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DECREASING TREND BUMPED PROBABILITY DISTRIBUTION: ITS PROPERTIES, SIMULATIONS AND APPLICATION

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ABSTRACT: *In this paper, a probability model is proposed which engaged the mathematical combination of Lindley distribution and a trigonometric component known as haversine function* δ_s. *The one parameter model prototype sustains the capacity to forecast multimodal decreasing trend sinusoidal outcomes. By nomenclature, the proposed probability model is called Modified Lindley Trigonometric Distribution (MLTD). Some statistical properties studied include the hazard function, mean residual life function, moments, conditional moments and moment generating function, Bonferroni and Lorenz curve, entropy, asymptotic distribution, order statistics, and parameter estimation; where the hazard function specially features a cyclic or periodic bathtub and inverted bathtub shape in chain format. The numerical behavior of the estimates of the average bias and mean square error were examined under Monte Carlo simulation approach; and an applicative simulation is experimented to underscore the parametric behavior of MLTD in data modeling. A real life flood data is used to illustrate the essence of the development.*

KEYWORDS: Lindley distribution, Trigonometry, Kernel, Sinusoids, Multimodal, Simulation.

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INTRODUCTION

Lindley Distribution (LD) was proposed according to Lindley (1958) to forecast for situations that mirror monotone decreasing trend, and with an increasing monotonic failure rate. It is a one parameter product of mixture model (Lindsay, 1995); which is a methodical approach for the development of a lifetime distribution and is given as:

$$g(u) = e^{-\gamma u} \frac{\gamma^2(1+u)}{\gamma+1}, \; u > 0, \; \gamma > 0 \quad (1)$$

The distribution was fitted to a lifetime waiting data collated from a sequenced customer queue management in a bank. Since then, many researchers have taken interest in the different forms of extensions and generalizations including the power LD due to Ghitany (2013), a two parameter generalized LD due to Ekhosuehi et al. (2018), Topp-Leone power LD due to Opone et al. (2022), two parameter extension of LD due to Gillariose and Tomy (2023), generalized power LD due to Guptha and Maruthan (2023), Shabeer et al. (2023) and the avalanche rest. As a matter of fact, each of these developments in literature has meaningfully addressed different profound limitations of their root model, and presented various possible modeling options.

Of course, the general motive for the various developments is simply to improve on the flexibility of the LD, ranging from left to right skewed density trends, monotone increasing density shapes, bathtub and inverted bathtub features, and symmetric characteristics. Others are monotone decreasing, bathtub and inverted bathtub hazard functions. It is observed that these flexibility characteristics were attained primarily due to the increased number of parameters as generated by the new compound structure. More so, some specific evaluations of their parameter combinations, is accountable for these improvements.

However, the motivation for this work stems from the fact that amidst the flexibility goodness, none of the Lindley family of distribution forecasts for heavily multimodal decreasing trend scenarios. These phenomenon is usually obtainable in countries' daily gross domestic product (GDP) rainfall data, crypto or forex trading, temperature data, climate change, flood data, sound and light wave data, to mention a few.

Figure 1: A pictorial example of multimodal decreasing trend

Furthermore, how can we describe the hazard rate of outcomes that are erratic? In the stock market, dwindling loss trend can be preceded by great improvement in profit margin; and the

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cycle continues. This is however suggestive of a hazard trend with continuous upward and downward movement. Obviously, the developments, so far, have not captured probability modeling in this order.

Consequently, this research aims at proposing a one parameter distribution which is a further development from LD that sustains the flexibility to reproduce as many modal structures as possible, in addition to the original shape of its root model and other possible shapes. Another objective will be to realize a sine wave hazard trend.

Howbeit, we construct the probability distribution using integration method and the idea of normalizing constant. This development will be achieved mathematically by primitively substituting δ_s for variable u in the component $(1+u)$, as contained in equation (1):

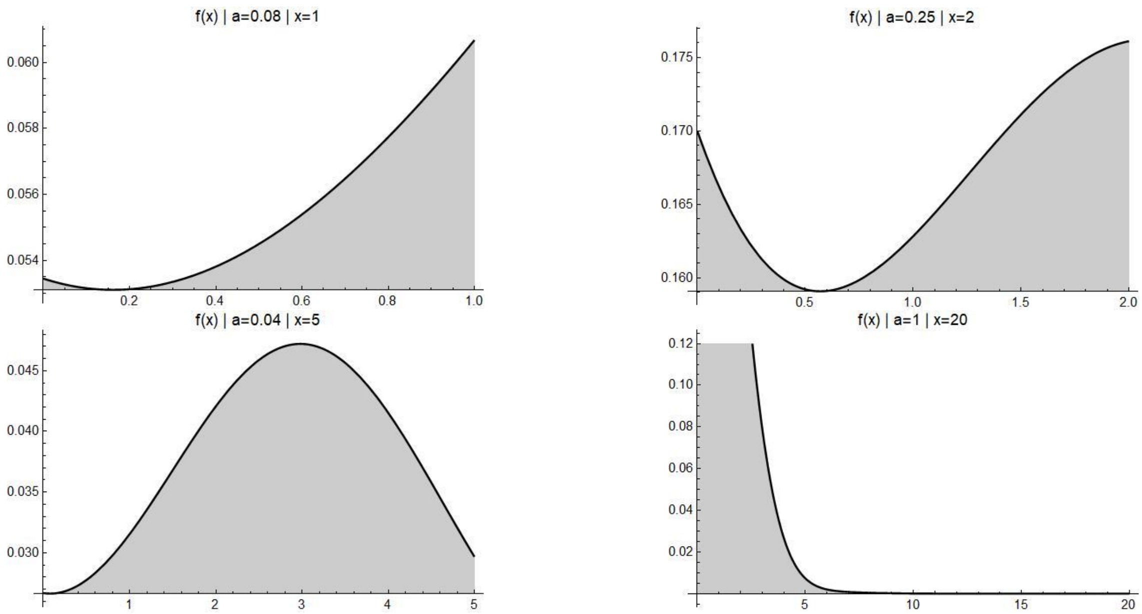
$$p_\tau = e^{-\gamma u}\ \frac{\gamma^2(1+\delta_s)}{\gamma+1}$$

where $\delta_s = Sin^2(u/2)$ and p_τ is a new kernel or arbitrary expression (*not a probability function any longer*). Hence, integrating p_τ for $u > 0$ and applying the normalizing constant, we obtain the density function as:

$$f(u,\gamma) = \ e^{-\gamma u}\ \frac{2\gamma(1+\delta_s)\,(1+\gamma+\gamma^2+\gamma^3)}{(1+\gamma)(3+2\gamma^2)},\ \ u > 0,\ \gamma > 0 \qquad (2)$$

$$= \ e^{-\gamma u}\ \frac{2\gamma(1+\gamma^2)(1+\delta_s)}{(3+2\gamma^2)}$$

Modified Lindley-Trigonometric Distribution (MLTD) could be another nomenclature for this development; where the shapes of the distribution are given thus:



