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OPTIMIZED INVESTMENT STRATEGY AND PROPORTIONAL REINSURANCE OF INSURANCE COMPANY UNDER ORNSTEIN–UHLENBECK AND GEOMETRIC BROWNIAN MOTION MODELS

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ABSTRACT: *In this work we investigated the optimization of an insurer's investment strategy and the proportional reinsurance rate of his portfolio under power utility preference in the cases of correlated and uncorrelated Brownian motions. The market in which the insurer traded two assets; a risky asset whose price process was governed by the geometric Brownian motion (GBM) and riskless asset that had its price driven by the Ornstein-Uhlenbeck stochastic model. We derived the required Hamilton-Jacobi-Bellman Equation (HJB) by applying the maximum principle of dynamic programming and the elimination of dependency on variables was used to obtain the analytic solutions of optimized investment strategy and the proportional reinsurance rate.*

KEYWORDS: Optimized investment strategy, Proportional reinsurance, Insurance company, Ornstein–Uhlenbeck model, Geometric Brownian motion model.

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INTRODUCTION

Insurance is one of the social sciences designed for risk taking. People are, on a daily basis, exposed to an uncountable number of risks affecting them and their properties. By paying a fixed cost called insurance premium to the reinsurer, the insurer transfers his risks to an insurer. The insurer who gives insurance to another insurer is called the reinsurer. To balance their profit and risk both the insurers and reinsurers use investment and reinsurance. A transaction where the reinsurer agrees to indemnify an insurer against all or some of the loss that the latter may sustain under a policy or policies that he has issued having been paid the required premium is called reinsurance.

Among the many contributors who have studied the maximization or minimizing the probability of ruin of the utility of terminal value for the insurer is Brown [1]. Brown contributed by giving an analytic solution to the problem of a firm that maximized her exponential utility of terminal wealth and minimized the probability of ruin where surplus process is given by Lundberg risk model.

The problem of optimal reinsurance-investment in a stock market where the risky asset followed the constant elasticity of variance and Jump-diffusion risk model was done by Zhibin and Bayraktar [2]. They obtained an explicit expression for the optimal strategies and value function which they demonstrated using numerical examples to express the impact of model parameters on the optimal strategies.

Yang and Jiaqin [3] studied the optimal investment-consumption-insurance problem where a random parameter was involved. They discussed the optimal investment, consumption, and life insurance purchase problem for a wage earner in the market, complete with a stream of Brownian motion.

The work of Deng et al. [4] gave a solution to the optimal proportional reinsurance and investment for a constant elasticity of variance model under variance principle. They assumed that the insurer's surplus process followed a jump-diffusion process. In this work it was assumed that the insurer can purchase proportional reinsurance through the variance principle and invest in a risk-free asset and a risky asset whose price is modeled by a CEV model. They obtain the techniques of stochastic control theory and closed—form expression for the value functions and optimal strategies.

The case of annuity contracts under the constant elasticity of variance (CEV) model was studied by Jianwei [5]. He derived the explicit solution for the power and exponential utility functions in two different periods, before and after retirement for the optimal investment strategy.

For the optimal portfolios of an insurer and a reinsurer under proportional reinsurance and power utility preference in which the insurer's and the reinsurer's surplus processes were approximated by geometric Brownian motion with drift was investigated by Ihedioha and Osu [6].

Li et al. [7] studied a time-consistent reinsurance-investment strategy for a mean-variance insurer under stochastic interest rate model and inflation risk and derived the time-consistent reinsurance-investment strategies as well as the corresponding value function for the mean-variance problem explicitly using numerical example.

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The case of optimal proportional reinsurance under the criteria of maximizing the expected utility and minimizing the value at risk was studied by Zhibin and Guo [8]. They proved the existence and uniqueness of the optimal strategies and Pareto optimal solution. They also obtained a relationship between the optimal strategies.

The case of proportional reinsurance and investment strategies in which the claim process is assumed to follow a Brownian motion with drift and the price process of the risky asset described by the constant elasticity of variance model was investigated and optimal solutions obtained for the reinsurance and investment strategies by Gu *et al*. [9].

