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THEORETICAL PROPERTIES OF THE LINDLEY EXPONENTIATED GUMBEL DISTRIBUTION

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ABSTRACT: *This study introduces a new statistical distribution, named the Lindley Exponentiated Gumbel (LEGu) distribution, aiming to enhance the flexibility and adaptability of statistical models for various environmental datasets. The distribution is constructed by combining elements of the Lindley density function with its corresponding cumulative density function (cdf) and probability distribution function (pdf), offering increased flexibility and versatility for statistical modelling and analysis in various fields of research and application. To derive insights into the newly proposed distribution, the study investigates its structural properties and characteristics and presents expansions for its probability density and cumulative density functions using generalized binomial expansion. Several important representations, such as the survival function, hazard function, quantile function, probability weighted moment, moment generating function, and distribution of order statistics, are provided for the LEGu distribution. The method of estimation involves maximum likelihood estimation, with the sample log-likelihood function derived. Due to the complex nature of the likelihood function, numerical optimization techniques like the Newton-Raphson method are proposed for estimating the distribution parameters effectively. The proposed distribution's versatility and robustness, coupled with the use of maximum likelihood estimators, pave the way for more accurate and reliable interpretations of environmental data, leading to valuable insights and potential applications in diverse environmental research and decision-making processes.*

KEYWORDS: Survival function, Binomial expansion, Environmental, Hazard, Quantile function.

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INTRODUCTION

This study presents a novel methodology focused on the creation of a new statistical distribution suitable for analysing environmental data. In this regard, a novel probability density function is introduced, which extends the exponentiated Gumbel distribution by incorporating an additional shape parameter through the application of the Lindley-G family of distributions as a link function. The aim is to establish a flexible and adaptable model capable of accommodating various environmental datasets.

To derive insights into the newly proposed distribution, the study investigates its structural properties and characteristics. The parameters of this distribution are estimated using well-established maximum likelihood estimation techniques, which are widely recognized for their effectiveness in statistical modelling.

By employing this approach, the researcher seeks to enhance the capability of statistical analysis and modelling for environmental data, enabling a deeper understanding of complex patterns and behaviours. The proposed distribution's versatility and robustness, coupled with the use of maximum likelihood estimation, pave the way for more accurate and reliable interpretations of environmental datasets, leading to valuable insights and potential applications in diverse environmental research and decision-making processes.

THE DISTRIBUTION

A. Lindley-G Family of Distributions

Cakmakyapan and Ozel (2017) extended the transformer (T-Y) generator introduced by Alzaatreh et al. (2013) to propose a novel and broader category of continuous probability distributions, which they named the "Lindley-G family of distributions. This family of distributions is constructed by combining the Lindley density function with its corresponding cumulative density function (cdf) and probability distribution function (pdf). The resulting Lindley-G family offers a wider range of distributions, providing increased flexibility and versatility for statistical modelling and analysis in various fields of research and application.

$$F(y)=1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-G(y)\right]\right]\right]\left[1-G(y)\right]^{\theta} \tag{1}$$

and

$$f(y)=\frac{\theta^{2}}{\theta+1}g(y)\left[1-\log\left[1-G(y)\right]\right]\left[1-G(y)\right]^{\theta-1} \tag{2}$$

B. Exponentiated Gumbel Distribution

Nadarajah (2006) proposed an extension of the Gumbel distribution, known as the exponentiated Gumbel (EG) distribution. The cumulative density function (cdf) and probability density function (pdf) of the EG distribution are defined as follows:

$$G(y)=1-\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha} \tag{3}$$

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and

$$g(x)=\frac{\alpha}{\sigma}\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}-\exp\left(\frac{-y-\mu}{\sigma}\right) \tag{4}$$

C. The Novel Distribution: Lindley Exponentiated Gumbel Distribution

The cumulative density function (cdf) of the Lindley Exponentiated Gumbel distribution is derived by substituting equation (3) into equation (1), and the probability density function (pdf) is obtained by substituting equation (4) into equation (2). The resulting expressions provide a complete characterization of the Lindley Exponentiated Gumbel distribution and allow for its application in various statistical analyses and modelling tasks.

$$F(y)=1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]^{\theta} \tag{5}$$

and

$$f(y)=\frac{\theta^2\alpha}{\sigma(\theta+1)}\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\left[1-\log\left[1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]\right]\right]$$
$$\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\exp(-y)\left[1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{-y-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]\right]^{\theta-1} \tag{6}$$

$-\infty<y<\infty$, $-\infty<\mu<\infty$ and $\alpha,\sigma,\theta>0$.