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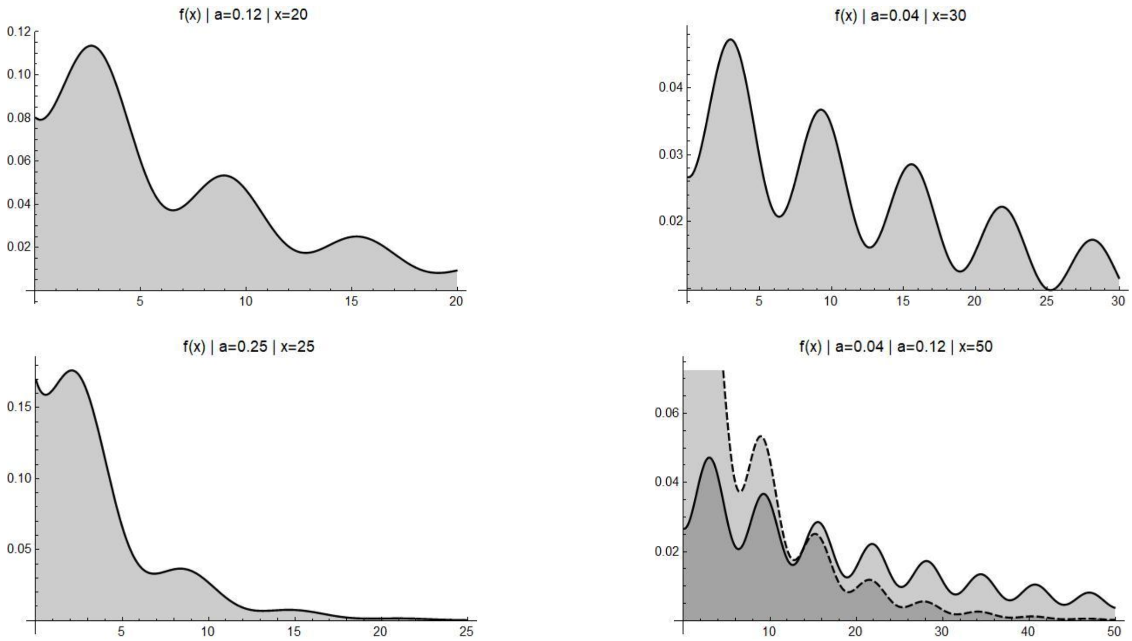

Figure 2: The MLTD PDF plots at different values of the parameter a

As revealed in Figure 2, MLTD exhibits different characteristics including monotone decreasing trend, which is the shape of the root distribution; monotone increasing, left skewed, bathtub and various regular and irregular k-modal decreasing trends.

It might be of concern to really validate MLTD as proper probability density function (PDF), since some of the trends are unconventional. Usually, $\int_0^{\infty} f(x)dx = 1$ serves as a more conventional approach to it; hence, we obtain:

$$\int_0^{\infty} e^{-\gamma u} \frac{2\gamma(1+\delta_S)\,(1+\gamma+\gamma^2+\gamma^3)}{(1+\gamma)(3+2\gamma^2)}\,du = \frac{2\gamma(1+\gamma^2)\left(\frac{1}{\gamma}+\frac{1}{2\gamma+2\gamma^3}\right)}{3+2\gamma^2} = 1$$

The cumulative distribution function (CDF) corresponding to equation (2) is derived as $\int_0^u f(x)dx$ and obtained thus:

$$F(u,\gamma) = 1 + \frac{e^{-\gamma u}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u]-\gamma\mathrm{Sin}[u]\}}{3+2\gamma^2} \quad (3)$$

Other properties of the proposed probability model will be studied in section (2); where simulation and real data application come in respectively at the remaining segments.

Hazard and Mean Residual Life Function (MRL)

Here, we present the hazard and MRL functions of MLTD using equations (2) and (3):

$$H(u,\gamma) = \frac{f(u,\gamma)}{1-F(u,\gamma)}$$

$$= -\frac{2\gamma(1+\gamma+\gamma^2+\gamma^3)(1+\delta_S)}{(1+\gamma)(-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u]-\gamma\mathrm{Sin}[u])}, u > 0, \gamma > 0$$

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The MLTD hazard plot in Figure 3 reveals a periodic trend characterized by cyclic bathtub and inverted bathtub chain. The plot shapes show to be similar at various parameter values, where the number of peaks is dependent on the evaluation of the variable u. The implication of the hazard outcome of MLTD is that the system it models is time risk based or has low reliability rate such that care has to be taken to know when to make purchases, sales or even withdrawals.

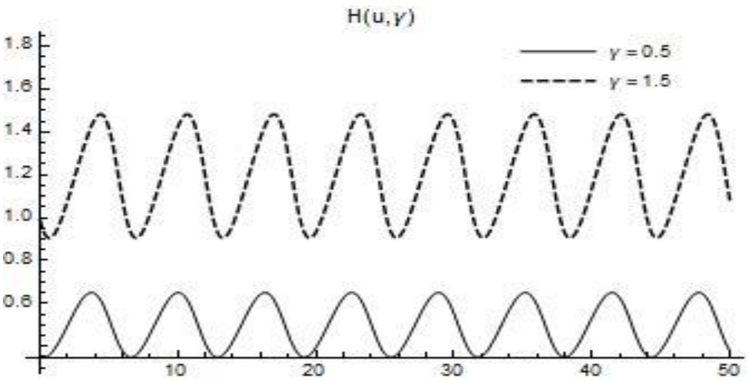

Figure 3: Hazard plot for MLTD for $\gamma = 0.5\ and\ 1.5$

Mean Residual Life Function (MRL) as a function under reliability theory, gives insight on the expected additional lifetime; on a condition that a system has survived or has been economically stable until time t. This is defined as:

$$m(x) = E[X - x \mid X > x] = \frac{1}{1 - G(x)} \int_x^{\infty} [\, 1 - G(t) \,]\, dt$$

Let $$\theta = \frac{1}{1 - G(u)} = -\frac{(3+2\gamma^2)e^{\gamma x}}{-3(1+\gamma^2)+\gamma^2\text{Cos}[u]-\gamma\text{Sin}[u]}$$

Hence, the MRL of a random variable $U \sim MLTD(u,\gamma)$ is obtained as

$$m(u) = \theta \int_u^{\infty} \left[1 - (1 + \frac{e^{-\gamma x}(-3(1+\gamma^2)+\gamma^2\text{Cos}[t]-\gamma\text{Sin}[t])}{3+2\gamma^2}) \right] dt$$

$$m(u) = -\frac{3(1+\gamma^2)^2+(\gamma^2-\gamma^4)\text{Cos}[u]+2\gamma^3\text{Sin}[u]}{(\gamma+\gamma^3)(-3(1+\gamma^2)+\gamma^2\text{Cos}[x]-\gamma\text{Sin}[x])}$$

where at $u = 0$, $$m(0) = -\frac{3\left(1+\gamma^2\right)^2+(\gamma^2-\gamma^4)}{(\gamma+\gamma^3)(-3(1+\gamma^2)+\gamma^2)} = \frac{3+7\gamma^2+2\gamma^4}{3\gamma+5\gamma^3+2\gamma^5} = \mu$$

Moments, Conditional Moments and Moment Generating Function

If X is a random variable with density function $g(x)$, then the r^{th} moment about the origin of X is defined by:

$$E(X^r) = \int_0^{\infty} x^r g(x) dx = \mu_r'$$

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The r^{th} raw moment for a distribution $U\sim MLTD(u,\gamma)$ about the origin is however derived as

$$E(U^r)=\frac{\gamma(1+\gamma^2)(6\gamma^{-1-r}-(-i+\gamma)^{-1-r}-(i+\gamma)^{-1-r})\Gamma[1+r]}{2(3+2\gamma^2)}$$

As a result, the first four r^{th} moments of the MLTD are further obtained at $r=1,2,3\ and\ 4$

$$\mu_1'=\frac{3+7\gamma^2+2\gamma^4}{3\gamma+5\gamma^3+2\gamma^5}=\mu; \qquad \mu_2'=\frac{2\gamma(1+\gamma^2)(3+9\gamma^2+12\gamma^4+2\gamma^6)}{(3+2\gamma^2)(\gamma+\gamma^3)^3}$$

$$\mu_3'=\frac{2\gamma(1+\gamma^2)(9+36\gamma^2+51\gamma^4+54\gamma^6+6\gamma^8)}{(3+2\gamma^2)(\gamma+\gamma^3)^4}; \qquad \mu_4'=\frac{24(3+15\gamma^2+30\gamma^4+25\gamma^6+25\gamma^8+2\gamma^{10})}{(3+2\gamma^2)(\gamma+\gamma^3)^4}$$

