score in n rolls of the m-sided die with turn-up side probabilities E_x satisfying the condition

$$q^m - p^m = q - p \qquad (1.7)$$

It was showed by Balakrishniam in [3] that $X_n^{(m)}$ has the familiar binomial distribution with index n and success probability p given by

$$f_B(x;p) = \sum_{s=0}^{\beta}(-1)^s\binom{n}{s}\binom{n-1+x-ms}{n-1}p^x q^{(m-1)n-x}; 0 \le x \le (m-1)n \qquad (1.8)$$

where $f_B(x;p) = P\left(X_n^{(m)} = x; p\right)$, $\beta = min\left\{n, \frac{x}{m}\right\}$. The combinatorial coefficient in (1.8) often denoted by $C_m(n,x)$ has been used extensively in probability studies [2,3,6,10,13,19-21] and related areas like reliability and inferential statistics [1,11,19]. For more properties on $C_m(n,x)$, generalized Pascal triangles or Pascal triangles of order m, we refer to [4,9,15,16,26-28] and the references therein.

In 2017, Okoli O. C. [24] modified the distribution of Balasubramanian in [3] by letting $m, k \in N$ (where m and k are not necessarily equal) and Δ be as defined in (1.0) above, and then take

$$T(x,y) = p^x q^y:\ x,y = 0,1,2,3,\ldots,m-1;\ x+y = k-1; 0 \le p,q \le 1 \qquad (1.9)$$

The following theorem was stated which is a consequence of several corollaries. (For more on this, see [24].)

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Theorem 1.1

Let $X_n^{(m,k)}$ be a random variable that counts the total score in n rolls of an m-sided die with range $x = 0,1,2, \ldots, m-1$ and turn-up side probabilities $T(x, k-1-x)$ $(x \in \{0,1,2,3, \ldots, m-1\})$ satisfying the condition $(q^m - p^m) = q^{m-k}(q-p)$; then the probability mass function (pmf) is given by

$$f_{O_1}(x;p) = \sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{n-1+x-ms}{n-1} p^x q^{(k-1)n-x}\,; 0 \le x \le (k-1)n \qquad (1.10)$$

where $f_{O_1}(x;p) = P\left(X_n^{(m,k)} = x; p\right),\ k \le m,\ \beta = min\left\{n, \frac{x}{m}\right\}$.

The distributions described by authors in $[3,24]$ are theoretically consistent with existing results; however, we worry about its accuracy in terms of practical and computational purposes. It is important to note that the above distribution(s) fail to give an accurate probability value when a fair dice is tossed for some given number of times. However, this abnormality was addressed with a detailed example in [25]; if we are dealing with a fair die (with $m = k$) then the results of the theorems of the author(s) in $[3, 24]$ reduce to

$$P\left(X_n^{(m,m)} = x; p\right) = \frac{1}{m^n}\sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{n-1+x-ms}{n-1}\,; 0 \le x \le (m-1)n \qquad (1.11)$$

In order to correct the computational inconsistency and accuracy abnormally associated with Equation (1.11), Okoli [25] redefined the turn-up sides (model) probabilities as

$$T(x,y) = p^x q^y\!:\ x, y = 1,2,3, \ldots, m;\ x + y = k; 0 = a \le p, q \le 1 = b. \qquad (1.12)$$

And the following theorem was proven:

Theorem 1.2

Let $X_n^{(m,k)}$ be a random variable that counts the total score in n rolls of an m-sided die with range $x = 1,2, \ldots, m$ and turn-up side probabilities $T(x, k-x)$ $(x \in \{1,2,3, \ldots, m\})$ satisfying the condition $p(q^m - p^m) = q^{m-k}(q-p)$; then, the probability mass function (pmf) is given by

$$P\left(X_n^{(m,k)} = x; p\right) = \sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{x-ms-1}{n-1} p^x q^{kn-x}\,; n \le x \le kn \qquad (1.13)$$

where $\beta = \left\{n, \left[\frac{x-n}{m}\right]\right\}$, $k \le m$.