Where α, θ are the shape parameters controlling the skewness and kurtosis, σ is the scale parameter and μ is the location parameter.

Without loss of generality, we assume $\mu=0$ and $\sigma=1$ then equation (5) and equation (6) become

$$F(y)=1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]^{\theta} \tag{7}$$

$$f(y)=\frac{\theta^2\alpha}{(\theta+1)}\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha-1}\left[1-\log\left[1-\left[1-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]\right]$$
$$\exp\left\{-\exp(-y)\right\}\exp(-y)\left[1-\left[1-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]^{\theta-1} \tag{8}$$

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THE SHAPE OF THE DISTRIBUTION

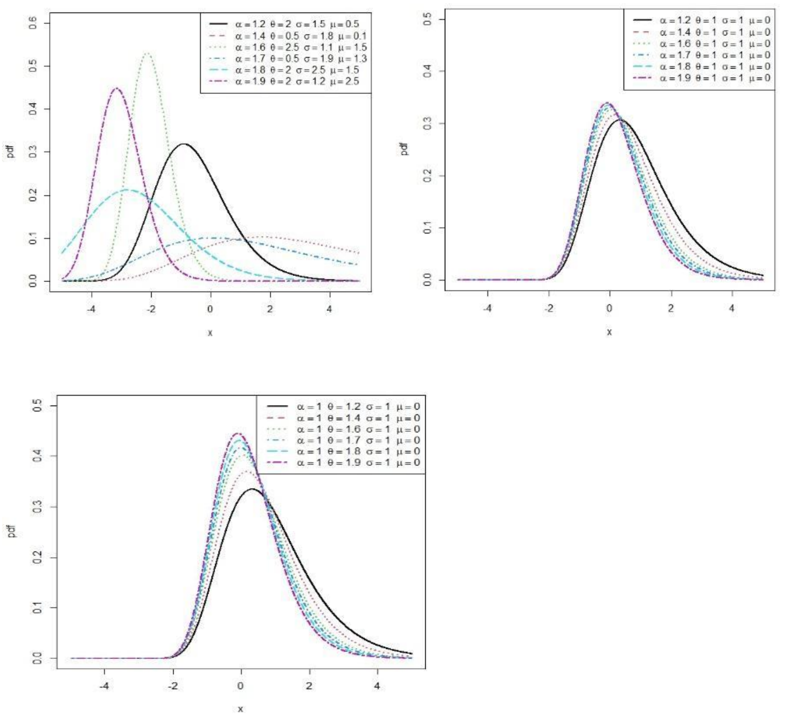

Figure 1: Plots of pdf of LEGu distribution with different parameter value

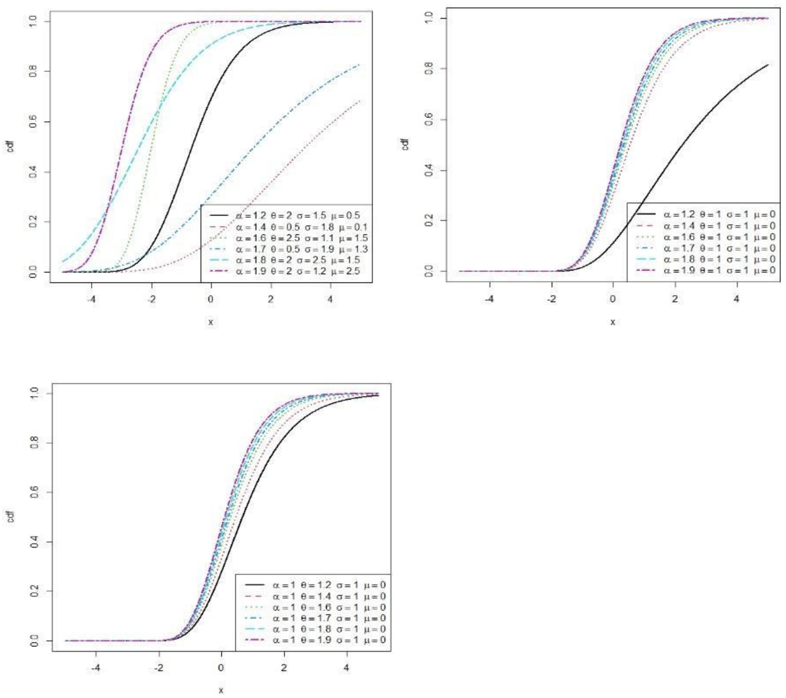

Figure 2: Plots of cdf of LEGu distribution when all the parameters varies

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IMPORTANT REPRESENTATION

In this section, we present an expansion for equation (7) and equation (8) using the generalized binomial expansion, which is given as follows:

$$(1-z)^t = \sum_{i=0}^{\infty}(-1)^i \binom{t}{i} z^i \tag{9}$$

and

$$\log(1-z) = -\sum_{i=0}^{\infty}(-1)^{i+1}\binom{g}{i+1} z^{i+1}, \quad |z|<1 \tag{10}$$

From equation (8), we have

