A PREDICTIVE MODEL FOR DIGITAL CURRENCIES PRICES USING GEOMETRIC BROWNIAN MOTION STOCHASTIC DIFFERENTIAL EQUATION: A CASE STUDY OF THE BITCOIN

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ABSTRACT: *In this research work, we developed a predictive model for digital currency prices, involving daily closing price as a function of time. We used the Geometric Brownian motion stochastic differential equation which was solved using inbuild functions in Microsoft Excel. While we used the Bitcoin as our case study, our model was able to predict the daily closing prices of Bitcoin to a reasonable degree of accuracy. We equally observe that the time dependent Geometric Brownian motion stochastic differential equation cannot give digital currency traders and investors a clue on when to trade off their digital assets. Thus, it become very risky using our model to make well informed trading decisions. We therefore, recommend that for minimum risk, trades and investors in digital currencies should consider a combination of other signal tools to take more informed and less risky trading decisions.*

KEYWORDS: Crypto Currency, Geometric Brownian motion, Bitcoin, Stochastic Modelling.

INTRODUCTION

In its early years, Bitcoin was known to a relatively narrow community of cryptography enthusiasts. The first time the currency made it into the mainstream media was probably in June 2011, when WikiLeaks started accepting donations in Bitcoin from its supporters (Halaburda and Sarvary (2015)). WikiLeaks started accepting donations in Bitcoin, while highlighting the flexibility of the currency, its anonymity and independence from traditional financial providers. By 2013, Bitcoin started appearing to be an increasing speculative investment opportunity (Parham (2017)). Its price (i.e. exchange rate to the US dollar) increased from under $15 in Jan-2013 to over $1,200 in Dec-2013. During this time, Bitcoin also started gaining foothold in electronic commerce, when the Chinese search engine Baidu (world's 5th most visited site at the time) started accepting Bitcoin for payments. However, restrictions were put by the US government on digital currencies when it was revealed that Bitcoin was being used for payments in the illicit activities like drug trade by illegal websites like Silk Road. FBI raided the offices of this website and seized over 26,000 Bitcoins from there. Subsequently, the Chinese website Baidu also stopped accepting Bitcoins. In 2011, Japan-based Mt. Gox, then the largest Bitcoin exchange, experienced a security breach in which 850,000 Bitcoins worth approximately $450 million were stolen. As digital signature of a Bitcoin purse is nearly impossible to crack using brute force method, it happened only because the digital signature, or the password was known to someone who was involved in the incidence (Yermack (2013)).

Bitcoin started gaining popularity as it was touted as an instantaneous and anonymous way to make transactions, defying national boundaries, with no central bank and country as authority. Because of its anonymous nature, Bitcoins have been used in past in the criminal money laundering and tax evasion schemes (Nabilou (2019).

Cryptographic currencies represent a growing asset class that has attracted much attention from financial communities. Cryptocurrencies are digital cash and payment systems that are encrypted in a blockchain system (Hayes 2016). The four main cryptocurrencies currently on the market are Bitcoin, Ethereum, Ripple, and Litecoin. The list is constantly changing as investors grow. Bitcoin, Etherum, and Litecoin use the same network of computers to store the same copies of all transactions. Therefore, the possibility of any anomalies is highly unlikely and the network is completely safe (Iwamura, Blomhøj & Kjeldsen, 2019). Bitcoin is currently trading at the top of the cryptocurrencies list. Moreover, Bitcoin's algorithm is used in most cryptocurrencies (Gandal & Halaburda, 2016). Each cryptocurrency has its own rules concerning the maximum amount of money, currency production, privacy, transaction rates added to the blockchain, and the various mechanisms used by miners to compete among each other and earn rewards.

(Indera, Alrasheedi, & Alghamdi. 2017). Bitcoin is a decentralized electronic exchange system and represents a major change in the global financial system. Its system is based on peer-to-peer and cryptographic protocols and is not managed by any government or bank (Vidal 2014). It operates on the basis of a collusive and uncertain system in which all transactions are placed in an open ledger called blockchain (Guo and Liang 2016). Due to limited resources, low transaction costs, and ease of transferring, Bitcoin has gained popularity rapidly in recent years across the globe. It has led to cryptocurrencies being recognized as an asset to the economy, and its reach extends to markets around the world (Hayes 2016).

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Stock market prediction is difficult due to its volatile and changeable nature (Kou et al. 2014; Kou et al. 2019); however, it has been extensively investigated by researchers. For example, Adebiyi et al. (2012) used a neural network to predict stock prices. It presents a hybridized approach which combines the use of the variables of technical and fundamental analysis of stock market indicators for prediction of future price of stock in order to improve on the existing approaches. Alrasheedi and Alghamdi (2012) used a linear discriminant and logit model to predict the SABIC (Saudi Basic Industries Corporation) price index, and Sathe et al. (2016) investigated share market prediction. More details can be found in other works, such as Cocianu and Grigoryan (2015) and Ma et al. (2010). Bitcoin is a fresh market that is still in its transition phase; therefore, a lot of fluctuations can still be observed (Briere et al. 2013). Due to its unstable nature, cryptocurrency prediction is not an easy task. Interestingly, based on the information provided from the website www.coindesk.com, Bitcoin has more than 50% of the market share in the cryptocurrency market at the time of this study. Therefore, studying its prediction is of great importance and researchers are becoming focused on it.

Dynamics of Bitcoin

Satoshi Nakamoto is the creator of Bitcoin (Nakamoto 2008). This name was used for the first time in 2008 and it is still unclear if this is a real name or nickname. In 2008, he published an article about cryptography on a mailing list of the website “www.metzdowd. com”. The article introduced a kind of digital currency that later became Bitcoin. In early 2009, he released Bitcoin’s source code, along with binary code compiled on “www.sourceforge.net”.