If $k = m$, Equation (1.13) becomes

$$P\left(X_n^{(m,m)} = x; p\right) = \sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{x-ms-1}{n-1} p^x q^{mn-x}\,; n \le x \le mn \qquad (1.14)$$

So that for an m-sided fair (balanced) die, Equation (1.14) becomes

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$$P\left(X_n^{(m,m)} = x; p\right) = \frac{1}{m^n}\sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{x - ms - 1}{n-1}; \ \ n \le x \le mn \qquad (1.15)$$

For computational purposes, Okoli [25] showed that Equation (1.15) gives an accurate answer and is more practically oriented when compared to Equation (1.11).

In order to unify and study the work done by the authors in [3,24,25] under one common probability distribution function, in 2017, Okoli [24] considered the following model.

Let $k \in N$ (where v is not necessarily equal to k) and Δ be as in (1.0) above and if we define the turn-up side probabilities as

$$T(x, y) = p^x q^y: \ x, y = u, u+1, u+2, u+3, \dots, v; \ x + y = k; 0 = a \le p, q \le b = 1. \ \ (1.16)$$

where $u < k \le v, u, k \in N \cup \{0\}$. The following theorem was proven:

Theorem 1.3
Let $X_n^{(v-u+1,k-u+1)}$ be a random variable that counts the total score in n rolls of an $(v - u + 1)$-sided die and turn-up side probabilities $T(x, k - x)$ satisfying the condition $p^u(q^{v-u+1} - p^{v-u+1}) = q^{v-k}(q - p)$, with range $x = u, u+1, u+2, u+3, \dots, v$. Then, the probability mass function (pmf) is given by

$$P\left(X_n^{(v-u+1,k-u+1)} = x; p\right) = \sum_{s=0}^{\beta}(-1)^s \binom{n}{s}\binom{n - 1 + x - (v - u + 1)s - un}{n-1} p^x q^{kn-x};$$

$$un \le x \le kn$$

where $\gamma = \left\{n, \left[\frac{x-un}{v-u+1}\right]\right\}, k \le v$.

Then Equation (1.16) describes a $(v - u + 1)$-sided die with turn-up side probability $T(x, y)$, so that if

(a) $u = 0$ and $k = v = m - 1, \Rightarrow v - u + 1 = m$, which is the m-sided die studied by the authors in [3].

(b) $u = 0$ and $k \le v = m - 1, \Rightarrow v - u + 1 = m$, which is the m-sided die studied by the author in [24].

(c) $u = 1$ and $k \le v = m, \Rightarrow v - u + 1 = m$, which is the m-sided die studied by the author in [25].

Motivated by the results of the authors above, it is on the basis of Equation (1.16) that we shall study the negative binomially distributed random variable of the sum of scores of n rolls of $(v - u + 1)$-sided die until the face marked l $(u \le l \le v)$ appears j times for a u truncated turn-up side probability, denoted by $T_u(x, k - x)$. The results obtain in this research shall include the result of the authors in [3] as a special case as we wish to demonstrate that in the next section of this research work.

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METHODOLOGY AND MAIN RESULTS

Consider consecutive rolls of this $(v-u+1)$-faced die with side probabilities $T(x,k-x)$ in (1.16), $x=u,u+1,u+2,u+3,\dots,v$. For a given positive integer j and the random variables $Z_{l,j}^{(v-u+1,k-u+1)}, X_{l,j}$ defined as

$$Z_{l,j}^{(v-u+1,k-u+1)} = total\ score\ until\ face\ marked\ l\ appears\ j\ times. \qquad (2.1)$$

$$X_{k,l,j} = number\ of\ rolls\ until\ face\ marked\ l\ appears\ j\ times \qquad (2.2)$$

