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A NON-STANDARD FINITE DIFFERENCE DISCRETIZATION SCHEME APPLIED TO A MALARIA MODEL

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ABSTRACT: *In this research work, a dynamically consistent non-standard finite difference (NSFD) scheme is developed to solve a continuous-time model of malaria transmission with herbal medicine as control strategy. We compared results from NSFD scheme with the standard finite difference methods (4^{th} order Runge-kutta and forward Euler methods). The numerical investigation showed that the proposed NSFD method remains consistent, preserves the positivity of solutions and converges to true equilibrium points of the continuous model independent of the step size h.*

KEYWORDS: Non-Standard Finite Difference, Herbal, Malaria, Runge-kutta, Uncomplicated.

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INTRODUCTION

Malaria is a vector–borne infectious disease which is caused by *Plasmodium* parasite and it is transmitted to humans through bites of infectious female anopheles mosquitoes (World Health Organization, 2022; Azuaba *et al*, 2020). This life–threatening disease is treatable and is usually classified into asymptomatic, mild (uncomplicated), and severe (complicated) malaria based on clinical symptoms. Asymptomatic malaria describes a case where a person harbours the parasite that can transmit the disease but not showing clinical symptoms due to naturally acquired immunity that develops slowly with age. Persons with mild (uncomplicated) malaria usually have fever and one or more of the following symptoms: chills, shivering, headache, vomiting, sweating, fatigue, and anemia but no clinical signs or laboratory findings of vital organ dysfunction. On the other hand, severe (complicated) malaria is characterized by at least one of the following clinical manifestation(s); coma (caused by cerebral malaria), convulsions, hypoglycemia, malarial anemia, metabolic acidosis (with difficulty in breathing), high fever, and/or spontaneous bleeding (World Health Organization, 2022; Bakary *et al*., 2018).

Conventional antimalarial drugs (such as chloroquine, artesunate, quinine, and amodiaquine) are used as first-line treatment for uncomplicated (mild) malaria in most malaria-endemic countries. However, resistance of *Plasmodium* parasites to these conventional antimalarial drugs has created a need for herbal therapy as alternative treatment for uncomplicated malaria infection. According to Collins and Duffy (2022), factors that increase the resistance of *Plasmodium* parasites to conventional antimalarial drugs include incomplete treatments of active malaria infections, overuse of antimalarial drugs, and use of counterfeit or substandard drugs. Uzor et al (2020) reported that the high cost and the adverse effects associated with some of these conventional antimalarial drugs also limit their usefulness in malaria control.

Traditionally, antimalarial herbal medicine has been used in the control of malaria for decades due to their efficacy, lower cost, safety and availability. Indeed, antimalarial herbal medicine is gaining popularity in developed as well as developing countries for treatment of uncomplicated malaria (Erhirhie *et al*, 2021; Oladeji *et al*, 2020; Adebayo & Krettli, 2010). In addition to the use of antimalarial herbal medicine, human behaviour (use of insecticides treated bed-nets and draining of mosquito breeding sites) and good housing condition have also been reported as preventive control measures in curbing the spread of malaria (Bala & Gimba, 2019; Oluwafemi & Azuaba, 2022; Witbooi *et al*, 2021). It is reported that malaria constitutes a significant constraint to economic growth; however it has been proven that economic development improved significantly in areas where malaria is eradicated.

Epidemiological models can be described by a system of first order non-linear ordinary differential equations (continuous-time model) in which exact solutions are generally difficult to obtain, hence the need to discretize the continuous-time model into a discrete scheme for numerical simulation. The results of the discrete–time models are more accurate and more suitable for describing infectious diseases (Liao & Yang, 2017). However, some discretization techniques do not always preserve the essential properties of the continuous-time models, such as; the positivity of solutions and numerical stability (Ndii *et al*, 2019). It is well known that in mathematical modeling of malaria transmission dynamics, the long-time behaviour of models converge to the steady state, thus, any numerical scheme used for numerical simulation of mathematical model arising from malaria must follow this phenomenon. Several discretization methods have been used in literature, including the

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explicit Euler methods, Runge-Kutta methods, and other Standard Finite Difference (SFD) methods. It has been shown that these standard finite difference methods implemented in a dynamical system can lead to negative solutions, numerical instabilities or converging to wrong equilibrium points. (Ndii *et al*, 2019; Lambert, 1991) One of the most important breakthroughs in this regard is the research made by Micken (Micken, 2000 & 2007) on the use of Non-Standard Finite Difference (NSFD) method which is based on the concept of exact difference scheme.

Therefore, in this research work we applied a non-standard finite difference scheme to a mathematical model of malaria with herbal medicine as control. We compared the NSFD methods with the SFD scheme and also tested their convergence properties at different step-lengths. The non-standard finite difference method is well known in literature and has been applied to many cases in recent years (Kocabiyik, 2022; Farago & Moslah, 2022; Ndii *et al*, 2019; Egbelowo, 2018; Rafiq, 2017).