In June 2009, Nakamoto launched the peer to peer Bitcoin network (Kaushal 2016) that allows individual members of the network to track all transactions, and started to mine Bitcoin. During the early days of crypto mining, there were few miners in the network. Therefore, the mining difficulty was low (Franco, 2014). These few miners were able to extract huge amounts of Bitcoin. Franco’s (2014) study used a Bitcoin data analysis and discovered that Nakamoto extracted nearly 1,000,000 Bitcoins. Interestingly, none of these Bitcoins had ever been spent, but the reason behind it is unknown. However, it is obvious that as soon as these Bitcoins are spent by Nakamoto, his identity will be known in the Blockchain.

The Concept of Digital Currency

The introduction of a digital currency would, in an extreme case described by Bindseil (2020), lead to a run on retail bank accounts, which would have disastrous effects for the system's stability and the ability to finance non-financial industries. Additionally, instantaneous cross-border transactions are made possible by digital currencies. If both parties are connected to the same network, an individual in the United States can, for example, send payments in digital currency to counterparty in Singapore. A general word that can be used to characterize various forms of currencies that are found in the electronic domain is "digital currency." Some categories of digital currency are:

Crypto Currency

Crypto currencies are virtual money that secure and validate network transactions through the use of encryption. These currencies are likewise managed and controlled through the use of cryptography. Among the crypto currency examples are Ethereum and Bitcoin. The regulation of crypto currency may vary based on the jurisdiction. Because they are solely digital and unregulated, cryptocurrencies are referred to as virtual currencies. The crypto currency market

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is extremely hard to understand because of the strong correlation between individual currencies and the rather weak correlation between crypto currencies and the equities market. The world's first peer-to-peer digital payments system, Bit coin was invented by an unknown developer (or group of developers) using the pseudonym Satoshi Nakamoto during the aftermath of the 2017–2018 financial crisis. It operates entirely without the involvement of a central entity (Nakamoto 2008,). Since its inception in 2009, Bit coin has consistently maintained its position as the most expensive crypto currency (Coingecko 2022). It was the first crypto currency to be created. The concept behind Bit coin was that it would operate digitally, just like real money or gold, allowing direct transactions between two parties to occur, anonymously and untraceably if desired. The fact that bit coin transactions are irreversible and permanent is another crucial aspect.

Stochastic Differential Equation

According to Herzog (2013) stochastic differential equation (SDE) represents the evolution of a stochastic process. Unlike ordinary differential equations (ODEs), which use deterministic functions, SDEs include random processes or noise in their formulation. However, solving SDEs often involves techniques from stochastic calculus, such as Itô calculus, to handle the differential terms involving the Wiener process. Analytical solutions to SDEs are rare, so numerical methods like Monte Carlo simulations or numerical integration techniques (e.g., Euler-Maruyama method) are commonly employed to approximate solutions. SDEs have many applications in pure mathematics and are used to simulate diverse behaviors of stochastic models such as stock prices, random growth models or physical systems that are prone to thermal fluctuations (Musiela, & Rutkowski, 2014).

Therefore, SDEs have a random differential, which is, at its most basic, random white noise calculated as the derivative of a Brownian motion or, more broadly, a semi martingale. Other sorts of random behavior are possible, such as Lévy processes or semi martingales with leaps. Random differential equations are equivalent to stochastic differential equations. Stochastic differential equations can also be extended to differential manifolds (Kesendal, 2013).

According to (Rogers and Williams, 2020) Stochastic differential equations originated in the theory of Brownian motion, specifically in the work of Albert Einstein and Marian Smoluchowski in 1905. Though Louis Bachelier was the first person credited with modeling Brownian motion in 1900, providing a very early example of a stochastic differential equation now known as the Bachelier model. Some of these early examples included linear stochastic differential equations, commonly known as Langevin equations after French physicist Langevin, which described the motion of a harmonic oscillator subjected to a random force.

This train of thought will serve as motivation for the following more rigorous analysis. To construct probability values, first define values for specific events and then extend them to a larger class of events in a consistent manner. This differs from counting state spaces, where probability values are first associated with single points. A comprehensive understanding of probability and random variables is key to addressing these challenges Arapostathis and Yuksel (2023)

Stochastic Differential Equations (SDEs) are a fundamental component of mathematical finance, offering a solid foundation for simulating the randomness inherent in financial markets. These equations allow for the measurement of a wide range of financial phenomena, from stock price movements to interest rate changes, by including random processes into their

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structure (Anderland, 2018). The use of SDEs in finance is more than a theoretical exercise; it has real-world ramifications for risk management, option pricing, and investment strategy. Understanding the stochastic character of financial instruments enables analysts and traders to make more educated judgments that take into account market unpredictability and volatility (Backhoff-Veraguas, Bartl, Beiglbock, and Eder, 2020). Stochastic Differential Equations (SDEs) have become an essential tool for comprehending the complicated and frequently unpredictable world of population dynamics and ecology.

The intrinsic randomness of environmental parameters, like as weather patterns, food availability, and predator-prey interactions, can be better described with SDEs than deterministic models. This technique helps ecologists to account for the unpredictability and uncertainty that are inherent in natural systems (Bar-Shalom and Tse, 2014). Conservationists use SDEs to assess the risk of extinction under various conditions, while resource managers use them to determine sustainable harvest levels. Theoretical ecology researchers use SDEs to investigate the nature of complexity and stability in ecosystems (Vivek and Mrinal, 2018). The classic logistic growth model can be extended into an SDE to account for random fluctuations in growth rate. SDEs are also used to model the movement of individuals in space, which is crucial for understanding species dispersal and habitat use.