Observe that the reduced die studied in [3] is equivalent to a zero truncated turn-up side E_x in (1.0), so that in general, if $T(x,k-x)$ is given by

$$T(x,k-x)=p^x q^{k-x};\ x=u,u+1,u+2,u+3,\dots,v.$$

then

$$T_u(x,k-x)=\frac{p^x q^{k-x}}{1-p^u q^{k-u}};\ x=u+1,u+2,u+3,\dots,v,k\le v. \qquad (2.3)$$

It is easy to see that the probability that a face marked l will appear j times in an n rolls is given by

$$P\left(X_{k,l,j}=n;p\right)=\binom{n+j-1}{j-1}p^{lj}q^{(k-l)j}(1-p^l q^{k-l})^n;\ 0\le n<\infty, k\le v \qquad (2.4)$$

Now adopting the basic theory on compounding of distributions in [13, pp. 44-45], the PGF of $Z_{l,j}^{(v-u+1,k-u+1)}$ can be written as $R(t)=E\left(t^{Z_{l,j}^{(v-u+1,k-u+1)}}\right)=G_{X_{k,l,j}}\left(G_{T_u}(t)\right)$.

where $G_{T_u}(t)$ is the PGF for the u truncated turn-up side probability of a $(v-u+1)$-sided die and $G_{X_{k,l,j}}(t)$ is the PGF for the number of rolls until the face marked l appears j times. We state the result for $G_{T_u}(t)$ and $G_{X_{k,l,j}}(t)$ in the lemma that follows:

Lemma 2.0

Let $G_{T_u}(t)$ and $G_{X_{k,l,j}}(t)$ be as defined above for a $(v-u+1)$-sided die, then

$$(i)\ \ G_{T_u}(t)=\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)$$

$$(ii)\ \ G_{X_{k,l,j}}(t)=\left(\frac{p^l q^{(k-l)}}{1-(1-p^l q^{k-l})t}\right)^j$$

$$(iii)\ \ R(t)=p^{lj}q^{(k-l)j}\left[1-(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j}$$

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Proof

$$G_{T_u}(t) = \sum_{x=u+1}^{v} t^x \frac{p^x q^{k-x}}{1-p^u q^{k-u}} = \frac{q^k}{1-p^u q^{k-u}} \sum_{x=u+1}^{v} \left(\frac{pt}{q}\right)^x$$

$$= \left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right) p^{u+1} t^{u+1} \left(\frac{q^{v-u} - p^{v-u} t^{v-u}}{q-pt}\right) \tag{2.5}$$

and

$$G_{X_{k,l,j}}(t) = \sum_{x=0}^{\infty} t^x P(X_{k,l,j} = x; p) = \sum_{x=0}^{\infty} t^x \binom{x+j-1}{x} p^{lj} q^{(k-l)j} (1-p^l q^{k-l})^x =$$

$$p^{lj} q^{(k-l)j} \sum_{x=0}^{\infty} \binom{x+j-1}{x} (1-p^l q^{k-l})^x t^x = p^{lj} q^{(k-l)j} \sum_{x=0}^{\infty} (-1)^x \binom{-j}{x} \left((1-p^l q^{k-l})t\right)^x$$

$$= p^{lj} q^{(k-l)j} [1-(1-p^l q^{k-l})t]^{-j} = \left(\frac{p^l q^{(k-l)}}{1-(1-p^l q^{k-l})t}\right)^j \tag{2.6}$$

Now by using Equation (2.5) and Equation (2.6), it follows that

$$H(t) = p^{lj} q^{(k-l)j} \left[1 - (1-p^l q^{k-l}) \left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right) p^{u+1} t^{u+1} \left(\frac{q^{v-u} - p^{v-u} t^{v-u}}{q-pt}\right)\right]^{-j} \tag{2.7}$$

It is important to note that the nature of the graph of a distribution function of this form is often studied by constructing an auxiliary function associated with Equation (2.7). For more on this, see [3, 12, 17, 23, 24, 25].