This study considered the case of an insurer who traded two assets in a complete market where the risky asset was governed by the geometric Brownian motion and the risk-less asset's rate of return driven by the Ornstein–Uhlenbeck model. The impact of the correlation of the Brownian motions on the insurer's optimal investment strategy and reinsured proportion rate shall be examined. We shall use the maximum principle of dynamic programming to obtain the Hamilton-Jacob-Bellman (HJB) equation from which we shall obtain the reinsurer's optimal investment in the risky and the optimal reinsured proportion rate and investigate the implications of the correlation of the Brownian motions.

PREAMBLES

Brownian Motion

Brownian motion is regarded as a simple continuous stochastic process that is widely used in finance and physics for modeling random behavior that evolves over time. An example of such behavior is the random movements of molecules of gas or fluctuation in an asset's price.

In mathematics, Brownian motion is described by the Wiener process as a continuous-time stochastic process named in honor of Norbert Wiener.

The Wiener process W_t is characterized by the following four facts;

1. $W_0 = 0$
2. $W_t \; is \; almost \; surely \; continuous$
3. $W_t \; has \; independent \; increments$
4. $W_t - W_s \sim N(0, t-s), (for \; 0 \leq s \leq t)$

$N(\mu, \sigma^2)$ denotes the normal distribution with expected value of μ and variance (σ^2).The condition that it has independent increment means that if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $W_{t_1} - W_{s_1} \; and \; W_{t_2} - W_{t_2}$ are independent random variables.

Geometric Brownian Motion

A stochastic process $S(t)$ is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation (SDE)

$$dS(t) = S(t)[\mu dt + \sigma dZ(t)]$$

where $Z(t)$ is a Wiener process or Brownian motion, μ and σ are constants.

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Ornstein-Uhlenbeck Model

The Ornstein-Uhlenbeck process is one of the several approaches used to model (with modifications) interest rate, and commodity price stochastically. An Ornstein-Uhlenbeck process $x(t)$, satisfies the following stochastic differential equation:

$$dx(t) = \theta\big(\mu - x(t)\big)dt + \sigma dZ(t)$$

where $\theta > 0, \mu \; and \; \sigma > 0$ are parameters and $dZ(t)$denotes the Wiener process. It is also mentioned as Vasicek model.

Dynamic Programming

Dynamic programming or recursive optimization is a technique that is used for obtaining solutions for multistage decision problems. There is no standard mathematical formulation of the dynamic programming for each problem depending on the variable given, and the objective of the problem, one has to develop a particular equation to fit for solution. Nowadays, applications of dynamic programming are done in almost day-to-day managerial problems, such as inventory problems, waiting line problems, resource allocation and so on. Dynamic programming may be classified depending on the nature of data available as deterministic and stochastic or probabilistic models. In deterministic models, the outcome at any decision stage is uniquely determined and known. This technique was developed by Richard Bellman in the early (1950) principle of optimality: this principle implies that a wrong decision taken at a stage does not prevent from taking optimal decision for the remaining stages. That principle is the firm base for dynamic programming technique.

Maximum Principle

Maximum principle is used in optimal control theory to find the base possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls. It was formulated in (1956) by the Russian mathematician Lev Pontryagin and his students. It has as a special case the Euler-Lagrange equation of the calculus variations. The principle states, informally, that the control Hamiltonian must take an extreme value over control in the set of all admissible controls. Whether the extreme value is maximum or minimum depends both on the problem and on the sign convention used to define the Hamiltonian. The normal convention, which is the one used in Hamiltonian, leads to a maximum hence maximum principle but the sign convention used in this article makes the extreme value a minimum.

If u is the set of values of permissible control, then the principle states that the optimal control u^* must satisfy: $H(x^*(t), u^*(t), \lambda^*(t), t) \le H(x^*(t), u, \lambda^*(t), t), \forall \, u \in U, t \in [t_o, t_f]$, where$x^* \in c^1[t_o, t_f]$ is the optimal state trajectory (a special type of optimization problem where the decision variables are functions rather than real numbers) and $\lambda^* \in B \vee [t_o, t_f]$ is the optimal costate trajectory.