Therefore, the variance, skewness and kurtosis of the MLTD can be obtained as

$$Variance(\mu_2)=\mu_2'-\mu^2=\sigma^2; \qquad Skewness\ (S_k)=\frac{\mu_3}{\sigma^3}=\frac{\mu_3'-3\mu_2'\mu+2\mu^3}{(\mu_2)^{3/2}}$$

$$Kurtosis\ (K_s)=\frac{\mu_4}{(\mu_2)^2}=\frac{\mu_4'-4\mu_3'\mu+6\mu_2'\mu^2-3\mu^4}{(\mu_2'-\mu^2)^2}$$

The r^{th} conditional moments of a density function $g(y)$ from a random variable Y is defined by

$$E(Y^r|Y>y)=\frac{1}{1-G(y)}\int_y^\infty x^r g(x)dx$$

For a distribution $U\sim MLTD(u,\gamma)$ the r^{th} conditional moments is derived thus:

$$E(U^r|U>u)=-\frac{\gamma(1+\gamma^2)\left(3(1+\gamma^2)^2(1+\gamma u)+\gamma^2(\mathrm{Cos}[u]-\gamma(\gamma+u+\gamma^2u)\mathrm{Cos}[u]+(u+\gamma(2+\gamma u))\mathrm{Sin}[u])\right)}{(\gamma+\gamma^3)^2(-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u]-\gamma\mathrm{Sin}[u])}$$

Finally, the MGF of MLTD as a derivative of $E(tu)=\int_0^\infty e^{tu}f(u)du$ is given by:

$$\mathcal{M}_t=E(tu)=\frac{\gamma(1+\gamma^2)(3+2(\gamma-t)^2)}{(3+2\gamma^2)(1+(\gamma-t)^2)(\gamma-t)}$$

Bonferroni and Lorenz Curve

Let X be a random variable from a probability distribution $g(x)\ or\ G(x)$, with non-negative and finite mean μ, then Bonferroni curve is obtained as $B(i)=\frac{1}{i\mu}\int_0^q x\,g(x)dx$; which can further be expressed as

$$B(i)=\frac{1}{i\mu}\left[\int_0^\infty x\,g(x)dx-\int_q^\infty xg(x)dx\right]=\frac{1}{i\mu}\left[\mu-\int_q^\infty x\,g(x)dx\right]$$

whereas the Lorenz curve is obtained as $(i)=\frac{1}{\mu}\int_0^q x\,g(x)dx$; and can be represented as

$$L(i)=\frac{1}{\mu}\left[\int_0^\infty x\,g(x)dx-\int_q^\infty xg(x)dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,g(x)dx\right]$$

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Now, $B(i) = \frac{1}{\mu}\int_0^i G^{-1}(x)dx = \frac{L(i)}{i}$ defines the relationship between the Bonferroni curve and Lorenz

where $\mu = E(X),\ q = G^{-1}(i)$ and $i \in [0,1]$

Now, let $U \sim MLTD(u,\gamma)$, then $B(i)\ and\ L(i)$ of MLTD are defined as

$$B(i) = (i)^{-1}\left[\frac{\begin{array}{c}(3\gamma+5\gamma^3+2\gamma^5)e^{-\gamma q}((3+7\gamma^2+2\gamma^4)\,e^{\gamma q}-3(1+\gamma^2)^2(1+\gamma q)\\ +\gamma^2((-1+\gamma(\gamma+q+\gamma^2 q))\mathrm{Cos}[q]-(q+\gamma(2+\gamma q))\mathrm{Sin}[q]))\end{array}}{\gamma(1+\gamma^2)(3+2\gamma^2)(3+7\gamma^2+2\gamma^4)}\right]$$

$$L(i) = \left[\frac{\begin{array}{c}(3\gamma+5\gamma^3+2\gamma^5)e^{-\gamma q}((3+7\gamma^2+2\gamma^4)\,e^{\gamma q}-3(1+\gamma^2)^2(1+\gamma q)\\ +\gamma^2((-1+\gamma(\gamma+q+\gamma^2 q))\mathrm{Cos}[q]-(q+\gamma(2+\gamma q))\mathrm{Sin}[q]))\end{array}}{\gamma(1+\gamma^2)(3+2\gamma^2)(3+7\gamma^2+2\gamma^4)}\right]$$

Where $B(i)\ and\ L(i)$ for MLTD are both increasing functions.

Randomness of MLTD

Here, we study the concept entropy, which is a statistical phenomenon that measures the uncertainty of a system or component; for example probability distribution. The Rényi entropy of a random X is given by:

$$E_R(\mathrm{x},s) = \frac{1}{1-s}\log\left(\int f^s(x)\,dx\right) \quad \text{where } s > 0\ and\ s \neq 1$$

Now, if a random variable $U \sim MLTD(u,\gamma)$, then the Renyi entropy of MLTD is derived thus:

$$E_R(\mathrm{u},s) = \frac{1}{1-s}\log\left(\int_0^\infty \left(\frac{2\gamma(1+\gamma+\gamma^2+\gamma^3)}{(1+\gamma)(3+2\gamma^2)}\right)^s (1+\delta_s)^s e^{-\gamma us}du\right)$$

But $(1+\mathrm{w})^n = \sum_{i=0}^{\infty}\binom{n}{i} w^i$

$$= \frac{1}{1-s}\log\left(\int_0^\infty \left(\frac{2\gamma(1+\gamma+\gamma^2+\gamma^3)}{(1+\gamma)(3+2\gamma^2)}\right)^s \sum_{i=0}^{\infty}\binom{s}{i}(\delta_s)^i e^{-\gamma us}du\right)$$

$$= \frac{1}{1-s}\left(\log \sum_{i=0}^{\infty}\binom{s}{i}\left(\frac{2\gamma(1+\gamma+\gamma^2+\gamma^3)}{(1+\gamma)(3+2\gamma^2)}\right)^s \int_0^\infty (\delta_s)^i e^{-\gamma us}du\right) \quad (4)$$

From equation (4), the Renyi entropy of MLTD can further be obtained numerically through software assistance. This is executed by evaluation of the upper limit of the integration (say n), and other parameters $s,\gamma\ and\ i$.