In the context of stochastic differential equations (SDEs), the gap between Ito and Stratonovich calculus is more than just a technicality; it represents various modeling philosophies and interpretations of unpredictability in systems. Both calculi offer frameworks for integrating SDEs, which are tools for modeling systems affected by random fluctuations Davis and Varaiya (2016).. They do, however, differ in their approach to the stochastic integral, which is a fundamental notion in SDE theory. Because of its non-anticipative nature, Ito Calculus is the most widely utilized in financial mathematics (Dobrushin, 2010). It is assumed that the existing state of a system does not predict future stochastic fluctuations. This makes it ideal for simulating random processes in markets where the future is intrinsically uncertain. The Ito integral is defined in such a way that it may be applied to filtrations, which are mathematical constructs that reflect the accumulation of information over time Douc, Fort, Moulines, and Soulier (2014).

Stratonovich SDEs can be solved using methods that are more akin to those used for ordinary differential equations (ODEs).

The physical interpretation of the system being represented, as well as the features of the stochastic processes involved, ultimately determine whether Ito or Stratonovich calculus is used. While Ito calculus may be more closely related to financial modeling due to its treatment of information flow, Stratonovich calculus provides a framework that is more consistent with classical calculus and thus more comprehensible in some physical applications (Hogeboom, 2023). The discussion between these two approaches demonstrates the richness and complexity of describing the stochastic environment (Georgiou and Lindquist 2018).

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MATERIALS AND METHODS

In this section we present the necessary tools for the stochastic analysis of the dynamics of the Crypto currency market. The data for the analysis was a secondary data collected from yahoo.com which can be made available on demand.

The Concept of Stochastic Modeling

Stochastic modeling is a form of financial modeling used to make investment decisions. This type of modeling predicts the likelihood of different outcomes under different conditions using random variables. Stochastic modeling presents data and forecasts outcomes that allow for a degree of unpredictability or randomness. Companies in many industries can use stochastic modeling to improve their business practices and increase profitability. In the financial services sector, planners, analysts, and portfolio managers use stochastic modeling to manage their assets and liabilities and optimize their portfolios.

Crypto Currency Model

Let $P(t)$ denote the price of a crypto currency at time t. We setup the model as follows.

Adapting the original model in Johansen, Ledoit, Sornette, (2000) our starting point is the equation

$$P(t) = \exp\left(X(t)\right)\left[1 - H(t - t_0)\right] \quad (2.1)$$

where t_0 denotes the time of the crash and $H(t)$ denotes the Heaviside function depicted in figure 2.1 and defined as follows;

$$H(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases} \quad (2.2)$$

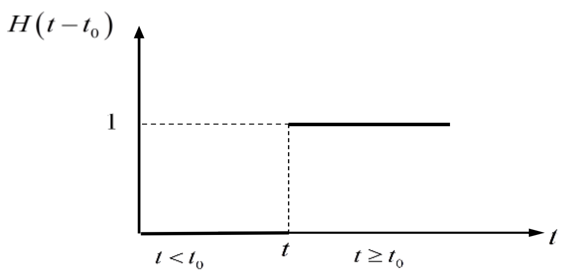

Fig. 2.1 The Heaviside function

When a crash occurs the asset price collapses completely. This follows qualitative features of past cryptocurrency crashes (White, 2014). The timing of the crash t_0 is assumed to be unknown but described by the probability density $f(t)$ and CDF. Further, $X(t)$ satisfies the stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \quad (2.3)$$

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where $W(t)$ is symmetric Cauchy noise (Samarodnitskiy and Taqqu, 1994). Set up in this way, the model also incorporates empirical evidence of extreme heavy-tails in empirical cryptocurrency prices (Gkillas and Katsiampa, 2018). Use of the Cauchy distribution as a financial model (Harris, 2017) is also motivated by analytical tractability (Samarodnitskiy and Taqqu, 1994), classical financial models (Mandelbrot, 1963) and the study of Black Swans and heavy-tailed phenomena (Taleb, 2007). The Cauchy parameters μ and 2σ also have a convenient interpretation as the median and the inter-quartile range respectively (Lee, 1997). These are often thought to be more robust summaries of financial risk and return (McNeil, Frey and Embrechts 2005) alongside further links to theoretically coherent measures of financial risk (Artzner, Delbaen, Eber and Heath, 1999).

Taking logarithms it follows that prior to the crash we have

$$\ln P(t) = \ln \exp\left(X(t)\right) + \ln\left[1 - H(t - t_0)\right] \tag{2.4}$$

$$= X(t) + \ln\left[1 - H(t - t_0)\right] = X(t) \tag{2.5}$$

Furthermore $X(t) = \ln P(t)$ satisfies the stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) - \frac{\delta(t - t_0)}{1 - H(t - t_0)} \tag{2.6}$$

The Geometric Brownian Motion (GBM)

The GBM has been the simplest model to describe stock- or asset-price dynamics. In this work, we model cryptocurrency closing price as a GBM. The essence of this asset price dynamics is that the relative price change dS/S can be split into a deterministic and a random component (Wilmott et al. 1995),

$$dS(t) = S(t)\mu(t)dt + S(t)\sigma(t)dW(t) \tag{2.7}$$

where μ is the expected rate of return over time, σ is the volatility (the amplitude of the noise), and $dW(t) = W(t + dt) - W(t)$ is the infinitesimal Wiener Process ($\langle dW(t)\rangle = 0$ and $dW^2(t) = dt$ Defining $P(t) \equiv \ln S(t)$ and using Itô's Lemma (Wilmott et al. 1995; Gardiner 1985; Itô 1944) one arrives at

$$dP(t) = \bar{\mu}dt + \sigma dW(t) \tag{2.8}$$

with $\bar{\mu}(t) = \mu - \frac{1}{2}\sigma^2$. Ito's stochastic calculus yields the exact solution (Wilmott et al. 1995; Gardiner 1985; Itô 1944)

$$P(t) = P(0) + \bar{\mu}t + \sigma\psi(t)\sqrt{t} \tag{2.9}$$

where $\psi(t)$ is a normal random variable with zero mean and unit variance.