Theorem 2.1

Let $Z_{l,j}^{(v-u+1,k-u+1)}$ be a random variable that count the total score from a negative binomial random number of rolls of a u truncated turn-up side probabilities $T_u(x, k-x)$ of $(v-u+1)$-sided die satisfying the condition

$$p^u q^{k-u} = 1 - q^{k-v} p^{u+1} \left(\frac{q^{v-u} - p^{v-u}}{q-p}\right)$$

with range $x = u, u+1, u+2, u+3, \dots, v$. Then the probability mass function (pmf) is given by

$$P\left(Z_{l,j}^{(v-u+1,k-u+1)} = r; p\right) =$$

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$$p^{lj}q^{(k-l)j}\sum_{x=0}^{\left[\frac{r}{(u+1)}\right]}\sum_{s=0}^{\beta}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x\binom{x-1+r-(v-u)s-(u+1)x}{x-1}$$

where $\beta=\left\{x,\left[\frac{r-(u+1)x}{v-u}\right]\right\}, k\le v.$

Proof

Now by using Equation (2.7), it follows that

$$p^{lj}q^{(k-l)j}\left[1-(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j}$$

$$=p^{lj}q^{(k-l)j}\left[1-\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)T^{u+1}\left(\frac{1-T^{v-u}}{1-T}\right)\right]^{-j};T=\frac{pt}{q}$$

Now

$$p^{lj}q^{(k-l)j}\left[1-\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)T^{u+1}\left(\frac{1-T^{v-u}}{1-T}\right)\right]^{-j}$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}(-1)^x\binom{-j}{x}\left(\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)T^{u+1}\left(\frac{1-T^{v-u}}{1-T}\right)\right)^x$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x\left(T^{u+1}\left(\frac{1-T^{v-u}}{1-T}\right)\right)^x=$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^xT^{(u+1)x}(1-T^{v-u})^x(1-T)^{-x}$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^xT^{(u+1)x}\left(\sum_{s=0}^{x}(-1)^s\binom{x}{s}T^{(v-u)s}\right)\sum_{v=0}^{\infty}(-1)^v\binom{-x}{v}T^v$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^xT^{(u+1)x}\left(\sum_{s=0}^{x}(-1)^s\binom{x}{s}T^{(v-u)s}\right)\left(\sum_{v=0}^{\infty}\binom{x-1+v}{v}T^v\right)$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x\sum_{v=0}^{\infty}\sum_{s=0}^{n}(-1)^s\binom{x}{s}\binom{x-1+v}{v}T^{(v-u)s+v+(u+1)x}$$

$$=p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\sum_{r=(u+1)x}^{\infty}\sum_{s=0}^{x}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x\times$$

$$\binom{x-1+r-(v-u)s-(u+1)x}{r-(v-u)s-(u+1)x}T^r$$

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$$= p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\sum_{r=(u+1)x}^{\infty}\sum_{s=0}^{x}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x \times$$

$$\binom{x-1+r-(v-u)s-(u+1)x}{r-(v-u)s-(u+1)x}p^rq^{-r}t^r$$

$$= p^{lj}q^{(k-l)j}\sum_{x=0}^{\infty}\sum_{r=(u+1)x}^{\infty}\sum_{s=0}^{x}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x$$
$$\times\binom{x-1+r-(v-u)s-(u+1)x}{x-1}\left(\frac{p}{q}\right)^r t^r$$

$$= p^{lj}q^{(k-l)j}\sum_{r=0}^{\infty}\sum_{x=0}^{\left[\frac{r}{(u+1)}\right]}\sum_{s=0}^{x}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x$$
$$\times\binom{x-1+r-(v-u)s-(u+1)x}{x-1}\left(\frac{p}{q}\right)^r t^r$$