MATERIAL AND METHODS

Elakhe *et al.*, (2023) formulated a mathematical model of malaria transmission with anti-malaria herbal therapy as control. The model is given by the following non-linear ordinary differential equations:

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \psi R_h - \left(\frac{b\beta_h I_m}{1+eI_m} + \mu_h\right) S_h \\
\frac{dE_h}{dt} &= \left(\frac{b\beta_h I_m}{1+eI_m}\right) S_h - (\alpha_h + \mu_h) E_h \\
\frac{dM_h}{dt} &= \alpha_h E_h - (\varepsilon + \tau + \mu_h) M_h \\
\frac{dC_h}{dt} &= \tau M_h - (\theta + \mu_h + \delta_h) C_h \\
\frac{dR_h}{dt} &= \varepsilon M_h + \theta C_h - (\psi + \mu_h) R_h \\
\frac{dS_m}{dt} &= \Lambda_m - b\left(\frac{\beta_{1m} M_h}{1+eM_h} + \frac{\beta_{2m} C_h}{1+eC_h}\right) S_m - \mu_m S_m \\
\frac{dE_m}{dt} &= b\left(\frac{\beta_{1m} M_h}{1+eM_h} + \frac{\beta_{2m} C_h}{1+eC_h}\right) S_m - (\alpha_m + \mu_m) E_m \\
\frac{dI_m}{dt} &= \alpha_m E_m - \mu_m I_m
\end{aligned}\right\} \quad (1)$$

with the initial conditions:

$$S_h(0) = S_{0h}, \quad E_h(0) = E_{0h}, \quad M_h(0) = M_{0h}, \quad C_h(0) = C_{0h}$$
$$R_h(0) = R_{0h} \quad S_m(0) = S_{0m}, \quad E_m(0) = E_{0m}, \quad I_m(0) = I_{0m}$$

The state variables and model parameters are shown below;

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Table 1: Description of state variables of our model

State variables	Description
S_h(t)	Number of susceptible humans at time t
E_h(t)	Number of exposed humans at time t
M_h(t)	Number of infectious human population with uncomplicated cases of malaria infection at time t
C_h(t)	Number of infectious human population with complicated cases of malaria infection at time t
R_h(t)	Number of humans with partial immunity at time t
N_h(t)	Total number of human population at time t
S_m(t)	Number of mosquitoes susceptible to malaria at time t
E_m(t)	Number of mosquitoes exposed to malaria at time t
I_m(t)	Number of infectious mosquitoes at time t
N_m(t)	Total number of mosquito population at time t

Table 2: Description of Model Parameters

Parameters	Descriptions
Λ_h	Recruitment term of susceptible humans
Λ_m	Recruitment term of mosquitoes
μ_h	Per capita natural mortality rate of humans
μ_m	Per capita natural mortality rate of mosquitoes
δ_h	Per capita disease–induced mortality rate of humans
b	Per capita biting rate of mosquitoes
β_h	Probability that a bite by an infectious mosquito on a susceptible human results in transmission of disease to the susceptible human
β_{1m}	Probability that a bite by a susceptible mosquito results in transmission of disease from a mild/ asymptomatic infectious human to the susceptible mosquito

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β_{2m}	Probability that a bite by a susceptible mosquito results in transmission of disease from a complicated infectious human to the susceptible mosquito
α_h	Per capita latent period in human
α_m	Per capita latent period in mosquito
ψ	Per capita loss of immunity by recovered human
e	Proportion of human compliance level to behaviour change (the use of insecticide treated bed-net, draining of mosquitoes breeding sites)
ε	Proportion of uncomplicated infectious human that recover due to antimalarial herbal drugs (g) and/or natural immunity (ϕ)
τ	Per capita rate of progression to infectious complicated human compartment
θ	Per capita recovery rate of complicated infection human due to treatment

Existence of Disease – Free Equilibrium Points: The disease-free equilibrium state of the model is given as:

$$E_0 = (S_h^*,\ E_h^*,\ M_h^*,\ C_h^*,\ R_h^*,\ S_m^*,\ E_m^*\ \ I_m^*) = \left(\frac{\Lambda_h}{\mu_h},\ \ 0,\ \ 0,\ \ 0,\ 0,\ \frac{\Lambda_m}{\mu_m},\ 0,\ \ 0\right) \qquad (2)$$

which represent the steady state in which there is no plasmodium parasite in the community.

Basic Reproduction Number R_0: The reproduction number of the model is given by:

$$R_0 = \sqrt{\frac{b^2\beta_h\Lambda_h\Lambda_m\alpha_h\alpha_m[\beta_{1m}(\theta+\mu_h+\delta_h)+\beta_{2m}\tau]}{\mu_h\mu_m^2(\alpha_h+\mu_h)(\varepsilon+\tau+\mu_h)(\theta+\mu_h+\delta_h)(\alpha_m+\mu_m)}} \qquad (3)$$

The focus of this research work is on the discretization of the continuous-time model (1) into discrete scheme employing the non-standard finite difference (NSFD) discretization method. however, we constructed some standard finite difference schemes (Forward Euler and 4th order Runge-Kutta methods) from the continuous-time model (1) as means of comparison with the non-standard finite difference method.