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From Taylor's formula to Ito's lemma

Let $f:\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable function; the Taylor's series expansion of f at (x_0,t_0) is written as;

$$\begin{aligned} f(x,t) &= f(x_0,t_0)+\frac{\partial f}{\partial x}(x_0,t_0)(x-x_0)+\frac{\partial f}{\partial t}(x_0,t_0)(t-t_0)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,t_0)(x-x_0)^2 \\ &+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(x_0,t_0)(t-t_0)^2+\frac{\partial^2 f}{\partial x\partial t}(x_0,t_0)(x-x_0)(t-t_0)+\varepsilon(x_0,t_0) \end{aligned} \quad (2.10)$$

Where $\varepsilon(x_0,t_0)\sim o((x-x_0)^2+(t-t_0)^2)$. The condition on ε shows that 3rd order terms are negligible with respect to 1st and 2nd order terms.

The case of when x is an Ito process which is written as a stochastic integral with respect to a Brownian motion. The variation of this process on a time-interval $t-t_0$ is of order $\sqrt{t-t_0}$. Consequently the 2nd order term $\frac{\partial^2 f}{\partial x\partial t}(x_0,t_0)(x-x_0)^2$ cannot be neglected because it is $O(t-t_0)$. It has magnitude $\frac{\partial f}{\partial t}(x_0,t_0)(t-t_0)$.

Suppose we set $df(x_0,t_0)=f(x,t)-f(x_0,t_0)$; $t-t_0=dt$ and $x-x_0=dx$ the above equation becomes;

$$\begin{aligned} df(x_0,t_0) &= \frac{\partial f}{\partial x}(x_0,t_0)dx+\frac{\partial f}{\partial t}(x_0,t_0)dt+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,t_0)(dx)^2+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(x_0,t_0)(dt)^2 \\ &+\frac{\partial^2 f}{\partial x\partial t}(x_0,t_0)dxdt+\varepsilon(x_0,t_0) \end{aligned} \quad (2.11)$$

Replace x by X_t with X a stochastic process evolving according to the SDE;

$$dX_t=\mu(X_t,t)dt+\sigma(X_t,t)dW_t \quad (2.12)$$

Hence from equation (3.34), giving up the arguments of the partial derivatives to simplify notations, for becomes, for (X_t,t); Applying now the calculation rules defined previously

(2.12) allows us to see that the coefficients $\frac{\partial^2 f}{\partial t^2}$ and $\frac{\partial^2 f}{\partial x\partial t}$ are negligible (o(dt)). It then follows that

$$\begin{aligned} df(X_t,t) &= \frac{\partial f}{\partial x}\left[\mu(X_t,t)dt+\sigma(X_t,t)dW_t\right]+\frac{\partial f}{\partial t}dt+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[\mu(X_t,t)dt+\sigma(X_t,t)dW_t\right]^2 \\ &+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2+\frac{\partial^2 f}{\partial x\partial t}\left[\mu(X_t,t)dt+\sigma(X_t,t)dW_t\right]dt+\varepsilon(X_t,t) \end{aligned} \quad (2.13)$$

Hence;

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$$df\left(X_t,t\right)=\frac{\partial f}{\partial x}\left[\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t\right]+\frac{\partial f}{\partial t}dt+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t\right]^2+\varepsilon'\left(X_t,t\right)$$

with $\varepsilon'=o\left(dt\right)$.

We note that the term $\left[\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t\right]^2$ can be expanded to give

$$\left[\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t\right]^2=\mu\left(X_t,t\right)^2\left(dt\right)^2+\sigma\left(X_t,t\right)^2\left(dW_t\right)^2+2\mu\left(X_t,t\right)\sigma\left(X_t,t\right)dtdW_t \quad (2.14)$$

The term $\left(dt\right)^2$ is $O\left(dt\right)^2$ and hence negligible. Also the term $dtdW_t$ is negligible because it is $O\left(\sqrt{dt}\right)^3$ and, finally the term $\left(dW_t\right)^2$ which is $O\left(dt\right)$ and hence not negligible. After the necessary simplifications, we arrive at;

$$df\left(X_t,t\right)=\frac{\partial f}{\partial x}\left[\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t\right]+\frac{\partial f}{\partial t}dt+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma\left(X_t,t\right)^2dt+\varepsilon''\left(X_t,t\right) \quad (2.15)$$

with $\varepsilon''\left(X_t,t\right)=o\left(dt\right)$.