Thus, this implies that

$$P\left(Z_{l,j}^{(v-u+1,k-u+1)}=r;p\right)=$$

$$p^{lj}q^{(k-l)j}\sum_{x=0}^{\left[\frac{r}{(u+1)}\right]}\sum_{s=0}^{x}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^k(1-p^lq^{k-l})}{1-p^uq^{k-u}}\right)^x\binom{x-1+r-(v-u)s-(u+1)x}{x-1}\left(\frac{p}{q}\right)^r;$$

$$0\le r<\infty$$

Corollary **2.2**

Let $Z_{l,j}^{(v-u+1,v-u+1)}$ be a random variable that counts the total score from a negative binomial random number of rolls of a u truncated turn-up side probabilities $T_u(x,v-x)$ of $(v-u+1)$-sided die satisfying the condition

$$p^uq^{v-u}=1-p^{u+1}\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)$$

with range $x=u,u+1,u+2,u+3,\dots,v$. Then the probability mass function (pmf) is given by

$$P\left(Z_{l,j}^{(v-u+1,v-u+1)}=r;p\right)=$$

$$p^{lj}q^{(v-l)j}\sum_{x=0}^{\left[\frac{r}{(u+1)}\right]}\sum_{s=0}^{\beta}(-1)^s\binom{x}{s}\binom{x+j-1}{x}\left(\frac{q^v(1-p^lq^{v-l})}{1-p^uq^{v-u}}\right)^x\binom{x-1+r-(v-u)s-(u+1)x}{x-1}\left(\frac{p}{q}\right)^r;$$

$$0\le r<\infty$$

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Corollary 2.3

Let $Z_{0,j}^{(v+1,v+1)}$ be a random variable that counts the total score from a negative binomial random number of rolls of a u truncated turn-up side probabilities $T_0(x, v-x)$ of (m)-sided die satisfying the condition

$$q^v = 1 - p\left(\frac{q^v - p^v}{q-p}\right)$$

with range $x = 0,1,2,3, \ldots, v$. Then the probability mass function (pmf) is given by

$$P\left(Z_{0,j}^{(v+1,v+1)} = r; p\right) =$$

$$\sum_{x=0}^{r} \sum_{s=0}^{\gamma} (-1)^s q^{(m-1)x} \binom{x}{s} \binom{x+j-1}{x} \binom{r-(v)s-1}{x-1} p^r q^{vj-r} \qquad 0 \le r < \infty$$

where $\gamma = \left\{x, \left[\frac{r-x}{v}\right]\right\}$.

We quickly note that Corollary 2.3 above is the result obtained by the authors in [3] if we put $v = m-1$.

Theorem 2.4

Let $Z_{l,j}^{(v-u+1,k-u+1)}$ be a random variable that counts the total score from a negative binomial random number of rolls of a $u-$ truncated turn-up side probabilities $T_u(x, k-x)$ of $(v-u+1)$-sided die satisfying the condition

$$p^u q^{k-u} = 1 - q^{k-v} p^{u+1} \left(\frac{q^{v-u} - p^{v-u}}{q-p}\right)$$

with range $x = u, u+1, u+2, u+3, \ldots, v$. Then the mean and variance are determined by

$$(i)\ E\left(Z_{l,j}^{(v-u+1,k-u+1)}\right) = R'(1)$$

$$(ii) Var\left(Z_{l,j}^{(v-u+1,k-u+1)}\right) = R''(1) + R'(1) - \left(R'(1)\right)^2$$

where

$$R'(1) = \frac{jp^{u+1}q^{k-v}(1-p^l q^{k-l})}{(1-p^u q^{k-u})p^l q^{(k-l)}} \left[(u+1)\left(\frac{q^{v-u} - p^{v-u}}{q-p}\right) + \left(\frac{p(q^{v-u} - p^{v-u})}{(q-p)^2} - \frac{(v-u)p^{v-u}}{q-p}\right)\right]$$

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$$R''(1) = j(j-1)[p^l q^{k-l}]^{-2}\left((1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}\right)^2 \left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right)\right]^2 + j(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}$$

$$\times [p^l q^{k-l}]^{-1}\left[u(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+(u+1)\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right) + (u+1)t^u\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right) + \left[\frac{-(v-u)(v-u-1)p^{v-u}}{q-p}+\frac{-2(v-u)p^{v-u+1}}{(q-p)^2} + \frac{2p^2(q^{v-u}-p^{v-u}t^{v-u})}{(q-p)^3}\right]\right]$$

