African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 7, Issue 4, 2024 (pp. 82-86)
www.abjournals.org

CONSTRUCTION OF FINITE 2-TRANSITIVE GROUPS BASED ON THE CONCEPT OF NEAR FIELD

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Cite this article:

Danbaba, A., Jelten, B. N., Momoh, U. S. (2024), Construction of Finite 2-Transitive Groups Based on the Concept of Near Field. African Journal of Mathematics and Statistics Studies 7(4), 82-86. DOI: 10.52589/AJMSS-HSL2VU8B

Manuscript History

Received: 12 Aug 2024

Accepted: 13 Oct 2024

Published: 18 Oct 2024

ABSTRACT: *Transitive permutation groups were studied basically based on the idea of a near field. The article mainly considered Dickson Near field to construct groups which are 2-transitive. To achieve the result, some restrictions were observed in relation to the field. The research went through the construction with the help of the classification scheme for the finite primitive group proposing that the subgroup of the group is the stabilizer of the group. The idea of this construction has relevance in coding theory.*

KEYWORDS: Transitive, p-groups, primitive, abelian, affine, field.

Article DOI: 10.52589/AJMSS-HSL2VU8B
DOI URL: https://doi.org/10.52589/AJMSS-HSL2VU8B

INTRODUCTION

It is obvious that 2-transitive groups are generally primitive. The concept of double transitivity is believed to be at the forefront in the classification of finite primitive groups. This paved the way for the emergence and the study of groups which are transitive to certain degrees. Much has been said and carried out on transitive groups in the later classification of finite simple groups. Further investigation shows that 2-transitive groups are rare. Momoh (1999) worked on transitive and primitive groups of prime degrees. Momoh (2003) further considered Uniprimitive permutation groups of degree p^2. Martin and Segan (2000) worked on the new notion of transitivity for groups and sets of permutations in which the O'Nan-Scott theorem was stated, but under the condition that the subgroup of the group is the socle of the group. In that work, he described the conditions for a group to be homogeneous and enumerated five steps of classification schemes that could be used for the determination of the degree of homogeneity.

A notable contribution was the work of Bomberg and Prager (2000) on finite permutation groups with a transitive minimal normal subgroup. Lieback (2000) worked on a regular subgroup with a primitive permutation group.

Cameron (2012) considered the classification of transitive groups in general up to some degrees. There was a lot of development on transitivity, especially regarding the 2-groups.

Alexander (2002) used the software GAP (Group Algorithm Programming) to view the order and transitivity of some of the groups. The software was further used by Ken (2013) where he developed an algorithm for the computation of transitive groups. Further investigation on transitivity showed that most of the sporadic groups were 2-transitive groups. The theory of near field given here will furnish us with an idea leading to the notion introduced in 1940 in order to create a response in practical communication problems and algebraic coding, as in the text written by Gallian (1990).

Preliminary Results

We begin with the necessary concepts which are of great need. These ideas are of eminent importance and will aid in the construction of the required groups via the near field. Therefore, we begin with definitions and theorems needed for this article.

Definition 2.1: A near field is a set F with at least two elements o and 1 and with two binary operation (+,.) satisfying

1. (F,+) is an abelian group
2. $(F^{"}, .)$ is a group with identity 1
3. There is a one-sided distributive law: $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ for all $\alpha,\beta,\gamma \epsilon F$

Theorem 2.2: Let $(D,+,.)$ be a division ring and let $\emptyset: D^* \to Aut(D)$ be a mapping from the group D to the automorphism group of D. Then, $\emptyset$ satisfies the condition $\emptyset\left(\alpha^{\emptyset(\beta)}.\beta\right) = \emptyset(\alpha)\emptyset(\beta)$ for all $\alpha,\beta \epsilon D^*$.

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Remark 2.3: Therefore, we specify the condition above by defining multiplication $\odot$ $on\ D^*$ by $\alpha \odot \beta = \alpha^{\emptyset(\beta)}.\beta\ for\ all\ \alpha, \beta \in D^*$ further identity exists and $\odot\ \beta = \emptyset(\alpha)\emptyset(\beta)$; thus, $(D, +, \odot)$ is a near field.

Remark 2.4: Dickson near field can be exemplified given any prime q and n, any integer such that each prime divides n and also divides $q - 1$ and $\not\equiv 0\ mod\ 4\ if\ q \equiv 1\ mod\ 4$; thus, n divides $(q^n - 1)/(q - 1)$.

Theorem 2.5: Let $|\Omega| \geq 2$ and let G be a sharply 2-transitive group which possesses a regular abelian normal subgroup K. Then there exists a near field F such that G is isomorphic to $G^{"}$.

Corollary 2.6: Every finite sharply 2-transitive group G is permutation isomorphic to a group $G^{"}$ obtained from a finite near field.

Remark 2.7: We further weaken the axioms of the near field as it is stronger in an effort to construct sharply 2-transitive groups. Therefore, we define a near domain.

Definition 2.8: A set with binary operation $(+, .)$ satisfying the following:

1. (F,+) has these properties

i. There is a zero element 0 such that $0 + \alpha = \alpha + 0\ for\ all\ \alpha \epsilon F$

ii. $\alpha + \beta = 0$ imply $\beta + \alpha = 0\ for\ all\ \alpha, \beta \epsilon F$

iii. Each equation $\alpha + \beta = \gamma$; any of the two elements determines the third.