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Hamilton-Jacobi-Bellman (HJB) Equation

This is a partial differential equation which is central to optimal control theory. The solution of the HJB equation is the value function which gives the minimum cost for a given dynamical system with an associated cost function. When solved locally, the Hamilton-Jacobi-Bellman is a necessary condition, but when solved over the whole of state space, the HJB equation is a necessary and sufficient condition for an optimum. The solution is open loop, but it also permits the solution of the closed loop problem. The Hamilton-Jacobi-Bellman equation can be generalized to the stochastic system as well. The equation is a result of the theory of dynamic programming which was pioneered in the year (1950's) by Richard Bellman and co-workers.

Insurance

Insurance is a contract, represented by a policy, in which an individual or entity receives financial protection or reimbursement against losses from an insurance company. The company pools clients" risks to make payments more affordable for the insured – (retrieved from http://www.investopedia.com/terms/i/insurance.asp)

Reinsurance

Reinsurance, often referred to as "insurance for insurance companies," is a contract between a reinsurer and an insurer. In this contract, the insurance company—the cedent—transfers risk to the reinsurance company, and the latter assumes all or part of one or more insurance policies issued by the cedent. Reinsurance contracts may be negotiated with a reinsurer or arranged through a third party; i.e., a reinsurance broker or intermediary. Reinsurers may also buy reinsurance protection, which is called "retrocession." This is done to reduce any further spread risk and the impact of catastrophic loss events-(retrieved from https://content.naic.org/cipr-topics/reinsurance)

Insurance Company

A company that provides coverage in the form of compensation resulting from loss, damages, injury, treatment or hardship in exchange for premium payments is called an insurance company. An insurance company calculates the risk of occurrence then determines the cost to replace (pay for) the loss to determine the premium amount.

THE MARKET AND MODEL FORMULATION

Assuming an insurance company has a claim process $C(t)$ at time t, described by

$$dC(t) = \alpha dt - \beta dZ_1(t), \tag{1}$$

where α and β are positive constants with $Z_1(t)$ a standard Brownian motion in a complete probability space $\{\Omega, F, (F_t), \wp\}$. If the premium rate is given as

$$p = (1+\theta)\alpha, \tag{2}$$

with security rise premium (safety loading) $\theta > 0$, using equation (1), the surplus process of the company is given by

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$$dS_p(t) = pdt - dC(t)$$
$$= \alpha\theta dt + \beta dZ_1(t). \quad (3)$$

Since the insurance company is permitted to purchase proportional reinsurance to reduce rise and pay reinsurance premium $r(t)$ continuously at the rate $(1+\lambda)\alpha\psi(t)$ with $\lambda > \theta > 0$ is the security risk premium of the reinsurer and $\psi(t)$ is the proportion measured at time t, the surplus of the company is then given as

$$dS_p(t) = \big(\theta - \lambda\psi(t)\big)\alpha dt + \beta\big(1-\psi(t)\big)dZ_1(t)\ , \quad (4)$$

Suppose the insurance company invests her surplus in a market consisting of; a risk–free asset (bond) and a risky asset (stock), with prices $P_b(t)$ and $P_s(t)$ at time t respectively and these price processes are driven by Ornstein–Uhlenbeck and geometric Brownian motion models, respectively, so that we have

$$P_b(t)\ (t) = r(t)P_b(t)\ dt, \quad (5)$$

for the risk–free asset, where

$$dr(t) = \sigma\big(\mu - r(t)\big)dt + \delta dZ_2(t), \quad (6)$$

where σ is the speed of mean reversion, μ the mean level attracting the interest rate, δ the constant volatility of the interest rate, $Z_2(t)$ is another standard Brownian motion, and

$$dP_s(t)\ = P_s(t)\ [\xi dt + \gamma dZ_3(t)], \quad (7)$$

where ξ and γ denote the appreciation rate (mean) and the volatility of the risky asset respectively $Z_3(t)$ is also a standard Brownian motion and,

$$Cov\big(Z_i(t), Z_i(t)\big) = \rho_{ij}t;\ \ i \neq j, i, j = 1,2,3. \quad (8)$$

The insurance company holds the risky asset as long as

$$\xi > r(t). \quad (9)$$

Let $X(t)$ be the amount of money invested in the risky asset by the company at time t, then the amount invested in the risk–free asset at time t is the difference $[W(t) - X(t)]$, where $W(t)$ is the total amount of money for investment.