Proof

Recall that

$$R(t) = p^{lj}q^{(k-l)j}\left[1-(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j}$$

$$R'(t) = -jp^{lj}q^{(k-l)j}\left[1-(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j-1} \times -(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}\left[(u+1)t^u\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right) + t^{u+1}\left(\frac{-(v-u)p^{v-u}t^{v-u-1}}{q-pt}+\frac{p(q^{v-u}-p^{v-u}t^{v-u})}{(q-pt)^2}\right)\right]$$

$$R'(1) = jp^{lj}q^{(k-l)j}(1-p^l q^{k-l}) \times \left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}\left[1-(1-p^l q^{k-l})\left(\frac{q^{k-v}}{1-p^u q^{k-u}}\right)p^{u+1}\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)\right]^{-j-1} \times \left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+\left(\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}-\frac{(v-u)p^{v-u}t^{v-u-1}}{q-p}\right)\right]$$

By applying the normalization condition, we have

$$E\left(Z_{l,j}^{(v-u+1,k-u+1)}\right) =$$

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$$\frac{jp^{u+1}q^{k-v}(1-p^lq^{k-l})}{(1-p^uq^{k-u})p^lq^{(k-l)}}\left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+\left(\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}-\frac{(v-u)p^{v-u}t^{v-u-1}}{q-p}\right)\right]$$

Now, by differentiating $H'(t)$, it implies that

$$R''(t)=j(j-1)p^{lj}q^{(k-l)j}\left((1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\right)^2\left[1-(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j-2}\times$$

$$\left[(u\ +1)t^u\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)+t^{u+1}\left(\frac{-(v-u)p^{v-u}t^{v-u-1}}{q-pt}+\frac{p(q^{v-u}-p^{v-u}t^{v-u})}{(q-pt)^2}\right)\right]^2$$

$$+jp^{lj}q^{(k-l)j}(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\times$$

$$\left[1-(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}t^{u+1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)\right]^{-j-1}\times$$

$$\left[u(u+1)t^{u-1}\left(\frac{q^{v-u}-p^{v-u}t^{v-u}}{q-pt}\right)+(u+1)t^u\left(\frac{-(v-u)p^{v-u}t^{v-u-1}}{q-pt}+\frac{p(q^{v-u}-p^{v-u}t^{v-u})}{(q-pt)^2}\right)+(u+1)t^u\left(\frac{-(v-u)p^{v-u}t^{v-u-1}}{q-pt}+\frac{p(q^{v-u}-p^{v-u}t^{v-u})}{(q-pt)^2}\right)+\left(\frac{-(v-u)(v-u-1)p^{v-u}t^{v-u-2}}{q-pt}+\frac{-2(v-u)p^{v-u+1}t^{v-u-1}}{(q-pt)^2}+\frac{2p^2(q^{v-u}-p^{v-u}t^{v-u})}{(q-pt)^3}\right)\right]$$

Thus,

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$$R''(1) = j(j-1)p^{lj}q^{(k-l)j}\left((1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\right)^2\Bigg[1 - (1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)\Bigg]^{-j-2}\Bigg[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right) + \left(\frac{-(v-u)p^{v-u}}{q-p} + \frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right)\Bigg]^2 + jp^{lj}q^{(k-l)j}(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\times$$

$$\Bigg[1-(1-p^lq^{k-l})\left(\frac{q^{k-v}}{1-p^uq^{k-u}}\right)p^{u+1}\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)\Bigg]^{-j-1}\Bigg[u(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right) + (u+1)\left(\frac{-(v-u)p^{v-u}}{q-p} + \frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right) + (u+1)\left(\frac{-(v-u)p^{v-u}}{q-p} + \frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right) + \left(\frac{-(v-u)(v-u-1)p^{v-u}}{q-p} + \frac{-2(v-u)p^{v-u+1}}{(q-p)^2} + \frac{2p^2(q^{v-u}-p^{v-u})}{(q-p)^3}\right)\Bigg]$$

The result follows by applying the normalization condition and the variance can be computed using the standard definition, $Var\left(Z_{l,j}^{(v-u+1,k-u+1)}\right) = R''(1) + R'(1) - \left(R'(1)\right)^2$. This completes the proof.