2. $(F^{"}, .)$ is a group

3. $\alpha. 0 = 0. \alpha = 0\ for\ all\ \alpha \epsilon F$

4. for all $\alpha, \beta \epsilon F$ there is an element $\delta_{\alpha\beta} \epsilon F^{"}$such that $\alpha + (\beta + \gamma) = (\alpha + \beta) + \delta_{\alpha\beta.\gamma}$

is called near a domain.

Next, we state a result of the version of the O'Nan-Scott theorem which is of importance in this work.

Remark 2.9: The conditions above suffice to furnish us with a better understanding on the procedures of applying the principle in relation to coding theory. In view of these conditions, we state some important results.

Definition 2.10: A group G is $\frac{1}{2}$ −transitive if all the orbits have the same size.

Definition 2.11: Let $G \leq sym(\Omega)$, then G is doubly transitive or 2-transitive if for any $\alpha_i, \beta_i \epsilon \Omega, i = 1,2$

$\alpha_i^g = \beta_i$ is implied for all $g \epsilon G$.

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Definition 2.12: A transitive group is said to be primitive if it contains no nontrivial blocks; otherwise, it is imprimitive.

We state the theorem which is basically of great importance to our result.

Theorem 2.13: If G is doubly transitive, then it is primitive.

The theorem that follows is a version of the classification scheme for finite primitive groups based on the socle type. In this work, emphasis will be less on the socle type but, rather, for a doubly transitive group, the stabilizer is a maximal subgroup. Hence,

Theorem 2.14: Let G be a group which acts primitively and faithfully on Ω with $|\Omega| = n$. Let $H = soc(G)$ and $\alpha \in \Omega$. Then, H is transitive of type T and isomorphic to group of affine, T which is abelian of order p, and $n = p^m \; and \; G_\alpha$ is a complement which acts on H and is simple.

Definition 2.15: let $G \leq sym(\Omega)$; the stabilizer for G is the subgroup of G such that $G_\alpha = \{\alpha \in \Omega, \alpha^g - \alpha\}$.

RESULTS

Theorem 3.1: Let G be a 2-transitive group of degree n and H a subgroup. If H is of prime order, then H is isomorphic to $AGL(1, F)$.

Proof: Suppose G is 2-transitive; it implies that for any $\alpha, \beta \epsilon \Omega$, G is transitive on $\Omega \backslash \alpha$. Therefore, by Theorem 2.1.5, G is an elementary abelian group. This also implies that G is isomorphic to subgroups of affine general linear group of the form $AGL(1, F)$. Hence, G is a near field.

Theorem 3.2: Let G be a primitive permutation group and F a near field; then G has a subgroup which is at most 2-transitive and isomorphic to $AGL(1, F)$.

Proof: If $G \leq sym(\Omega)$ is primitive, then it is also abelian. This shows that G is regular and so by Theorem 2.5. Therefore, this forces G to be isomorphic to $AGL(1, F)$, by Theorem 2.2 in relation to Remark 2.1. Thus, it follows from Theorem 3.1; G is 2-transitive.

Theorem 3.3: Let G be a group and H a subgroup of G. If G is 2-transitive, then the stabilizer of the group G is a maximal subgroup.

Proof: Since G is 2-transitive, then it is certainly primitive and so H is a maximal subgroup of G. Therefore, by the first condition of Theorem 2.14, it implies that H is isomorphic to $AGL(1, F)$ and abelian. Since the stabilizer is a maximal subgroup, thus $H = AGL(1, F)$.

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CONCLUSION

Based on the idea of near field, we were able to construct finite 2-transitive groups which are elementary abelian and isomorphic to the affine groups of the form $AGL(1, F)$. Also, the work was also able to relate the idea of the results obtained due to the construction of the groups to the field F. Most of the groups constructed were elementary abelian and showed isomorphism with the groups of affine type. The result obtained has direct consequences in coding theory, particularly in the transmission of messages.

REFERENCES

Alexander, H. (2002). Constructing transitive permutation groups; *Journal of symbolic Computation.10(3).369-381.*

Bomberg ,J., & Praeger, C.E (2000). Finite permutation groups with a transitive minimal normal subgroup. *Journal of Mathematics Subject Classification* 5-15.

Cameron, P.J (2012). Permutation groups and regular semi groups. London Math Subject classification.1273-1285

Dixion, J., & Mortimer, B (1996). *Permutation groups Graduate texts in mathematics.* Springer New York`.

Frobenius ,F.G. (1904). Uber die charaktere der mehch trasitiven gruppen.mouton de grupter 558-571. *Berlin.*

Gallian A.J (1990). Contemprorary Abstract Algebra. Lexinton Massachusetts Toronto

Ken, B (2013). Computation in Permutation Groups. *Cornell University*, USA.

Liebeck, W., & Praeger C.E, & Sax, I .J. (2000). Primitive permutation groups with a regular subgroup. *Journal of Mathematics subject classification.* 20B15, 05C25.

Martin, W.J., & Segan, B (2000). New notion of transitivity for groups and set of permutations. *Journal of Math Soc. Math Subject Classification 20B20*.

Momoh, S.U (1999). On Transitive and Primitive Permutation Groups of Prime Degree. *Nigerian Journal of mathematics association (*12), 183-189.

Momoh, S.U (2003). On Uniprimitive Groups of Degree p^2. *Abacus (* 30)28, 293-296.

O′Nan , M.E., & Scott, L (1979). Finite groups. Santa Cruz conference. London *Math Soc* (2)32.

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