$$\left[1-\log\left[1-\exp\{-\exp(-y)\}\right]\right]^{\alpha} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha}{i}\log\left[1-\exp\{-\exp(-y)\}\right]^i \tag{11}$$

$$\log\left[1-\exp\{-\exp(-y)\}\right]^i = -\sum_{j=0}^{\infty}(-1)^{j+1}\left[\exp\{-\exp(-y)\}\right]^{j+1} \tag{12}$$

$$\left[1-\left[1-\left[1-\exp\{-\exp(y)\}\right]^{\alpha}\right]\right]^{\theta-1} = \sum_{k}^{\infty}(-1)^k\binom{\theta-1}{k}\left[1-\left[1-\exp\{-\exp(y)\}\right]^{\alpha}\right]^k \tag{13}$$

$$\left[1-\left[1-\exp\{-\exp(y)\}\right]^{\alpha}\right]^k = \sum_{d=0}^{\infty}(-1)^d\binom{k}{d}\left[1-\exp\{-\exp(y)\}\right]^{\alpha d} \tag{14}$$

$$\left[1-\exp\{-\exp(y)\}\right]^{\alpha d} = \sum_{a=0}^{\infty}(-1)^a\binom{\alpha d}{a}\left[\exp\{-\exp(y)\}\right]^a \tag{15}$$

$$\left[\exp\{-\exp(y)\}\right]^{a+j+2} = \sum_{p=0}^{\infty}(-1)^p\binom{a+j+2}{p}\exp(-y)^p \tag{16}$$

On combining all the expansions together, we have

$$f(y) = \frac{\theta^2\alpha}{\theta+1}\sum_{i,j,k,d,a,p=0}^{\infty}(-1)^{i+j+k+d+a+p}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p}\exp(-y)^{p+1} \tag{17}$$

Equation (17) can be re-written as

$$f(y) = \sum_{i,j,k,d,a,p=0}^{\infty}\psi_v \exp(-y)^{p+1} \tag{18}$$

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Where $\psi_v = \frac{\theta^2\alpha}{\theta+1}(-1)^{i+j+k+d+a+p}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p}$ (19)