We can now write the process $f\left(X_t,t\right)$ as a stochastic differential by grouping dt terms on the one hand and dW_t on the other hand;

$$\begin{aligned}df\left(X_t,t\right)&=\frac{\partial f}{\partial x}\mu\left(X_t,t\right)dt+\frac{\partial f}{\partial x}\sigma\left(X_t,t\right)dW_t+\frac{\partial f}{\partial t}dt+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma\left(X_t,t\right)^2dt+\varepsilon''\left(X_t,t\right)\\&=\left(\frac{\partial f}{\partial x}\mu\left(X_t,t\right)+\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\left(X_t,t\right)\right)dt+\sigma\left(X_t,t\right)\frac{\partial f}{\partial x}dW_t+\varepsilon''\left(X_t,t\right)\end{aligned} \quad (2.16)$$

This brief heuristics gives the intuition of the result, however a rigorous proof of Ito's lemma requires more precautions. Especially the terms $\varepsilon\left(X_t,t\right)$ are stochastic and claiming "ε is negligible with respect to dt" is not sufficiently precise. However our aim in this thesis is not to prove Ito's lemma but to demonstrate it importance in solving the Brownian Motion SDE.

We now formally state the Ito's lemma, which is nothing else than a Taylor formula in a specific stochastic environment.

Proposition

Let X be an Ito process satisfying the SDE

$$dX_t=\mu\left(X_t,t\right)dt+\sigma\left(X_t,t\right)dW_t \quad (2.17)$$

Furthermore, let $f:\mathbb{R}^2\to\mathbb{R}$ be a function with continuous partial derivatives up to order 2. The process Y defined by $Y_t=f\left(X_t,t\right)$ is an Ito process satisfying the SDE given by

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$$dY_t = \left(\frac{\partial f}{\partial x} \mu(X_t, t) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2(X_t, t) \right) dt + \sigma(X_t, t) \frac{\partial f}{\partial x} dW_t \quad (2.18)$$

If we write the SDE of Y in the following form:

$$dY_t = \mu_Y(Y_t, t)\, dt + \sigma_Y(Y_t, t)\, dW_t \quad (2.19)$$

We get;

$$\mu_Y(Y_t, t) = \frac{\partial f}{\partial x} \mu(X_t, t) + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2(X_t, t) \quad (2.20)$$

$$\sigma_Y(Y_t, t) = \sigma(X_t, t) \frac{\partial f}{\partial x} \quad (2.21)$$

APPLICATIONS OF THE ITO'S LEMMA

Let W denote a Wiener process with parameters μ and σ which are constant for W and Y the process defined by $Y_t = f(W_t) = \exp(W_t)$. Y is then the transformation of the Brownian motion by the exponential function. We observe that t does enter the transformation implying $\frac{\partial f}{\partial t} = 0$. The dynamics of Y is obtained by applying Ito's lemma.

$$\mu_Y(Y_t, t) = \frac{\partial f}{\partial x} \mu + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 = \exp(W_t) \left(\mu + \frac{\sigma^2}{2} \right) = Y_t \left(\mu + \frac{\sigma^2}{2} \right) \quad (2.22)$$

$$\sigma_Y(Y_t, t) = \sigma \frac{\partial f}{\partial x} = \sigma Y_t \quad (2.23)$$

Or equivalently:

$$\frac{dY_t}{Y_t} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dW_t \quad (2.24)$$

Y, which may represent the price of a stock or some other financial instrument, is called a geometric Brownian motion.

Symmetrically, let Y denote a price process satisfying the SDE;

$$Y_0 = 1$$

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t \quad (2.25)$$

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Let X be defined as $X_t = g(Y_t) = \ln(Y_t)$. In this case, we get;

$$\frac{\partial g}{\partial x}\mu(Y_t,t)+\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}\sigma^2(Y_t,t)=\frac{\partial g}{\partial x}\mu Y_t+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}\sigma^2 Y_t$$

$$=\mu-\frac{\sigma^2}{2} \tag{2.26}$$

$$\sigma Y_t\frac{\partial g}{\partial x}=\sigma$$

These equalities lead to:

$$dY_t=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma dW_t \tag{2.27}$$

Existence and Unicity of Solutions of SDE

In this section we concern ourselves with the existence and uniqueness of solution of stochastic differential equations. In particular we wish to know what assumptions must be imposed on the mean μ and drift σ for the SDE to define a stochastic process having nice properties. We assume that the filtration $\Im$ on the probability space $(\Omega,\Im,P)$ is the natural filtration of a standard Brownian motion W.

Definition

A stochastic differential equation is given by a stochastic differential associated with a boundary condition, i.e.,

$$X_0=c \qquad \text{(SDE)}$$

$$dX_t=\mu(X_t,t)dt+\sigma(X_t,t)dW_t. \tag{2.28}$$

In the general case c may be a random variable. However in most financial models, c is a constant, for example the initial price of a financial asset or the initial short-term rate of interest.

Definition

A stochastic process X is a solution of the SDE (2.28) on $[0,T]$ if

i) X is adapted to $\Im$

ii) The functions μ and σ satisfy respectively;

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$\int_0^T |\mu(X_t,t)|dt < +\infty$ and $\int_0^T \sigma^2(X_t,t)dt < +\infty$ (2.29)

iii) X satisfies;

$$X_t = X_0 + \int_0^t \mu(X_s,s)ds + \int_0^t \sigma(X_s,s)dW_s \quad (2.30)$$

The following proposition provides conditions on μ and σ for a stochastic differential equation to have a solution.

Proposition

Assuming conditions (i) and (ii) in **definition (2.7.2)** are satisfied, then equation (2.30) has a unique solution (P-a.s), which is a stochastic process X adapted to $\mathfrak{I}$, with continuous paths, satisfying $E\left(\int_0^T X_t^2 dt\right) < +\infty$.