The investment strategy $[\psi(t); X(t)]$ is said to be admissible if it is $F - progressive$ and satisfies $0 \leq \psi(t) \leq 1$. That is

$$E\left[\int_0^1 X^2(t)dt\right] < \infty. \quad (10)$$

For the admissible strategy $\big(\psi(t), X(t)\big)$, the wealth process of the company is driven by the stochastic differential equation (SDE)

$$dW(t) = \frac{X(t)dP_s(t)}{P_s(t)} + [W(t) - X(t)]\frac{dP_b(t)}{P_b(t)} + dS_p(t). \quad (11)$$

Applying equations (4), (5), and (7) to (11) we get,

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$$dW(t) = X(t)[\xi dt + \gamma dZ_3(t)] + [W(t) - X(t)]r(t)dt + \big(\theta - \lambda\psi(t)\big)\alpha dt + \beta\big(1 - \psi(t)\big)dZ_1(t)\,, \tag{12}$$

Equation (12) simplifies to

$$dW(t) = \big[\big(\xi - r(t)\big)X(t) + r(t)W(t) + \big(\theta - \lambda\psi(t)\big)\alpha\big]dt + \gamma X(t)dZ_3(t) + \beta\big(1 - \psi(t)\big)dZ_1(t), \tag{13}$$

where

$$dt.dt = dt \cdot dZ_1(t) = dt \cdot dZ_3(t) = dZ_1 \cdot dZ_3 = 0\ dZ_1 \cdot dZ_1 = dZ_3 \cdot dZ_3 = dt\ \}\,. \tag{14}$$

The quadratic variation of equation (13) is given by

$$< dW(t) > = [\gamma^2 X^2(t) + \beta^2(1-p)^2 + \rho_{13}\gamma\beta(1-\psi)]dt. \tag{15}$$

Other quadratic variations are

$$\langle dP_s(t)\rangle = \gamma^2 {P_s}^2 dt\,,\quad \langle dP_s dW(t)\rangle = [\gamma^2 X + \rho_{13}\gamma\beta(1-\psi)]P_s dt\ \langle dr(t)\rangle \geq \delta^2 dt\ \langle dr(t)dP_s(t)\rangle \geq \rho_{23}\delta\gamma p_s dt\ \langle dr(t)dW(t)\rangle = [\rho_{12}\delta\beta(1-\psi) + \rho_{23}\delta\gamma X]dt\ \}. \tag{16}$$

Considering the Arrow–Pratt measure of relative risk aversion (RRA) or the coefficient of relative risk aversion for power utility function which is defined as

$$R(w) = \frac{-wV"(w)}{V'(w)}, \tag{17}$$

where w is the investment wealth level, we continue our problem investigating the special case

$$V(w) = \frac{w^{1-\phi}}{1-\phi}, \phi \neq 1, \tag{18}$$

with ϕ as the constant relative risk aversion parameter.

Theorem

The insurance company's optimal investment in the risky asset (stock) is

$$P^*{}_s(t) = \frac{\xi p_s}{\gamma^2\phi} + \frac{\rho_{23}\delta p_s H r}{\gamma^2\phi H} + \frac{(1-\phi)\gamma w p_s}{\phi}$$

with reinsured portion rate of value

$$\psi^*(t) = \frac{\lambda\alpha w}{\beta^2\phi} + \frac{(1-\phi)\rho_{12}\delta w}{\beta\phi r}$$

when the Brownian motions correlate and

$${p_s}^* = \frac{p_s}{\phi}\left[\frac{\xi}{\gamma^2} + (1-\phi)\gamma w\right],$$

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as the optimal investment in the risky asset (stock) and the optimized proportional reinsurance rate is given as

$$\psi^* = \frac{\lambda\alpha w}{\beta^2\phi},$$

when the Brownian motions do not correlate.

Proof:

We derive the Hamilton–Jacobi–Bellman (HJB) equation beginning with the Bellman equation.

$$V\big(t, r(t), W(t)\big) = E[V(t+\Delta t,)r'(t), P'_s(t), W'(t)]\,, \quad (20)$$

where $W'(t)$ denotes the insurance company is wealth a time $(t+\Delta t)$.