Corollary 2.5

Let $Z_{l,j}^{(v-u+1,v-u+1)}$ be a random variable that counts the total score from a negative binomial random number of rolls of a u truncated turn-up side probabilities $T_u(x, v-x)$ of $(v-u+1)$-sided die satisfying the condition

$$p^uq^{v-u} = 1 - p^{u+1}\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)$$

with range $x = u, u+1, u+2, u+3, \ldots, v$. Then the mean and variance are determined by

$$(i)\quad E\left(Z_{l,j}^{(v-u+1,v-u+1)}\right) = R'(1)$$

$$\frac{jp^{u+1}(1-p^lq^{v-l})}{(1-p^uq^{v-u})p^lq^{(v-l)}}\left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right) + \left(\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2} - \frac{(v-u)p^{v-u}}{q-p}\right)\right]$$

$$(ii) Var\left(Z_{l,j}^{(v-u+1,k-u+1)}\right) = R''(1) + R'(1) - \left(R'(1)\right)^2$$

$$R'(1) = E\left(Z_{l,j}^{(v-u+1,v-u+1)}\right) =$$

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$$\frac{jp^{u+1}(1-p^l q^{v-l})}{(1-p^u q^{v-u})p^l q^{(v-l)}}\left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+\left(\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}-\frac{(v-u)p^{v-u}}{q-p}\right)\right]$$

$$R''(1)=j(j-1)[p^l q^{v-l}]^{-2}\left(\left(\frac{1-p^l q^{v-l}}{1-p^u q^{v-u}}\right)p^{u+1}\right)^2\left[(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)\right.$$
$$\left.+\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right)\right]^2+j\left(\frac{1-p^l q^{k-l}}{1-p^u q^{k-u}}\right)p^{u+1}$$
$$\times[p^l q^{v-l}]^{-1}\left[u(u+1)\left(\frac{q^{v-u}-p^{v-u}}{q-p}\right)+(u+1)\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right)\right.$$
$$+(u+1)\left(\frac{-(v-u)p^{v-u}}{q-p}+\frac{p(q^{v-u}-p^{v-u})}{(q-p)^2}\right)$$
$$\left.+\left(\frac{-(v-u)(v-u-1)p^{v-u}}{q-p}+\frac{-2(v-u)p^{v-u+1}}{(q-p)^2}+\frac{2p^2(q^{v-u}-p^{v-u})}{(q-p)^3}\right)\right]$$

Corollary 2.6

Let $Z_{0,j}^{(v+1,v+1)}$ be a random variable that counts the total score from a negative binomial random number of rolls of a u truncated turn-up side probabilities $T_0(x,v-x)$ of $(v+1)$-sided die satisfying the condition

$$q^v=1-p\left(\frac{q^v-p^v}{q-p}\right)$$

with range $x=0,1,2,3,\ldots,v$. Then the mean $E(Z_{0,j}^{(v+1,v+1)})$ is given by

$$E\left(Z_{0,j}^{(v+1,v+1)}\right)=\frac{jp}{q^v}\left(\frac{1-(v+1)p^v}{q-p}\right)$$

It is important to note that if we take $v=m-1$ in Corollary 2.3 and Corollary 2.6, we obtain the result of the authors in [3].

In conclusion, however, we can go on to obtain several other results (corollaries) by varying and adjusting the parameters k,l,u,v in the theories stated above. This is an indication that the probability distribution in [3] for $Z_{0,j}^{(v+1,v+1)}$, a random variable that counts the total score from a negative binomial random number of rolls of a zero-truncated turn-up side probabilities $T_0(x,v-x)$ of $(v+1)$-sided die, is domiciled in the probability distribution we defined in this paper, hence an improvement and a generalization of the result obtained in [3].

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