(a) $\exists M > 0$ such that $\forall t \in [0,T], \forall(x,y) \in \mathbb{R}^2$

$$\max\left(|\mu(x,t)-\mu(y,t)|;|\sigma(x,t)-\sigma(y,t)|\right) \le M\,|x-y|$$

$$\mu(x,t)^2 + \sigma(x,t)^2 \le M(1+x^2), \forall(x,y) \in \mathbb{R}^2$$

(b) X_0 is square integrable, independent of $\mathfrak{I}_t$ for any t.

A detailed proof of this result may be found in Oksendal (2000), page 66. (Stoch. Processes for finance).

Oksendal, B. (2000); Stochastic Differential Equations: An introduction with applications 5th ed., Springer.

Remark

The condition in (a) is called a Lipschitz condition. It limits the slopes of μ and σ which must be finite and bounded by a constant which doesn't depend on t. The second part of condition (a) puts some restrictions on the growth of μ and σ. As μ is the instantaneous expectation of X variations, the condition means that the LHS must be of order $\left(1+x^2\right)^{\frac{1}{2}}$. Thus, we cannot have a drift growing out of proportion. If the condition were not satisfied, the drift would grow too rapidly with the level reached by the process. This would be the case for example if $\mu(x,t) = e^x$.

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ANALYSIS AND RESULTS

Introduction

Our setting here is a probability space $(\Omega, \mathfrak{I}, P)$ endowed with a filtration $F = \{\mathfrak{I}_t, t \geq 0\}$ that satisfies the usual conditions of right-continuity and completeness. On the given probability space, we consider a main market in which heterogeneous agents buy or sell Bitcoin and denote by $S = \{S_t : t \geq 0\}$ the price process of the crypto currency. We assume that the Bitcoin price dynamics is described by the following geometric Brownian motion stochastic differential equation:

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t, \quad S_0 = s_0 \in \mathbb{R}^+ \tag{3.1}$$

where$\mu_S \in \mathbb{R}\backslash\{0\}, \sigma_S \in \mathbb{R}$represent model parameters; $W = \{W_t : t \geq 0\}$ is a standard Brownian motion on $(\Omega, \mathfrak{I}, P)$. We recall that equation (4.1) was introduced in the previous chapter and a formal solution provided.

Solution of the Geometric Brownian Motion SDE

A key assumption of the Black-Scholes option pricing model is that the instantaneous crypto currency price movements can be characterized by the geometric Brownian motion SDE given by

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t \tag{3.2}$$

Here, S_t is the crypto currency price process, μ_S and σ_S, are constants, t is time, and dW_t follows a stochastic process called a Wiener process under which $dW_t = \varepsilon\sqrt{dt}$ where ε is a random number draw from the standardized normal distribution. Equation (3.2) is commonly known as geometric Brownian motion (GBM), with μ_S and σ_S called the drift parameter and the volatility parameter, respectively.

Equation (3.2) can be written

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t \tag{3.3}$$

$$\Rightarrow \quad d\ln S_t = \left(\mu_S - \frac{\sigma_S^2}{2}\right)dt + \sigma_S dW_t \tag{3.4}$$

The stochastic process as characterized by equation (4.4) indicates that $\ln S_t$ is normally distributed. Equivalently, S_t is lognormally distributed. With S_0 and S_T denoted as the crypto prices at time 0 and time T respectively, with $dW_t = \varepsilon\sqrt{dt}$ and integrating both sides equation (4.4) leads to

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$$S_T = S_0 \exp\left[\left(\mu_S - \frac{\sigma_S^2}{2}\right)T + \sigma_S \varepsilon \sqrt{T}\right] \quad (3.5)$$

Furthermore, the expected value and the variance of the crypto price S_t are given by

$$E[S_t] = S_0 \exp(\mu_S T) \text{ and } Var[S_t] = S_0^2 [\exp(2\mu_S T)]\left[\exp(\sigma_S^2 T)\right] \quad (3.6)$$

Simulation of daily Bitcoin price movements

Although geometric Brownian motion is a stochastic process in continuous time, its implementation in simulation scenario requires that it be approximated in a discrete time setting. We assume for now that a day as a proportion of a year is short enough for such an approximation to work well. For simplicity our simulation will be based in MS Excel. It could however be implemented in Python or some other software.

Now in order to simulate the time paths of daily crypto prices, from and initial price to a closing price, we need an explicit expression of the crypto price on each day in terms of the crypto price a day earlier. Such an expression is a recursive version of equation (3.5).

Specifically, if we use t and $t + \Delta t$; instead of 0 and T > 0; to indicate two successive points in time, equation (3.5) can be written as

$$S_{t+\Delta t} = S_0 \exp\left[\left(\mu_S - \frac{\sigma_S^2}{2}\right)\Delta t + \sigma_S \varepsilon \sqrt{\Delta t}\right] \quad (3.6)$$

Now, let n be the number of days in a year. Here, the number can be based on calendar days or trading days; however, the latter is more common in practice. The time interval Δt between two adjacent days is the proportion $1/n$ of a year. For notational convenience, let S_t and S_{t+1} be the stock prices on two adjacent days, for $t = 0,1,2,3,\ldots$; until the closing of the crypto trading. Provided that μ_S and σ_S are stated in annual terms, we can write equation (3.6)

$$S_{t+1} = S_t \exp\left[\left(\mu_S - \frac{\sigma_S^2}{2}\right)\frac{1}{n} + \frac{\sigma_S}{\sqrt{n}}\varepsilon\right] \quad (3.7)$$

For a given initial price S0 and given constant values of μ_S and σ_S ; equation (3.7) will enable $S_1, S_2, S_3, \ldots$ to be generated. The idea is to use equation (3.7) recursively, starting from day 0; for each trading day, we generate a new random draw of ε from the standardized normal distribution for the equation to simulate the crypto price of the next day. It is important to note that in Excel, Brownian motion can be simulated using the RAND ( ) and NORM.INV( ) functions.