Equation (20) can be rewritten as

$$E[V(t+\Delta t,)r'(t), P'_s(t), W'(t) - V(t, r(t), P_s(t), W(t))] = 0\,. \quad (21)$$

The division of equation (21) by Δt and taking limit to zero gives the Bellman equation

$$E\left[\frac{dV}{dt}\right] = 0\,. \quad (22)$$

The dynamic programming maximum principle states that: $\frac{d^2V}{dp_s{}^2}(dp_s)^2$

$$dV = \frac{\partial V}{\partial r}dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial W}dW + \frac{1}{2}\left\{\frac{\partial^2 V}{\partial r^2}(dr)^2 + \frac{\partial^2 V}{\partial W^2}(dW)^2\right\} + \frac{\partial V}{\partial P_s}dP_s\frac{\partial^2 V}{\partial r\partial W}(drdW) + \frac{\partial^2 V}{\partial r\partial P_s}(drdP_s) + \frac{\partial^2 V}{\partial P_s\partial W}(dP_s dW). \quad (23)$$

The application of equations (6), (7), (13), (15) and (16) in (23) gives

$$dV = V_t + V_r[\alpha(\mu - r)dt + \delta dZ_2] + V_w\{[(\xi - r)x + wr + (\theta - X\psi)\alpha]dt + \gamma x dZ_3 + \beta(1-\psi)dZ_1\} + Vp_s[p_s(\xi dt + \gamma dZ_3)] + V_{rw}[\rho_{12}\delta\beta(1-\psi) + \rho_{23}\delta\gamma X]dt + Vrp_s[\rho_{23}\delta\gamma p_s dt] + V_w p_s[\gamma^2 p_s x dt] + \frac{1}{2}\{V_{rr}(\delta^2 dt) + V_{ww}(\gamma^2x^2 + \beta^2(1-\psi)^2)dt + Vp_sp_s(\gamma^2 p_s{}^2 dt)\}. \quad (24)$$

Substituting for dV in equation (22) using equation (24) and taking expectation yields

$$V_t + V_r[\alpha(\mu - r)] + V_w\{[(\xi - r)x + wr + (\theta - X\psi)\alpha]\} + Vp_s[\xi p_s] + V_{rw}[\rho_{12}\delta\beta(1-\psi) + \rho_{23}\delta\gamma X] + Vrp_s[\rho_{23}\delta\gamma p_s] + Vp_s w[\gamma^2 p_s x] + \frac{1}{2}\{V_{rr}(\delta^2) + V_{ww}(\gamma^2x^2 + \beta^2(1-\psi)^2) + Vp_sp_s(\gamma^2 p_s{}^2)\} = 0. \quad (25)$$

Equation (25) is the Hamilton–Jacob–Bellman (HJB) equation.

Differentiating equation (25) with respect to p_s gives

$$\xi V_{p_s} + \delta\gamma V_{r_{p_s}} + \gamma^2{}_{p_s} x V_{wp_s} + p_s\gamma^2 V_{p_sp_s} = 0, \quad (26)$$

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that modifies to

$$p_s{}^* = -\left[\frac{\xi V_{p_s}+\rho_{23}\delta\gamma V_{rp_s}+\gamma^2 X V_{wp_s}}{\gamma^2 V_{p_sp_s}}\right]. \tag{27}$$

Equation (27) is the optimal investment in the risky asset (stock)

Also, differentiation (25) with respect to $\psi(t)$ yields

$$-\lambda\alpha V_w - \beta^2\psi V_{ww} - \rho_{12}\delta\beta V_{rw} = 0, \tag{28}$$

which on simplification becomes

$$\beta^2\psi V_{ww} = -\rho_{12}\delta\beta V_{rw} - \lambda\alpha V_w\ ,$$

from which obtain

$$\psi^* = \frac{-[\lambda\alpha V_w+\rho_{12}\delta\beta V_{rw}]}{\beta^2 V_{ww}}. \tag{29}$$

Equation (29) is the optimal proportional reinsurance rate.