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Table 1 (a-d): *Numerical analysis on MLTD entropy* (a)

n	$i=0$ $a=0.1$ $s=0.1$	$i=0$ $a=0.2$ $s=0.2$	$i=0$ $a=0.3$ $s=0.3$	$i=0$ $a=0.4$ $s=0.4$	$i=0$ $a=0.5$ $s=0.5$
10	2.2028	2.1360	2.0166	1.8287	1.5716
20	2.9188	2.7773	2.5040	2.1352	1.7294
50	3.7799	3.3413	2.7462	2.2039	1.7430
75	4.1059	3.4593	2.7604	2.2045	1.7430
100	4.3066	3.4999	2.7619	2.2045	1.7430
150	4.5358	3.5199	2.7621	2.2045	1.7430

(b)

n	$i=0$ $a=0.6$ $s=0.6$	$i=0$ $a=0.7$ $s=0.7$	$i=0$ $a=0.8$ $s=0.8$	$i=0$ $a=0.9$ $s=0.9$	$i=0$ $a=0.99$ $s=0.99$
10	1.2489	0.8455	0.2661	-1.0394	-21.2805
20	1.3164	0.8072	0.2744	-1.0363	-21.2749
50	1.3183	0.8704	0.2744	-1.0363	-21.2749
75	1.3182	0.8704	0.2744	-1.0363	-21.2749
100	1.3182	0.8704	0.2744	-1.0363	-21.2749
150	1.3182	0.8704	0.2744	-1.0363	-21.2749

(c)

n	$i=1$ $a=0.6$ $s=0.6$	$i=1$ $a=0.7$ $s=0.7$	$i=1$ $a=0.8$ $s=0.8$	$i=1$ $a=0.9$ $s=0.9$	$i=1$ $a=0.99$ $s=0.99$
10	-0.79152	-2.18667	-4.91953	-13.0178	-157.927
20	-0.72188	-2.15767	-4.90787	-13.0124	-157.914
50	-0.71927	-2.15738	-4.90784	-13.0124	-157.914
75	-0.71927	-2.15738	-4.90784	-13.0124	-157.914
100	-0.71928	-2.15738	-4.90784	-13.0124	-157.914
150	-0.719275	-2.15738	-4.90784	-13.0124	-157.914

(d)

n	$i=2$ $a=0.1$ $s=0.1$	$i=2$ $a=0.2$ $s=0.2$	$i=2$ $a=0.3$ $s=0.3$	$i=2$ $a=0.4$ $s=0.4$	$i=2$ $a=0.5$ $s=0.5$
10	1.9956	0.9845	0.6583	0.1745	-0.52937
20	1.7722	1.5022	1.0587	0.4339	-0.38877
50	2.6945	2.1149	1.3309	0.5165	-0.37095
75	3.0199	2.2317	1.3449	0.5170	-0.37094
100	3.2201	2.2719	1.3464	0.5170	-0.37094
150	3.4483	2.2915	1.3465	0.5170	-0.37094

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The numerical analysis for MLTD entropy is presented in Table 1(a-d). We observe explicitly that entropy can be positive or negative. More so, for any two consecutive values of the parameter s, say (s_i *and* s_j), (as we have, horizontally across the table) the Renyi entropy of MLTD $E_R(\mathrm{u}, s)$, satisfies the proposition according to Golshani and Pasha (2010):

$$\rightarrow \qquad s_i < s_j \quad \rightarrow \quad E_{Ri}(\mathrm{u}, s) \geq E_{Rj}(\mathrm{u}, s)$$

Asymptotic Distribution of MLTD

The concept of extremum is adopted in the approximations of the CDFs of a statistical estimator.

- If $X_i,\ i = 1,2, \dots, m$ is a sequence of random variables with CDF $G(x)$
- If $G_m(x) \rightarrow G(x)$ as n increases

then, there exist a limiting distribution. Now, if a sequence of random variables X_i are independent, identically distributed with means zero and unit variances; and $S_m = X_1 + X_2 + \cdots + X_m$, then $\lim \mathrm{sum}_{m\to\infty} \frac{|S_m|}{\sqrt{2mloglogm}} = 1$. Bensid and Zeghdoudi (2017) derived the asymptotic distributions of the sample extremum of the family of Lindley distributions using l'Hopital's rule. In the same manner, we examine the extreme behavior for a MLTD; where the asymptotic distribution of sample extremum $U_{1;m} = \min(U_1, .., U_m)$ and $U_{m;m} = \max(U_1, .., U_m)$ can respectively be derived as the $\lim_{t\to 0} \frac{F(tu)}{F(t)}$, and $\lim_{t\to\infty} \frac{1-F(t+u)}{1-F(t)}$.

$$\lim_{t\to 0} \frac{F(tu)}{F(t)} = \lim_{t\to 0} \left\{\left[1 + \frac{e^{-\gamma tu}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[tu]-\gamma\mathrm{Sin}[tu]\}}{3+2\gamma^2}\right] \Big/ \left[1 + \frac{e^{-\gamma t}\{-3(1+t^2)+\gamma^2\mathrm{Cos}[t]-\gamma\mathrm{Sin}[t]\}}{3+2\gamma^2}\right]\right\}$$

$$= \lim_{t\to 0} \frac{(3+2\gamma^2)+(e^{-\gamma tu}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[tu]-\gamma\mathrm{Sin}[tu]\})}{(3+2\gamma^2)+(e^{-\gamma t}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[t]-\gamma\mathrm{Sin}[t]\})}$$

$$= 1$$

Now, for $\quad \lim_{t\to 0} \frac{F(tu)}{F(t)} = x \lim_{t\to 0} \frac{f(tu)}{f(t)}$

$$\lim_{t\to 0} \frac{F(tu)}{F(t)} = x \lim_{t\to 0} e^{-\gamma tu}\ \frac{2\gamma(1+\gamma^2)(1+Sin^2(tu/2))}{(3+2\gamma^2)} \Big/ e^{-\gamma t}\ \frac{2\gamma(1+\gamma^2)(1+Sin^2(t/2))}{(3+2\gamma^2)}$$

$$= x \lim_{t\to 0} 1$$

$$= x,\ \sim X_{1;n}\ \text{ minima}$$

$$\lim_{t\to\infty} \frac{1-F(t+u)}{1-F(t)} =$$

$$\lim_{t\to\infty} \left[\left(-\frac{e^{-\gamma(t+u)}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[(t+u)]-\gamma\mathrm{Sin}[(t+u)]\}}{3+2\gamma^2}\right) \Big/ \left(-\frac{e^{-\gamma t}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[t]-\gamma\mathrm{Sin}[t]\}}{3+2\gamma^2}\right)\right]$$

$$= \frac{e^{-\gamma(t+u)}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[(t+u)]-\gamma\mathrm{Sin}[(t+u)]\}}{e^{-\gamma t}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[t]-\gamma\mathrm{Sin}[t]\}}$$

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$$= e^{-\gamma u}, \; X_{n;n} \text{ maxima}$$

By implication, the asymptotic distribution of MLTD extremum for $0 < x < b$ is given by

$$P\{\alpha_n(X_{n:i} - \beta_n) \leq x\} \xrightarrow{d} 1 - \exp(-x)$$
$$P\{\varepsilon_n(X_{n:n} - \tau_n) \leq x\} \xrightarrow{d} \exp(-\exp(-\gamma x))$$

where $\alpha_n, \beta_n, \varepsilon_n \; and \; \tau_n > 0$ are the normalizing constants.

Distribution of Order Statistics

The distribution of order statistics refers to an arrangement of sample values in the ascending order. These statistics depend on the sequential arrangement of values, but not on the values themselves; where the special cases include sample median, minimum and maximum values and quantile. Within the subject of continuous probability distribution, the CDF is used in the study of random samples; and the objective is to achieve the uniform distribution by reducing the analysis to order statistics, see Casella and Berger (2021).