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Remark

Given the stochastic nature of price movements as characterized by geometric Brownian motion, each set of simulated time paths of stock and option prices will inevitably differ from any other set as generated repeatedly in simulation runs. From a statistical perspective, we are interested in knowing what simulated prices can be expected and how widely dispersed are such prices. Equations (3.6) can be used directly to compute the expected crypto price and the standard deviation of crypto prices, respectively, on each day until the close of trading. With t being a day label, we simply substitute T on the right-hand side of each of the two equations with $1/n$; for $t = 0, 1, 2, 3, \ldots$; until the close of trading, that is,

$$E[S_t] = S_0 \exp\left(\frac{\mu_S t}{n}\right) \tag{3.8}$$

and

$$\sqrt{Var[S_t]} = E[S_t]\sqrt{\left[\exp\left(\frac{\sigma_S^2 T}{n}\right)\right]} \tag{3.9}$$

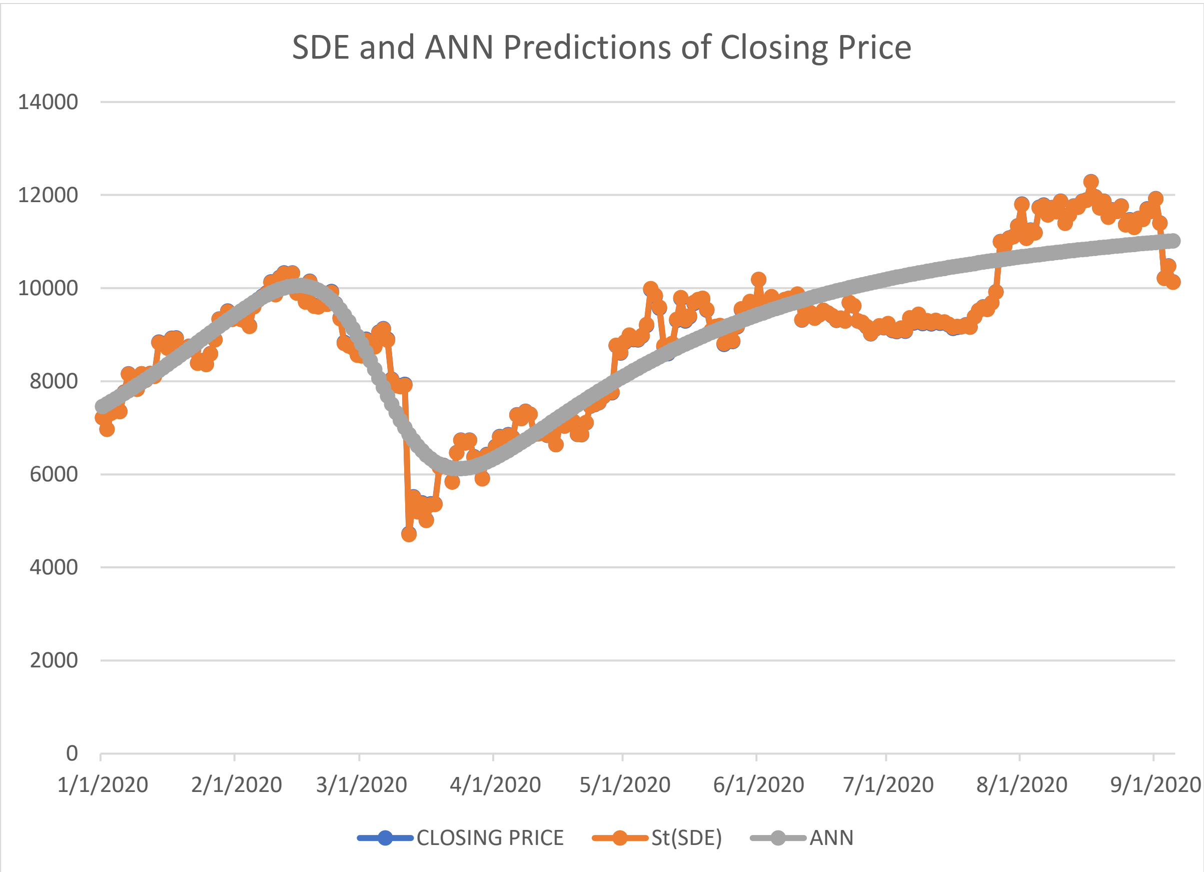

Fig. 4.1 SDE Simulation and ANN Prediction of digital currency prices for 250 days.

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Above is 250 days prediction of Bitcoin daily closing price using our model. Also shown in the chart is ANN prediction of Bitcoin closing price for the same period under review.

CONCLUSION

Based on the findings of the study, the following conclusion were consequently reached. The study concluded that SDE time dependent model can be used to both predict digital currency prices. But it is important to note that our model cannot be use to forecast the future price of digital currencies beyond one day due to the high volatility nature of digital currency. The above is also in agreement with Stephen, Abhishek, James, Boleslaw, Szymanski, and Korniss (2020).

RECOMMENDATION

Based on the findings of the study the following recommendations were made.

1. By modeling the underlying stochastic processes, traders can develop predictive models to forecast future price movements and identify profitable trading opportunities. Thus, we recommend that traders combine our model with other signals tool to make a more informed and accurate trading decision.