Also, expansion for the cdf is obtained from equation (7) as

$$[F(x)]^h = \left[1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-x)\}\right]^{\alpha}\right]\right]\right]^h \left[\left[1-\exp\{-\exp(-x)\}\right]^{\alpha\theta}\right]^h \tag{20}$$

$$\left[\left[1-\exp\{-\exp(-y)\}\right]\right]^{\alpha\theta h} = \sum_{w=0}^{h}(-1)^w\binom{\alpha\theta h}{w}\left[\exp\{-\exp(-y)\}\right]^w \tag{21}$$

$$\left[1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\right]^h = \sum_{m=0}^{h}(-1)^m\binom{h}{m}\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]^m$$

$$\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]^m = \sum_{q=0}^{\infty}(-1)^q\binom{m}{q}\frac{\theta}{\theta+1}\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha q} \tag{22}$$

$$\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha q} = -\sum_{b=0}^{\infty}(-1)^{b+1}\binom{\alpha q}{b+1}\left[\exp\{-\exp(-y)\}\right]^{b+1} \tag{23}$$

$$\left[\exp\{-\exp(-y)\}\right]^{b+1+w} = \sum_{z=0}^{\infty}(-1)^z\binom{b+1+w}{z}\exp(-y)^z \tag{24}$$

On combining all the expansions (21 -24) together, we have

$$[F(y)]^h = \frac{\theta}{\theta+1}\sum_{w,m=0}^{h}\sum_{q,b,z=0}^{\infty}(-1)^{w+m+q+b+z}\binom{\alpha\theta h}{w}\binom{h}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z}\exp(-y)^z\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{25}$$

Equation (25) can be re-written as

$$[F(y)]^h = \sum_{w,m=0}^{h}\xi_t\exp(-y)^z \tag{26}$$

Where $\zeta_t = \sum_{q,b,z=0}^{\infty}\frac{\theta}{\theta+1}(-1)^{w+m+q+b+z}\binom{\alpha\theta h}{w}\binom{h}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z}$ (27)

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PROPERTIES OF LINDLEY EXPONENTIATED GUMBEL (LEGU) DISTRIBUTION

In this section, we present some important properties of the LEGu distribution:

a. Survival function

The survival function, denoted as S(y), represents the probability that an item will not fail before a specific time t. It can be mathematically defined as follows:

$$s(y) = P(Y > y) = 1 - F(y) \tag{28}$$

S(y) is the survival function at time y

P(Y > y) is the probability that the random variable Y (representing the time to event or failure) is greater than y

The survival function of LEGu distribution is given as

$$s(y) = \left[1 - \frac{\theta}{\theta+1}\left[\log\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]^{\theta} \tag{29}$$

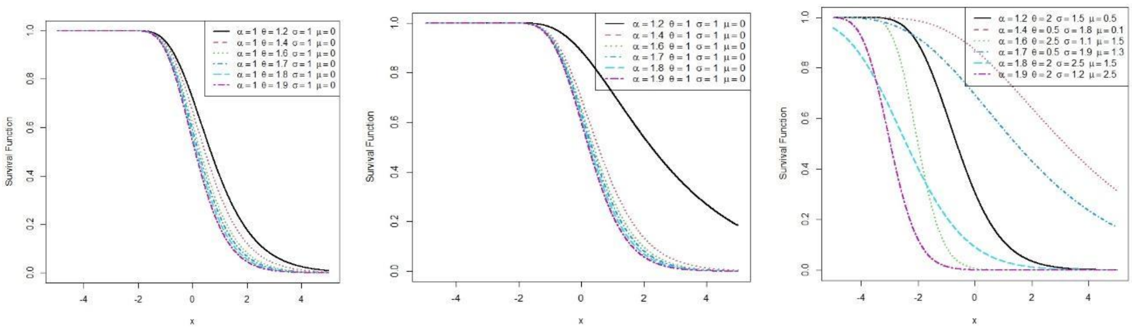

Figure 3: Plots of survival function of LEGu distribution when θ varies and other parameters are constant.