If $U_i, i = 1,2,\ldots,m$ are random variables of size n from a distribution, then the order statistics $X_1 < X_2 <,\ldots,< X_m$ are defined random variables, from the ascending order. The order statistics for MLTD is thus given:

$$G(u) = \sum_{j=k}^{m} \binom{m}{j} [F(u)]^j \; [1 - F(u)]^{m-j}$$

$$= \sum_{j=k}^{m} \sum_{l=0}^{m-j} \binom{m}{j} \binom{m-j}{l} (-1)^l \; F^{j+l}(u)$$

where the corresponding PDF is derived as

$$g(x) = \frac{m!}{(k-1)!(m-k)!} f(u) \, F^{k-1}(u) \, \{1 - F(u)\}^{m-k}$$

But $$\{1 - F(u)\}^{m-k} = \sum_{l=0}^{\infty} \binom{m-k}{l} (-1)^l [F(u)]^l$$

$$\therefore \quad g(x) = \frac{m!\, 2\gamma(1+\gamma^2)}{(3+2\gamma^2)(k-1)!(m-k)!} \{(1 + \delta_s)e^{-\gamma u}\} \left\{1 + \frac{e^{-\gamma u}\{-3(1+\gamma^2) + \gamma^2 \text{Cos}[u] - \gamma \text{Sin}[u]\}}{3+2\gamma^2}\right\}^{k-1}$$
$$\left\{\sum_{l=0}^{\infty} \binom{m-k}{l} (-1)^l \left[1 + \frac{e^{-\gamma u}\{-3(1+\gamma^2) + \gamma^2 \text{Cos}[u] - \gamma \text{Sin}[u]\}}{3+2\gamma^2}\right]^l\right\} \quad (5)$$

By implication, the PDF of smallest order statistics is obtained evaluating $j = k = 1$ in equation (5):

$$g_{1:n} = \frac{2\gamma(1+\gamma^2)m}{(3+2\gamma^2)} [(1 + \delta_s)e^{-\gamma u}] \left\{\sum_{l=0}^{\infty} \binom{m-1}{l} (-1)^l \left[1 + \frac{e^{-\gamma u}\{-3(1+\gamma^2) + \gamma^2 \text{Cos}[u] - \gamma \text{Sin}[u]\}}{3+2\gamma^2}\right]^l\right\}$$

where the corresponding PDF of maximum order statistics is obtained at $j = k = n$ in equation (5)

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$g_{n:n} = \frac{2\gamma(1+\gamma^2)m}{(3+2\gamma^2)}[(1+\delta_s)e^{-\gamma u}]\left\{1+\frac{e^{-\gamma u}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u]-\gamma\mathrm{Sin}[u]\}}{3+2\gamma^2}\right\}^{n-1}\left\{\sum_{l=0}^{\infty}\binom{0}{l}(-1)^l\left[1+\frac{e^{-\gamma u}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u]-\gamma\mathrm{Sin}[u]\}}{3+2\gamma^2}\right]^l\right\}$

More so,

$$G_{U_{(1)}}(u) = P\{min[X_i] \leq x\} = 1-[1-F_U(u)]^n$$

$$G_{U_{(n)}}(u) = P\{max[X_i] \leq x\} = [F_U(u)]^n$$

are other special cases of order statistics.

Different Parameter Estimation Approach

Some methods for parameter estimation are presented here; which include Cramer-von Mises (CVM) estimator, Minimum distance estimation (MDE), maximum spacing estimation (MS_pE) and maximum likelihood estimation (MLE).

Let a distribution $G(x)$ be defined for an independent and identical random sample $X_i, i = 1,2,\dots,m$ from a population; where $G(x,\theta)\colon \theta \in \Theta$ and $\Theta \subseteq R^j \ for\ j \geq 1$. Let empirical distribution function (EDF) be $G_m(x)$ on the premises of the sample; then

$$C_{vm}(\hat{\theta}) = (12m)^{-1} + \sum_{i=1}^{m}\left[G\left(x_{(i)}\right) - \frac{2i-1}{2m}\right]^2 \quad (6)$$

defines the Cramer-von Mises (CVM) estimator for parameter θ; where the goal is to minimize the function $C_{vm}(\hat{\theta})$, see Bee (2023). Now, by substituting the CDF of MLTD as in equation (3) in (6), we obtain

$$C_{vm}(\hat{\gamma}) = (12m)^{-1} + \sum_{i=1}^{m}\left[\left(1+\frac{e^{-\gamma u_{(i)}}\{-3(1+\gamma^2)+\gamma^2\mathrm{Cos}[u_{(i)}]-\gamma\mathrm{Sin}[u_{(i)}]\}}{3+2\gamma^2}\right) - \frac{2i-1}{2m}\right]^2 \quad (7)$$

Furthermore, we can minimize equation (7) using software algorithm to obtain the estimates of CVM; or we can maximize it by solving $(\partial\ C_{vm}(\hat{\gamma}))/\partial\gamma = 0$.

Nombebe et al. (2023) carried out a comparative study among estimation methods with emphasis on the Minimum Distance Estimation. Although the MDE is found to be consistent and asymptotically normal, it is shows to be statistically inefficient when compared to the MLE. However, the omission of the Jacobian, which is consistent with likelihood functions, is observed to be accountable for this limitation. Now, if there exist a $\hat{\theta} \in \Theta$ such that

$$d\{G(x,\hat{\theta}),\ G_m(x)\} = \inf[d\{G(x,\theta),\ G_m(x)\}\theta \in \Theta$$

where $d\{.,.\}$denotes any distance function, and $G_m(x) = \frac{1}{m}\sum_{i=1}^{m} 1_{[t,\infty]}(x)$; where $1_{[t,\infty]}$ is a characteristic function for $[t,\infty]$; then $\hat{\hat{\theta}}$ is called the MDE, see Drossos and Philippou (1980).

Maximum spacing estimation as originally applied by Cheng and Amin (1983) is based on the probability integral transform; where derived random samples that are independent from

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any random variable are uniformly distributed owing to the CDF of the random variable. In other words, MSE makes selection of parameter values that possibly render the observed data in definite quantitative measure of uniformity.

Let a corresponding ordered sample be given as $[x_{(1)}, \dots, x_{(n)}]$, and spacing be defined as $A_i(\theta)$; where maximum spacing estimator of $\hat{\theta}$ is a value that maximizes the function:

$$\hat{\theta} = \underbrace{\text{argmax}}_{\theta \in \Theta} \mathcal{M}_{sp}$$

$$\to \qquad \mathcal{M}_{sp} = \frac{1}{m+1} \sum_{i=1}^{m+1} ln\{ A_i(\theta)\}$$

where $A_i(\theta) = G\left(x_{(i)}\right) - G\left(x_{(i-1)}\right)$; $G\left(x_{(0)}\right) = 0$; $G\left(x_{(m+1)}\right) = 1$; see [Almetwally et al. (2023)].

In terms of MLTD, we obtain

$$\mathcal{M}_{sp}(\gamma) = \frac{1}{m+1} \sum_{i=1}^{m+1} ln\left[F\left(u_{(i)}\right) - F\left(u_{(i-1)}\right)\right]$$

$$= \frac{1}{m+1} \sum_{i=1}^{m+1} ln\left[\left(1 + \frac{e^{-\gamma u_{(i)}}\left\{\begin{matrix} -3(1+\gamma^2)+\gamma^2 \text{Cos}[u_{(i)}] \\ -\gamma \text{Sin}[u_{(i)}] \end{matrix}\right\}}{3+2\gamma^2}\right) - \left(1 + \frac{e^{-\gamma u_{(i-1)}}\left\{\begin{matrix} -3(1+\gamma^2)+\gamma^2 \text{Cos}[u_{(i-1)}] \\ -\gamma \text{Sin}[u_{(i-1)}] \end{matrix}\right\}}{3+2\gamma^2}\right)\right] \quad (8)$$

The estimates of MS$_p$E can further be obtained by maximizing equation (8) at $\frac{\partial \mathcal{M}_{sp}(\gamma)}{\partial \gamma} = 0$.