Equations (27) and (29) contain first and second partial differentials, $V_w, V_{p_s}, V_{rp_s}, V_{wp_s}, V_{p_sp_s}$, and V_{ww} that are to be eliminated.

Therefore, we adopt a solution structure of the form

$$V(t,r,p_s,w) = \frac{w^{1-\varphi}}{1-\varphi}G(t,r,p_s), \tag{30}$$

such that at the terminal time T

$$G(T,r,p_s) = 1. \tag{31}$$

This conjectured will help in eliminating the dependency on w. So, we obtain

$$V_{rp_s} = \frac{w^{1-\phi}}{1-\phi}G_{rp_s}, V_{rw} = w^{-\phi}G_r, V_{p_s} = \frac{w^{1-\varphi}}{1-\varphi}G_{p_s}, V_{p_sp_s} = \frac{w^{1-\phi}}{1-\phi}G_{p_sp_s},\ V_{rr} = \frac{w^{1-\phi}}{1-\phi}G_{rr}, V_w = w^{-\phi}G, V_{ww} = -\phi w^{-\phi-1}G, V_{wp_s} = w^{-\phi}G_{p_s}\ \}\ . \tag{32}$$

Applying (30) and (32) in (27) and (29) respectively, we get

$$p_s{}^* = -\left[\frac{\xi\frac{w^{1-\varphi}}{1-\varphi}G_{p_s}+\rho_{23}\delta\gamma\frac{w^{1-\phi}}{1-\phi}G_{rp_s}+\gamma^2 x w^{-\phi}G_{p_s}}{\gamma^2\frac{w^{1-\phi}}{1-\phi}G_{p_sp_s}}\right].$$

$$= -\left[\frac{\xi G_{p_s}}{\gamma^2 G_{p_sp_s}} + \frac{\rho_{23}\delta G_{rp_s}}{\gamma G_{p_sp_s}} + \frac{(1-\phi)xwG_{p_s}}{G_{p_sp_s}}\right], \tag{33}$$

for the investment in the risky asset and

$$\psi^* = \frac{-[\lambda\alpha w^{-\phi}G+\rho_{12}\delta\beta w^{-\phi}G_r]}{-\beta^2\phi w^{-\phi-1}G}$$

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$$= \frac{\lambda\alpha w}{\beta^2\phi} + \frac{\rho_{12}\delta w G_r}{\beta\phi G}, \tag{34}$$

for the optimal proportional reinsurance rate.

Equation (33) and (34) have the first and second partial derivatives, G_{p_s}, G_{rp_s}, and $G_{p_sp_s}$, so we conjecture a solution structure of the form

$$G(t,r,p_s) = \frac{{p_s}^{1-\phi}}{1-\phi} H(t,r), \tag{35}$$

such that at the terminal time, T

$$H(T,r) = \frac{1-\phi}{{p_s}^{1-\phi}}. \tag{36}$$

We get from Equation (35), that

$$G_{p_s} = {p_s}^{-\phi} H,\; G_{p_sp_s} = -\phi {p_s}^{-\phi-1} H, G_r = \frac{{p_s}^{1-\phi}}{1-\phi} H_r\; G_{rp_s} = {p_s}^{-\phi} H_r \;\}.$$
(37)

Applying (35) and (36) in (33), we get

$$p_s{}^* = -\left[\frac{\xi {p_s}^{-\phi} H}{\gamma^2(-\phi {p_s}^{-\phi-1})H} + \frac{\rho_{23}\delta {p_s}^{-\phi} Hr}{\gamma(-\phi {p_s}^{-\phi-1})H} + \frac{(1-\phi)xw {p_s}^{-\phi} H}{(-\phi {p_s}^{-\phi-1})H}\right]$$

$$= \frac{\xi p_s}{\gamma^2\phi} + \frac{\rho_{23}\delta p_s Hr}{\gamma^2\phi H} + \frac{(1-\phi)\gamma w p_s}{\phi}. \tag{38}$$

for the optimal investment in the risky asset.