b. Hazard function

The hazard rate function (hrf) is given as

$$h(y) = \frac{f(y)}{1-F(y)} = \frac{f(y)}{s(y)} \tag{30}$$

The hazard function of LEGu distribution is given as

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$$h(y)=\frac{\frac{\theta^4\alpha}{\theta+1}\left[1-\exp\{-\exp(-y)\}\right]^{\alpha-1}\exp\{-\exp(-y)\}\exp(-y)\left[1-\log\left[1-\left[1-\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\right]\times\left[1-\left[1-\left[1-\exp\{-\exp(y)\}\right]^{\alpha}\right]\right]^{\theta-1}\times\left[1-\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]}{\left[\theta+1-\theta\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]} \tag{31}$$

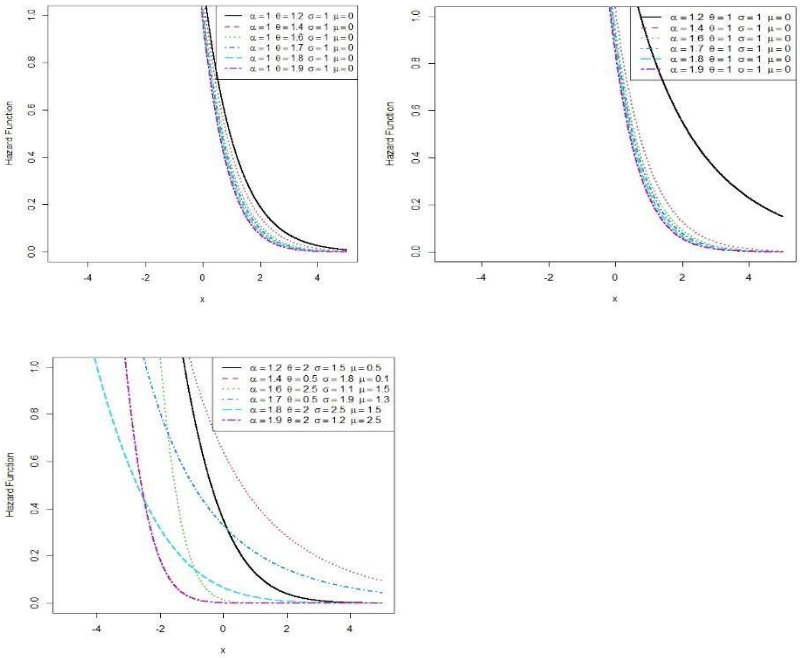

Figure 4: Plots of hazard function of LEGu distribution when θ varies and other parameters are constant.

c. Quantile function

The quantile function, Q(u), 0 < u < 1, for the T-Y family of distributions is computed by using the formula of Alzaatreh et al. (2013) as

$$Q(u)=F^{-1}\left\{1-e^{R^{-1}(u)}\right\} \tag{32}$$

$$F(y)=1-\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]^{\theta}=U \tag{33}$$

$$\left[1-\frac{\theta}{\theta+1}\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\left[\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]^{\theta}=1-U \tag{34}$$

$$\frac{\theta+1-\theta\left[\log\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\left[\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]^{\theta}}{\theta+1}=1-U \tag{35}$$

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$$\theta+1-\theta\left[\log\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\left[\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]^{\theta}=(1-U)(\theta+1) \tag{36}$$

$$-\theta\left[\log\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\left[\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]^{\theta}=(1-U)(\theta+1)-\theta+1 \tag{37}$$

$$-\left[\log\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\left[\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]=\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}} \tag{38}$$

$$-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{2\alpha}=\exp\left[\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right] \tag{39}$$

$$-1-\exp\left\{-\exp(-y)\right\}=\exp\left[\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right]^{\frac{1}{2\alpha}} \tag{40}$$

$$\exp\left\{-\exp(-y)\right\}=\exp\left[\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right]^{\frac{1}{2\alpha}}+1 \tag{41}$$

$$-\exp(-y)=\log\left[\exp\left[\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right]^{\frac{1}{2\alpha}}+1\right] \tag{42}$$

$$y=-\log\left[-\log\left[\exp\left[\left[(1-U)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right]^{\frac{1}{2\alpha}}+1\right]\right] \tag{43}$$