Lastly, the goal of MLE is to obtain the parameter values of a probability function that optimizes the likelihood function over the parameter space. This can be studied under different data conditions, which include uncensored data and or censoring, see details: Fang et al. (2015) and Kinaci et al. (2014). The concept of censoring is what ensures that data observations are correctly extracted; in the sense that truncated events are not treated like exhaustive investigations. In general, the likelihood functions for the both data conditions are respectively given as:

$$L(x, \theta) = \prod_{i=1}^{n} [g(x_i)]$$

$$L(x, \theta) = \frac{M!}{(M-m)!} \{\prod_{i=1}^{n} g(x_i)\}\{1 - G(x_T)\}^{M-m} \quad (9)$$

where $g(x_i)$ and $G(x_i)$ are the PDF and CDF of a distribution with an independent random observations $x_i, i = 1, 2, \dots, m$; M is the number of specimens being investigated or number of trials. Now, if the fixed time or cycle or count to an event (say failure time) is x_0, then for Type 1 censoring according to equation (9), the time of termination $x_T = x_0$; and $x_T = x_n$ for Type 2 case. However, for this study, we have our emphasis only on the complete or uncensored investigations.

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Now, let $u_i, i = 1,2, \dots, m$ be a vector of observations from MLTD, and then the log-likelihood for the complete data is defined by:

$$l(u,\gamma) = logL(u,\gamma) = \sum_{i=1}^{n} log\{f(u,\gamma)\}$$

$$l(u,\gamma) = \sum_{i=1}^{n} \log\left\{ \frac{2\gamma(1+\gamma^2)}{(3+2\gamma^2)} \left[1 + Sin^2\left(\frac{u}{2}\right)\right] e^{-\gamma u}\right\}$$

$$= log\left[\frac{2\gamma(1+\gamma^2)}{(3+2\gamma^2)}\right]^m + \sum_{i=1}^{m} \log\left[1 + Sin^2\left(\frac{u_i}{2}\right)\right] - \gamma\sum_{i=1}^{m} u_i$$

$$= mlog\,(2\gamma) + mlog\,(1+\gamma^2) - mlog\,(3+2\gamma^2) + \sum_{i=1}^{m} \log\left[1 + Sin^2\left(\frac{u}{2}\right)\right] - \gamma\sum_{i=1}^{m} u_i \qquad (10)$$

The score function for equation (10) is defined by

$$\frac{\partial l}{\partial \gamma} = \frac{m}{\gamma} + \frac{2\gamma m}{1+\gamma^2} - \frac{4\gamma m}{3+2\gamma^2} - \sum_{i=1}^{m} u_i$$

$$\frac{m}{\gamma} + \frac{2\gamma m}{1+\gamma^2} - \frac{4\gamma m}{3+2\gamma^2} - \sum_{i=1}^{m} u_i = 0$$

$$\frac{m}{\gamma} + \frac{2\gamma m}{1+\gamma^2} - \frac{4\gamma m}{3+2\gamma^2} - \sum_{i=1}^{m} u_i = 0$$

$$\frac{3+7\gamma^2+2\gamma^4}{3\gamma+5\gamma^3+2\gamma^5} = \frac{\sum_{i=1}^{n} u_i}{m}$$

$$(3+7\gamma^2+2\gamma^4) - \bar{u}(3\gamma+5\gamma^3+2\gamma^5) = 0 \qquad (11)$$

The polynomial structure of the equation in (11) suggests that multiple roots will be obtained; hence we resort to numerically optimization such as Newton's method, to realize the estimates of the estimator $\hat{\gamma}$. More so, we might be interested in studying the following properties of MLE:

- The estimator $\hat{\gamma}$ of γ is bias if $E[\hat{\gamma}] - \gamma \neq 0$
- The estimator $\hat{\gamma}$ of γ is consistent if $\hat{\gamma} \overset{p}{\rightarrow} \gamma \ as\ m \rightarrow \infty$. This also implies that

$$\lim_{m\to\infty} P(\,|\hat{\gamma} - \gamma| > \epsilon) = 0$$

- The estimator $\hat{\gamma}$ of γ is asymptotically normal:

$$\sqrt{m}(\hat{\gamma} - \gamma) \overset{D}{\rightarrow} N\left(0, \frac{1}{I(\gamma)}\right)$$

These will be investigated under Monte Carlo simulation.

Average Bias and Mean Square Error (MSE) under Monte Carlo Approach

In this section, the stability of the MLE of MLTD is investigated, through Monte Carlo simulation study. For different sample sizes $n = 15,\ 30,\ 55,\ 80, 100, 200\ and\ 350$, the experiment engages a 10000 times repeated trials. In executing the algorithm for the biasedness, consistency and asymptotic normality objectives, a quantile function from MLTD

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comes handy; as it is the kernel in the whole sequence of the data codes. The inverse cumulative function of MLTD is a derivative from $F^{-1}(p) = u$, where $F(u) = p,\ 0 < p < 1$. This is given as

$$[(3+2\gamma^2)(1-p)] + [e^{-\gamma x}\{-3(1+\gamma^2) + \gamma^2 \mathrm{Cos}(u) - \gamma \mathrm{Sin}(u)\}] = 0$$

Opone (2021) detailed the Monte Carlo algorithm, where the average bias and mean square error of the estimator $\hat{\gamma}$ is given by

- $Average\ Bias = \left[\ \frac{1}{M}\sum_{i=1}^{M}(\hat{\gamma}_i - \gamma)\right]$
- $MSE = \left[\ \frac{1}{M}\sum_{i=1}^{M}(\hat{\gamma}_i - \gamma)^2\ \right]$

Table 2: *Average Bias and MSE of the Estimator $\hat{\gamma}$*

Parameter	N	Average Bias (γ)	MSE (γ)
	15	0.31626	5.27540
	30	0.19005	3.57610
$\gamma = 0.1$	55	0.10085	2.24623
	80	0.09485	2.05691
	100	0.05852	1.35571
	200	0.03523	1.08446
	350	0.01861	0.45169
	15	0.06401	0.25472
	30	0.03385	0.12843
$\gamma = 0.5$	55	0.01876	0.07529
	80	0.01232	0.04211
	100	0.01029	0.03573
	200	0.00621	0.02137
	350	0.00305	0.01258
	15	-0.00161	0.03749
	30	-0.00182	0.01141
$\gamma = 1.5$	55	-0.00250	0.00905
	80	-0.00414	0.00884
	100	-0.00560	0.00546
	200	-0.01141	0.00288
	350	-0.02391	0.00199
	15	-0.00279	0.15275
	30	-0.00635	0.08667
$\gamma = 2.5$	55	-0.01125	0.03772
	80	-0.01708	0.02354
	100	-0.02033	0.02418
	200	-0.04072	0.01381
	350	-0.07141	0.00554

The estimates for the average bias and mean square error are presented in Table 2, and at different selected values of the parameter. Apparently, from the Table, we deduce that the

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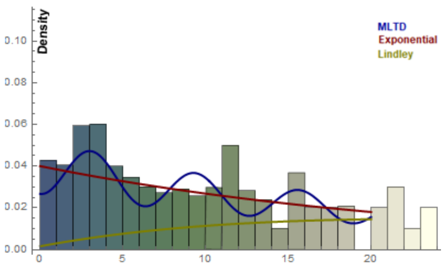
estimates of the average bias and mean square error decrease as the sample size n increases. This simply indicates that the estimator of MLTD is consistent and asymptotically stable.

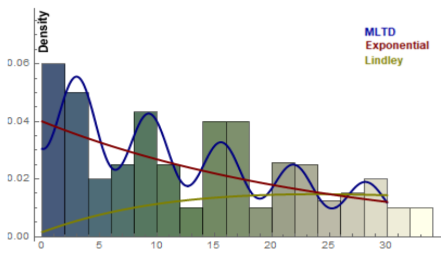
Simulative and Real Life Application

In this part of the research, we engage fixed simulation of 100 sample data from MLTD over different range of supports $x \in R$; where $x \leq n$ for $n = 20, 30, 40$ and 50. This will in essence, x-ray the sinusoidal behavior of MLTD in the modeling of unimodal, bimodal, tri-modal and multimodal decreasing trend. More so, we take in to consideration the tendencies of MLTD at different levels of the selected parameters. For brevity purposes, we consider these parameters $\gamma = 0.04, 0.12$ and 0.25 alongside some inferential criteria; which include: Anderson Darling (A^{d*}), Cramer-von Mise (C^{v*}), Kolmogrove (K^{s*}), Kuiper(K^{u*}), Pearson X^2 (P^{s*}), Watson U^2 (W^{u*}) and the p-value. These statistics project the comparative fitness of distributions mirroring a particular data; where lesser measures indicate better fit, apart from the p-value that reads otherwise. The Exponential and Lindley distributions as monotone decreasing probability functions are also juxtaposed in the analogy.