2. Stochastic differential equations are essential for modeling the complex and volatile nature of digital currency markets. They enhance the understanding of price dynamics, improve risk management, facilitate option pricing, support algorithmic trading, and aid in portfolio optimization. By incorporating randomness into the models, SDEs provide a realistic and robust framework for analyzing and trading digital currencies. We suggest that digital currency investors or traders can actually combine SDEs and ANN for a more accurate prediction and forecast.

REFERENCES

Alrasheedi, R, and Alghamdi, M. (2012). Price discovery on Bitcoin markets. Digital Finance 1: 139–61.

Arapostathis A. and S. Yuksel (2023). Convex analytic method revisited: Further optimality results and performance of deterministic ¨ policies in average cost stochastic control. Journal of Mathematical Analysis and Applications, 517(2):126567.

Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. Math. Financ, 9, 203-228.

Backhoff-Veraguas, D. Bartl, M. Beiglbock, and Eder, M. (2020). Adapted Wasserstein distances and stability in mathematical ¨ finance. Finance and Stochastics, 24(3):601–632.

Bar-Shalom, Y. and Tse, E. (2014). Dual effect, certainty equivalence, and separation in stochastic control. IEEE Transactions on Automatic Control, 19(5):494–500, October.

Bindseil, L. (2020), Informational efficiency of Bitcoin—An extension. Economics Letters 163: 106–9.

Dobrushin, R. L. (2010). Central limit theorem for nonstationary Markov chains. i. Theory of Probability & Its Applications, 1(1):65– 80.

Article DOI: 10.52589/AJMSS-LZLNQMN8
DOI URL: https://doi.org/10.52589/AJMSS-LZLNQMN8

Douc, G. F., Moulines, E., and Soulier, P. (2014). Practical drift conditions for subgeometric rates of convergence. Ann. Appl. Probab, 14:1353–1377.
Franco, V. (2014). The Impact of Futures Trading on Intraday Spot Volatility and Liquidity: Evidence from Bitcoin Market. Available online: https://ssrn.com/abstract=3094647 (accessed on 8 January 2020).
Georgiou, T. T. and Lindquist, A. (2018). The separation principle in stochastic control, redux. IEEE Transactions on Automatic Control, 58(10):2481–2494.
Guo, S and Liang, U. (2016). Efficient capital markets: A review of theory and empirical work. The Journal of Finance 25: 383–417.
Harris, D. E. 2017. The distribution of returns. Journal of Mathematical Finance 7, 769-804.
Hayes, A. (2016) "Cryptocurrency Value Formation: An Empirical Analysis Leading to a Cost of Production Model for Valuing Bitcoin", Telematics and Informatics, Vol. 34, Pages 1308-1321.
Hudaib, A. Z. (2014) ''E-payment security analysis in depth,'' Int. J. Comput. Sci. Secur., vol. 8, no. 1, p. 14,
Indera, T., Alrasheedi, R. and Alghamdi, M. (2017). Asymmetric multifractal cross-correlations between the main world currencies and the main cryptocurrencies. Physica A: Statistical Mechanics and its Applications 523: 1057–71.
Iwamura, R, Blomhøj, M., and Kjeldsen, T.H. 2(019). Time-varying long-term memory in Bitcoin market. Finance Research Letters 25: 280–84.
Johansen, A., Ledoit, O. and Sornette, D. (2000). Crashes as critical points. Int. J. of Theor.and Appl. Financ. 3, 219-255.
Kaushal, H. (2016). The inefficiency of Bitcoin revisited: A high-frequency analysis with alternative currencies. Finance Research Letters 28: 68–73
Oksendal, B. K. (2000). Stochastic Differential Equations: An Introduction with Applications. Berlin: Springer. ISBN 3-540-04758-1
Lee, P. M. (1997). Bayesian statistics: an introduction, second ed. Arnold, London.
Mandelbrot, B. (1963). The variation of certain speculative prices. J. of Bus. 36, 394-419.
McNeil, A. J., Frey, R., and Embrechts, P. (2005). Quantitative risk management. Princeton University Press, Princeton.
Musiela, M., and Rutkowski, M. (2014), Martingale Methods in Financial Modelling, 2nd Edition, Springer Verlag, Berlin.
Nabilou, E. (2019). Network causality structures among Bitcoin and other financial assets: A directed acyclic graph approach. The Quarterly Review of Economics and Finance 70: 203–13.
Nakamoto, S. (2008) "Bitcoin: A Peer-to-Peer Electronic Cash System." https://bitcoin.org/bitcoin.pdf.
Parham, Y. (2017). Network models to enhance automated cryptocurrency portfolio management. Frontiers in Artificial Intelligence 3: 22.
Rogers, L.C.G. and Williams, D. (2020). Diffusions, Markov Processes and Martingales, Vol 2: Ito Calculus (2nd ed., Cambridge Mathematical Library ed.). Cambridge University Press. doi:10.1017/CBO9780511805141. ISBN 0-521-77594-9. OCLC 42874839.
Samarodnitskiy, G. and Taqqu, M. S. (1994). Stable non-Gaussian random processes: Stochastic Models with Infinite Variance: Stochastic Modeling. Champman & Hall. New York.
Stephen, D., Abhishek C., James F., Boleslaw K. Szymanski I. and Korniss G. (2020). Using correlated stochastic differential equations to forecast cryptocurrency rates and social media activities. Applied Network Science; 5:17. https://doi.org/10.1007/s41109-020-00259-1
Taleb, N. N. (2007). The Black Swan: the impact of the highly improbable. Random House.
Vidal, I. (2014). BitCoin meets Google Trends and Wikipedia: Quantifying the relationship between phenomena of the Internet era. Scientific Reports 3: 3415.

 Article DOI: 10.52589/AJMSS-LZLNQMN8
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