Equation (43) is the quantile function of LEGu distribution. To obtain the median, we set $u=0.5$ in equation (43). Then, the median is given as

$$y_{median}=-\log\left[-\log\left[\exp\left[\left[(1-0.5)(\theta+1)-\theta+1\right]^{\frac{1}{\theta}}\right]^{\frac{1}{2\alpha}}+1\right]\right] \tag{44}$$

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d. Probability Weighted Moment

$$\tau_{r,s}=E\left[Y^{r}F(y)^{s}\right]=\int_{0}^{\infty}y^{r}f(y)F(y)^{s}dy \tag{45}$$

replacing h with s in equation (45), we have

$$\tau_{r,s}=\int_{0}^{\infty}y^{r}\sum_{i,j,k,d,a,p=0}^{\infty}\sum_{w,m=0}^{s}\psi_{v}\xi_{t}\exp(-y)^{p+z+1}dy \tag{46}$$

$$=\sum_{i,j,k,d,a,p=0}^{\infty}\sum_{w,m=0}^{s}\psi_{v}\xi_{t}\int_{0}^{\infty}y^{r}\exp(-y)^{p+z+1}dy \tag{47}$$

Where $\int_{0}^{\infty}y^{r}\exp(-y)^{p+z+1}dy=\Gamma(r+p+z+1)$ (48)

Therefore

$$\tau_{r,s}=\sum_{i,j,k,d,a,p=0}^{\infty}\sum_{w,m=0}^{s}\psi_{v}\xi_{t}\Gamma(r+p+z+1) \tag{49}$$

where

$$\zeta_{t}=\sum_{q,b,z=0}^{\infty}\frac{\theta}{\theta+1}(-1)^{w+m+q+b+z}\binom{\alpha\theta s}{w}\binom{s}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z} \tag{50}$$

and

$$\psi_{v}=\frac{\theta^{2}\alpha}{\theta+1}(-1)^{i+j+k+d+a+p}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p} \tag{51}$$

e. Moment

$$E(Y^{r})=\int_{0}^{\infty}y^{r}f(y)dy \tag{52}$$

$$E(Y^{r})=\sum_{i,j,k,d,a,p=0}^{\infty}\psi_{v}\int_{0}^{\infty}y^{r}\exp(-y)^{p+1}dy \tag{53}$$

Where $\int_{0}^{\infty}y^{r}\exp(-y)^{p+1}dy=\Gamma(r+p+1)$ (54)

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Therefore

$$E(Y^r) = \sum_{i,j,k,d,a,p=0}^{\infty} \psi_v \Gamma(r+p+1) \tag{55}$$

The mean of the distribution is obtained by setting r = 1 in equation (55)

$$E(Y) = \sum_{i,j,k,d,a,p=0}^{\infty} \psi_v \Gamma(p+2) \tag{56}$$

f. Moment Generating Function (MGF)

$$M_y(t) = \int_0^{\infty} e^{ty} f(y) dy$$

(57)

Consider the expansion

$$e^{ty} = \sum_{m=0}^{\infty} \frac{(ty)^m}{m!},$$

(58)

then the MGF of the LEGu distribution follows from the moment as

$$M_y(t) = \frac{\sum_{i,j,k,d,a,p=0}^{\infty} \sum_{m=0}^{\infty} t^m \psi_v \Gamma(m+p+1)}{m!} \tag{59}$$

g. Distribution of order Statistics

Let $Y_1, Y_2, \ldots Y_n$ be a random sample and its ordered values are denoted as $Y_{(1)}, Y_{(2)}, \ldots Y_{(n)}$. The pdf of order statistics is obtained using the below function

$$f_{r:n}(y) = \frac{f(y)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(y)^{v+r-1} \tag{60}$$

The pdf of rth order statistic for LEGu distribution is obtained by replacing h with $v+r-1$ in cdf expansion. We have

$$f_{r:n}(y) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j,k,d,a,p=0}^{\infty} \sum_{w,m=0}^{v+r-1} (-1)^v \binom{n-r}{v} \psi_v \zeta_t \exp(-y)^{p+z+1} \tag{61}$$

where

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$$\zeta_t = \sum_{q,b,z=0}^{\infty} \frac{\theta}{\theta+1}(-1)^{w+m+q+b+z}\binom{\alpha\theta(v+r-1)}{w}\binom{v+r-1}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z} \tag{62}$$

and

$$\psi_v = \frac{\theta^2\alpha}{\theta+1}(-1)^{i+j+k+d+a+p}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p} \tag{63}$$