Table 3: Performance comparison at $\gamma = 0.04$

$x \leq n$	Model	A^{d*}	C^{v*}	K^{s*}	K^{u*}	P^{s*}	W^{u*}	P-value
	MLTD	26.629	5.5546	0.4549	0.4612	69.78	1.3306	1.8e-13
20	Lindley	147.53	22.503	0.7949	0.8037	464.9	4.8516	0
	Exp	26.629	5.5454	0.4493	0.4662	75.76	1.3857	1.9e-13
	MLTD	11.370	2.1318	0.2973	0.3068	39.88	0.5885	6.47e-6
30	Lindley	100.33	17.066	0.6487	0.6587	281.94	3.3221	0
	Exp	11.225	2.1331	0.3011	0.3093	43	0.6284	6.43e-6
	MLTD	6.3006	1.0982	0.2040	0.2043	27.66	0.2585	0.00145
40	Lindley	83.631	13.933	0.5153	0.5247	204.98	2.3441	0
	Exp	6.3382	1.0993	0.2018	0.2063	32.6	0.2720	0.00144
	MLTD	2.8644	0.4054	0.1373	0.1489	16.74	0.2085	0.0699
50	Lindley	63.577	11.127	0.4819	0.4909	167.02	1.7477	0
	Exp	2.9380	0.4099	0.1394	0.1637	17.52	0.2208	0.068

(a) 

(b) 

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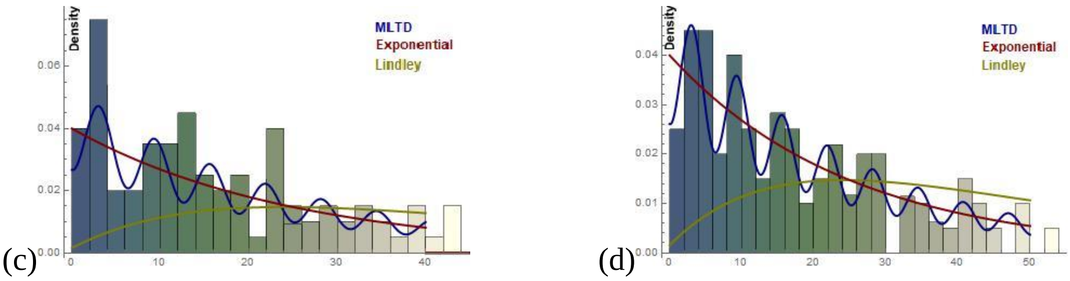

Figure 4: Density fit for MLTD simulations at $\gamma = 0.04$

Table 4: Performance comparison at $\gamma = 0.12$

$x \leq n$	Model	A^{d*}	C^{v*}	K^{s*}	K^{u*}	P^{s*}	W^{u*}	P-value
	MLTD	2.9974	0.4473	0.1387	0.1857	37.54	0.2319	0.0544
20	Lindley	51.83	9.2086	0.4039	0.4128	145.7	1.45.7	3.3e-16
	Exp	3.3986	0.4933	0.1454	0.2229	51.32	0.3636	0.0414
	MLTD	1.4761	0.2197	0.1262	0.1548	22.46	0.1693	0.2320
30	Lindley	27.362	4.2595	0.3139	0.3200	103.32	0.5829	1.2e-10
	Exp	1.7068	0.2409	0.1339	0.1623	25.84	0.1478	0.2
	MLTD	1.0105	0.1846	0.0926	0.0937	7.38	0.0673	0.2990
40	Lindley	43.494	7.2195	0.3805	0.3889	141.28	1.1983	0
	Exp	0.6824	0.1303	0.0836	0.1036	6.42	0.0665	0.4550
	MLTD	0.4004	0.0462	0.0583	0.1037	7.9	0.0462	0.899
50	Lindley	32.331	5.2724	0.3265	0.3362	99.94	0.8335	7.7e-13
	Exp	0.2459	0.0349	0.0505	0.0861	9.2	0.0285	0.957

Plots

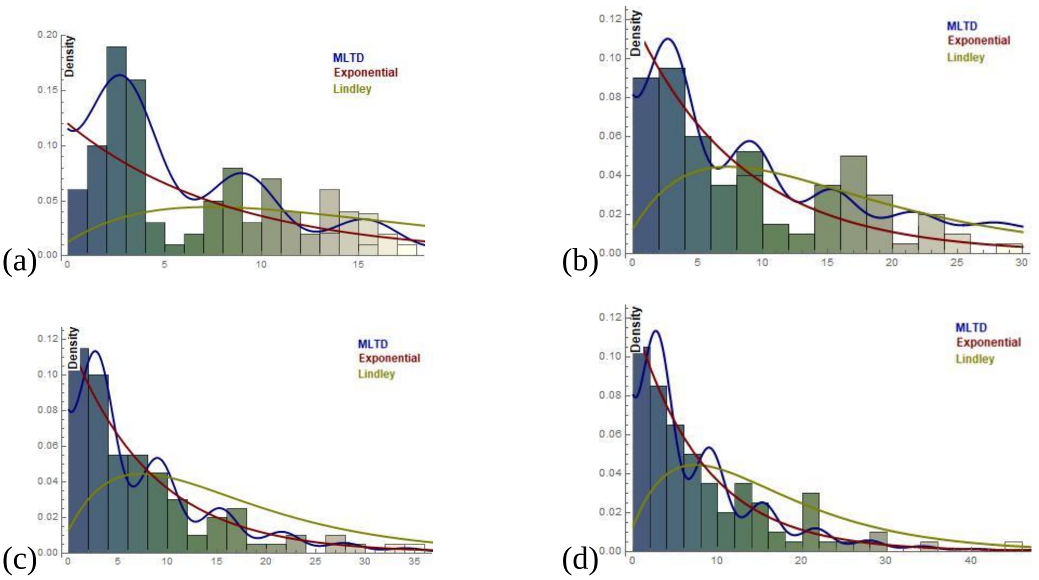

Figure 5: Density fit for MLTD simulations at $\gamma = 0.12$

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Table 5: Performance comparison at $\gamma = 0.25$

$x \leq n$	Model	A^{d*}	C^{v*}	K^{s*}	K^{u*}	P^{s*}	W^{u*}	P-value
20	MLTD	1.2391	0.4059	0.1329	0.1448	8.94	0.1359	0.1697
	Lindley	34.286	6.6444	0.4412	0.4463	98.1	1.4552	7.7e-16
	Exp	1.1558	0.2682	0.1304	0.1290	8.65	0.2189	0.1770
30	MLTD	0.5505	0.0788	0.0642	0.0908	14.66	0.0388	0.699
	Lindley	26.956	5.2050	0.3435	0.3451	65.62	0.9564	1.1e-12
	Exp	0.6751	0.0798	0.0592	0.1001	12.06	0.0772	0.693
40	MLTD	0.6270	0.0745	0.0644	0.1023	10.76	0.0654	0.724
	Lindley	25.385	4.9039	0.3246	0.3305	61.72	0.8413	4.9e-12
	Exp	1.6559	0.2273	0.1132	0.1462	18.3	0.1477	0.151
50	MLTD	1.2725	0.2521	0.1118	0.1238	21.16	0.1046	0.186
	Lindley	32.579	6.4987	0.4189	0.4266	79.14	1.3661	1.2e-15
	Exp	1.5402	0.2077	0.1048	0.1199	19.86	0.2746	0.197