The optimal proportional reinsurance rate now becomes

$$\psi^* = \frac{\lambda\alpha w}{\beta^2\phi} + \frac{\rho_{12}\delta w \frac{{p_s}^{1-\phi}}{1-\phi} H_r}{\beta\phi \frac{{p_s}^{1-\phi}}{1-\phi} H}, \tag{39}$$

And simplifies to

$$\psi^* = \frac{\lambda\alpha w}{\beta^2\phi} + \frac{\rho_{12}\delta w H_r}{\beta\phi H}. \tag{40}$$

Further equation (38) and (40) contain H_r, a first partial derivative. Therefore to eliminate the dependency on r we conjecture that

$$H(t,r) = \frac{r^{1-\phi}}{1-\phi} K(t), \tag{41}$$

such that

$$K(T) = \frac{1-\phi}{r^{1-\phi}}, \tag{42}$$

at the terminal time T.

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We obtain from (41) that

$$H_r = r^{-\phi}K. \tag{43}$$

Using (39) and (41) in (38) we get

$$p_s{}^* = \frac{\xi p_s}{\gamma^2\phi} + \frac{\rho_{23}\delta p_s r^{-\phi}K(t)}{\gamma\phi\frac{r^{1-\phi}}{1-\phi}K(t)} + \frac{(1-\phi)\gamma w p_s}{\phi}$$

$$= \frac{p_s}{\phi}\left[\frac{\xi}{\gamma^2} + \frac{(1-\phi)\rho_{23}p_s\delta}{\gamma\phi r} + (1-\phi)\gamma w\right]. \tag{44}$$

This is the optimized amount for investment in the risky asset (stock) which is a function of the total amount of money available for investment, the price of the risky asset.

For the optimal proportional reinsurance rate, we have

$$\psi^* = \frac{w}{\phi\beta}\left[\frac{\lambda\alpha}{\beta} + \frac{(1-\phi)\rho_{12}\delta}{\phi r}\right], \tag{45}$$

which is a fraction of the total amount of money available for investment in both assets.

In the case where the Brownian motions do not correlate we have

$$p_s{}^* = \frac{p_s}{\phi}\left[\frac{\xi}{\gamma^2} + (1-\phi)\gamma w\right], \tag{46}$$

as the optimal amount of money available for investment in the risky asset (stock). Also the optimized proportional reinsurance rate is given as

$$\psi^* = \frac{\lambda\alpha w}{\beta^2\phi}. \tag{47}$$

FINDINGS

1. The case of investment strategies: we found the investment strategy in the risky asset is of the fraction available for investment in the risky asset and is also dependent on the correlation coefficient of second and third Brownian motions.

2. The reinsurance proportion rate is a function of the amount of money available for investment in both assets.

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CONCLUSION

The optimal investment problem for an insurer who takes insurance cover is investigated by this work. The insurer makes investments in two assets: a risk-free (bond) asset and a risky (stock) asset aims at obtaining an optimal investment strategy and optimal proportion rate of reinsurance considering the correlation and none correlation of the Brownian motions, when power utility preference is adopted

The application of the maximum principle of dynamic programming led to obtaining the required Hamilton-Jacobi-Bellman Equations (HJB). Suitable conjectures towards eliminating dependency on variables based on the homogeneity of the objective function, the restriction and the terminal condition helped us to find the closed-form solutions to the optimal investment strategy and optimal proportional reinsurance rate.

We found that $$p_s^* = \frac{p_s}{\phi}\left[\frac{\xi}{\gamma^2} + \frac{(1-\phi)\rho_{23}p_s\delta}{\gamma\phi r} + (1-\phi)\gamma w\right].$$

This is the optimized amount for investment in the risky asset (stock) which is a function of the total amount of money available for investment, the price of the risky asset.

For the optimal proportional reinsurance rate, we have

$$\psi^* = \frac{w}{\phi\beta}\left[\frac{\lambda\alpha}{\beta} + \frac{(1-\phi)\rho_{12}\delta}{\phi r}\right],$$

which is a fraction of the total amount of money available for investment in both assets.

In the case where the Brownian motions do not correlate we have

$$p_s^* = \frac{p_s}{\phi}\left[\frac{\xi}{\gamma^2} + (1-\phi)\gamma w\right],$$

as the optimal amount of money available for investment in the risky asset (stock). Also the optimized proportional reinsurance rate is given as

$$\psi^* = \frac{\lambda\alpha w}{\beta^2\phi}.$$

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