The pdf of the minimum order statistic of the LEGu distribution is obtained by setting r = 1 in equation (63)

$$f_{1:n}(y) = n\frac{\theta^3\alpha}{(\theta+1)^2}\sum_{v=0}^{n-1}\sum_{i,j,k,d,a,p=0}^{\infty}\sum_{w,m=0}^{v}\sum_{q,b,z=0}^{\infty}(-1)^{w+m+q+b+z}(-1)^{i+j+k+d+a+p}(-1)^{v}\binom{n-1}{v}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}$$
$$\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p}\binom{\alpha\theta v}{w}\binom{v}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z}\exp(-y)^{p+z+1} \tag{64}$$

Also, the pdf of the maximum order statistic of the LEGu distribution is obtained by setting r = n in equation (62)

$$f_{n:n}(y) = n\frac{\theta^3\alpha}{(\theta+1)^2}\sum_{i,j,k,d,a,p=0}^{\infty}\sum_{w,m=0}^{v+n-1}\sum_{q,b,z=0}^{\infty}(-1)^{w+m+q+b+z}(-1)^{i+j+k+d+a+p}(-1)^{v}\binom{n-1}{v}\binom{\alpha}{i}\binom{i}{j+1}\binom{\theta-1}{k}$$
$$\binom{k}{d}\binom{\alpha d}{a}\binom{a+j+2}{p}\binom{\alpha\theta(v+n-1)}{w}\binom{(v+n-1)}{m}\binom{\alpha q}{b+1}\binom{b+1+w}{z}\exp(-y)^{p+z+1} \tag{65}$$

METHOD OF ESTIMATION

Maximum Likelihood Estimation

Since maximum likelihood estimators give the maximum information about the population parameters, therefore this section presents the maximum likelihood estimates (MLEs) of the parameters that are inherent within the distribution function.

Let $Y_1, Y_2, \ldots Y_n$ be random variables of the LEGu distribution of size n. Then sample log-likelihood function of the LEGu distribution is obtained as

$$\log L = 2n\log\theta - n\log(\theta+1) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\{-\exp(-y)\}\right] - \sum_{i=1}^{n}\exp(-y) - \sum_{i=1}^{n}y_i$$
$$+\sum_{i=1}^{n}\log\left[1-\log\left[1-\left[1-\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right]\right] + \theta - 1\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\exp\{-\exp(-y)\}\right]^{\alpha}\right]\right] \tag{66}$$

Differentiating equation (66) with respect to each parameter and equating to zero, we have

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$$\frac{d\log L}{d\theta}=\frac{2n}{\theta}-\frac{n}{\theta+1}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]$$
(67)

$$\frac{d\log L}{\alpha}=\sum_{i=1}^{n}\log\left[1-\exp\left\{-\exp(-y)\right\}\right]-\sum_{i=1}^{n}\frac{\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\log\left[1-\exp\left\{-\exp(-y)\right\}\right]}{1-\log\left[1-\left[1-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]}$$
$$-\theta-1\sum_{i=1}^{n}\frac{\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\log\left[1-\exp\left\{-\exp(-y)\right\}\right]}{1-\left[1-\left[1-\left[1-\exp\left\{-\exp(-y)\right\}\right]^{\alpha}\right]\right]} \tag{68}$$