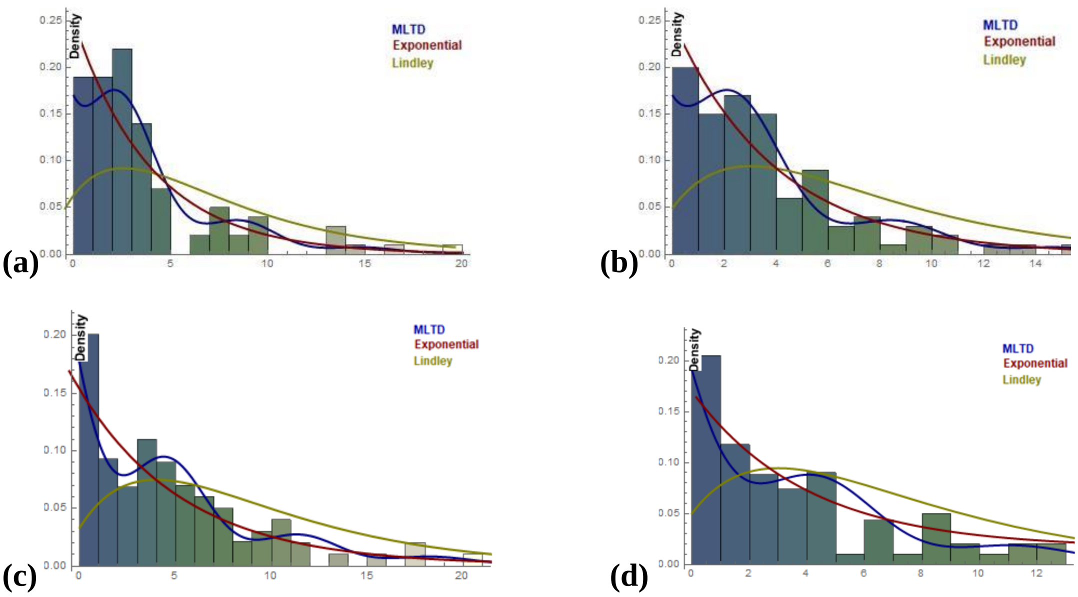

Figure 6: Density fit for MLTD simulations at $\gamma = 0.25$

Tables 3-5; and Figures 4-6, present the numerically and graphical analysis of the simulation density fit. Firstly, the graphical study is indicative of the appropriate choice of parameters, in quest to fit a real life data exhibiting different decreasing trend modal outcomes. Of course it is expected that the proposed distribution superiorly fits the simulated samples at different levels of the parameters. However, by careful observation, we underscore the different scenarios where MLTD may not stand the chance of optimum data fitness. This is seen in:

- Table 4 and or Figure 5c at parameter $\gamma = 0.12$, for $x \leq 40$
- Table 4 and or Figure 5d at parameter $\gamma = 0.12$, for $x \leq 50$
- Table 5 and or Figure 6a at parameter $\gamma = 0.25$, for $x \leq 20$

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- Table 5 and or Figure 6d at parameter $\gamma = 0.25$, for $x \leq 50$

This implies that the proposed distribution is a good completion, conditionally for outcomes that exhibit *obvious* sinusoidal trends.

Finally, we fit the MLTD and already mentioned one parameter existing lifetime distributions to a real data which denotes the exceedance of Wheaton River flood applied in Akinsete et al. (2008).

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0

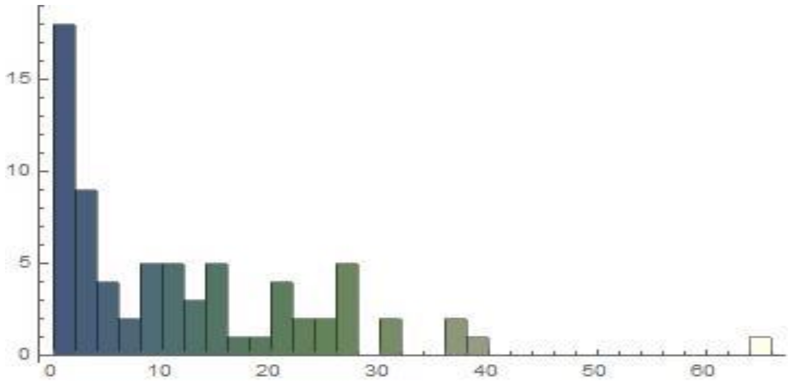

Figure 7: Graphical plot showing the histogram of the flood data

The data showed to have approximately 3-4 modes as observed in Figure 7. As a result, we make an appropriate choice of parameter which could limit the modal structure of the density fit, sizeable to that of data. This is notably one edge Mathematica has over R software in model fitting of this order. Akaike information criterion (A^{ic*}) and log-likelihood *(-log L)* will be used alongside some of the above mentioned inferential measures.

Table 7: *Parameter estimates and corresponding standard error statistics*

Model	γ	Std error
MLTD	0.0823	0.00972
Lindley	0.1530	0.01281
Exp	0.0819	0.00966

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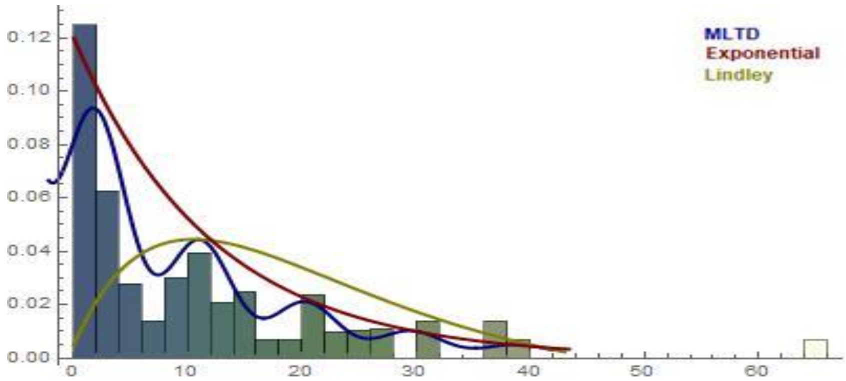
Table 8: *Distribution fit inference*

Model	A^{d*}	C^{v*}	K^{s*}	A^{ic*}	$-logL$
MLTD	1.6961	0.2523	0.1414	510.11	-254.055
Lindley	7.4215	0.8183	0.2408	530.42	-264.211
Exp	222.03	22.226	0.9996	506.25	-252.128



Based on the inferential measures, Table indicates that the proposed Modified Lindley Trigonometric Distribution is a good completion compared to some one parameter distributions considered. More so, Figure which supports the empirical facts evidently shows that the proposed distribution fits outcomes that exhibit sinusoidal decreasing trends; which is a common characteristic of some real events.

CONCLUSION

In this development, a class of sinusoidal family of distributions termed Modified Lindley Trigonometric Distribution (MLTD) was proposed and some statistical properties were derived. Among the properties, generating multimodal features with just one parameter stands out; where the trigonometric interjected function is accountable for this novelty. An applicative simulation study was carried out to demonstrate the likelihood of the proposed distribution. Finally, a real life data was employed to give essence to the applicability of MLTD and the outcome of the result showed that the proposed distribution is a model prototype for decreasing trend sinusoidal phenomena.

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