$$\frac{d\log L}{d\mu}=(\alpha+1)\sum_{i=1}^{n}\frac{\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}\frac{1}{\sigma}\exp\left(\frac{-y_i}{\sigma}\right)\exp\left(\frac{\mu}{\sigma}\right)}{1-\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}}-\sum_{i=1}^{n}\frac{1}{\sigma}\exp\left(\frac{-y_i}{\sigma}\right)\exp\left(\frac{\mu}{\sigma}\right)-\sum_{i=1}^{n}\left(\frac{y_i-1}{\sigma}\right)$$
$$+\sum_{i=1}^{n}\frac{\alpha\left[1-\exp\left\{-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}\frac{1}{\sigma}\exp\left(\frac{-y_i}{\sigma}\right)\exp\left(\frac{\mu}{\sigma}\right)}{1-\log\left[1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]\right]}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{\alpha\left[1-\exp\left\{-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}\frac{1}{\sigma}\exp\left(\frac{-y_i}{\sigma}\right)\exp\left(\frac{\mu}{\sigma}\right)}{1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]} \tag{69}$$

$$\frac{d\log L}{\sigma}=\frac{-n}{\sigma}+(\alpha-1)\sum_{i=1}^{n}\frac{\exp\left[-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right]\exp\left(\frac{-y_i-\mu}{\sigma}\right)\frac{y_i-\mu}{\sigma^2}}{1-\exp\left[-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right]}-\sum_{i=1}^{n}\exp\left(\frac{-y_i-\mu}{\sigma}\right)\frac{y_i-\mu}{\sigma^2}-\sum_{i=1}^{n}\frac{y_i-\mu}{\sigma^2}$$
$$+\sum_{i=1}^{n}\frac{\alpha\left[1-\exp\left\{-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}\exp\left(\frac{-y_i-\mu}{\sigma}\right)\frac{y_i-\mu}{\sigma^2}}{1-\log\left[1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]\right]}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{\alpha\left[1-\exp\left\{-\exp\left(\frac{-y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha-1}\exp\left\{-\exp(\frac{-y_i-\mu}{\sigma})\right\}\exp\left(\frac{-y_i-\mu}{\sigma}\right)\frac{y_i-\mu}{\sigma^2}}{1-\left[1-\left[1-\exp\left\{-\exp\left(\frac{y_i-\mu}{\sigma}\right)\right\}\right]^{\alpha}\right]} \tag{70}$$

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Due to the complex form of equations (67), (68), (69), and (70), obtaining an analytical solution for the Maximum Likelihood estimator of the unknown parameters may not be feasible. In such cases, an alternative approach is to use the nonlinear Newton-Raphson technique to numerically maximize the likelihood function. The Newton-Raphson method is an iterative optimization algorithm that can efficiently approximate the solutions to complex equations, enabling us to find the optimal values for the unknown parameters that maximize the likelihood function. By employing the Newton-Raphson technique, we can obtain accurate numerical solutions for the Maximum Likelihood estimator, allowing us to effectively estimate the parameters of interest for the LEGu distribution and make informed statistical inferences based on the data at hand.

SUMMARY AND CONCLUSION

This study introduces a novel statistical distribution tailored for environmental data analysis, aiming to enhance the flexibility and adaptability of statistical models for various environmental datasets. The methodology involves incorporating an additional shape parameter into the exponentiated Gumbel distribution using the Lindley-G family of distributions as a link function.

The distribution's properties and characteristics are investigated, and its parameters are estimated using maximum likelihood estimation techniques. By employing this approach, the study seeks to improve statistical analysis and modeling for environmental data, facilitating a deeper understanding of complex patterns and behaviours.

The novel distribution, termed Lindley Exponentiated Gumbel (LEGu) distribution, combines elements of the Lindley density function and the exponentiated Gumbel distribution, offering increased flexibility for statistical modeling. The study explores the structural properties of the LEGu distribution and presents expansions for its probability density and cumulative density functions using generalized binomial expansion.

Several important representations, such as the survival function, hazard function, quantile function, probability weighted moment, moment, moment generating function, and distribution of order statistics, are provided for the LEGu distribution.

The method of estimation involves maximum likelihood estimation, with the sample log-likelihood function derived for the LEGu distribution. Due to the complex nature of the likelihood function, numerical optimization techniques like the Newton-Raphson method are proposed for estimating the distribution parameters effectively.

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