Forward Euler Scheme

Let us represent S_h^n, E_h^n, M_h^n, C_h^n, R_h^n, S_m^n, E_m^n and I_h^n respectively as the numerical approximations of $S_h(t)$, $E_h(t)$, $M_h(t)$, $C_h(t)$, $R_h(t)$, $S_m(t)$, $E_m(t)$ and $I_m(t)$ at $t = nh$, $n = 0,1,2,\ldots$, where h is the step-size. Then we have the discrete model for system (1) given by:

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$$\left.\begin{aligned}
\frac{S_h^{n+1}-S_h^n}{h} &= \Lambda_h + \psi R_h^n - \left(\frac{b\beta_h I_m^n}{1+eI_m^n} + \mu_h\right) S_h^n \\
\frac{E_h^{n+1}-E_h^n}{h} &= \left(\frac{b\beta_h I_m^n}{1+eI_m^n}\right) S_h^n - (\alpha_h + \mu_h) E_h^n \\
\frac{M_h^{n+1}-M_h^n}{h} &= \alpha_h E_h^n - (\varepsilon + \tau + \mu_h) M_h^n \\
\frac{C_h^{n+1}-C_h^n}{h} &= \tau\, M_h^n - (\theta + \mu_h + \delta_h) C_h^n \\
\frac{R_h^{n+1}-R_h^n}{h} &= \varepsilon\, M_h^n + \theta C_h^n - (\psi + \mu_h) R_h^n \\
\frac{S_m^{n+1}-S_m^n}{h} &= \Lambda_m - \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n} + \mu_m\right) S_m^n \\
\frac{E_m^{n+1}-E_m^n}{h} &= \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n}\right) S_m^n - (\alpha_m + \mu_m) E_m^n \\
\frac{I_m^{n+1}-I_m^n}{h} &= \alpha_m E_m^n - \mu_m I_m^n
\end{aligned}\right\} \tag{4}$$

where $S_h^0 > 0,\ E_h^0 \geq 0,\ M_h^0 \geq 0,\ C_h^0 \geq 0,\ R_h^0 \geq 0,\ S_m^0 > 0,\ E_m^0 \geq 0$, and $I_m^0 \geq 0$

Rearranging equation (4), we have the forward Euler scheme for the continuous model (1) given below:

$$\left.\begin{aligned}
S_h^{n+1} &= S_h^n + h\left[\Lambda_h + \psi R_h^n - \left(\frac{b\beta_h I_m^n}{1+eI_m^n} + \mu_h\right) S_h^n\right] \\
E_h^{n+1} &= E_h^n + h\left[\left(\frac{b\beta_h I_m^n}{1+eI_m^n}\right) S_h^n - (\alpha_h + \mu_h) E_h^n\right] \\
M_h^{n+1} &= M_h^n + h[\alpha_h E_h^n - (\varepsilon + \tau + \mu_h) M_h^n] \\
C_h^{n+1} &= C_h^n + h[\,\tau\, M_h^n - (\theta + \mu_h + \delta_h) C_h^n] \\
R_h^{n+1} &= R_h^n + h[\,\varepsilon\, M_h^n + \theta C_h^n - (\psi + \mu_h) R_h^n] \\
S_m^{n+1} &= S_m^n + h\left[\Lambda_m - \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n} + \mu_m\right) S_m^n\right] \\
E_m^{n+1} &= E_m^n + h\left[\left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n}\right) S_m^n - (\alpha_m + \mu_m) E_m^n\right] \\
I_m^{n+1} &= I_m^n + h[\alpha_m E_m^n - \mu_m I_m^n]
\end{aligned}\right\} \tag{5}$$

where $S_h^0 > 0,\ E_h^0 \geq 0,\ M_h^0 \geq 0,\ C_h^0 \geq 0,\ R_h^0 \geq 0,\ S_m^0 > 0,\ E_m^0 \geq 0$, and $I_m^0 \geq 0$

Explicit 4th Order Runge-Kutta (RK-4) Method

In a similar fashion with the Euler Forward method, we develop the RK-4 scheme for the system (1) as:

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$$\left.\begin{aligned}
\frac{S_h^{n+1}-S_h^n}{h} &= \Lambda_h + \psi R_h^n - \left(\frac{b\beta_h I_m^n}{1+eI_m^n} + \mu_h\right) S_h^n \\
\frac{E_h^{n+1}-E_h^n}{h} &= \left(\frac{b\beta_h I_m^n}{1+eI_m^n}\right) S_h^n - (\alpha_h + \mu_h) E_h^n \\
\frac{M_h^{n+1}-M_h^n}{h} &= \alpha_h E_h^n - (\varepsilon + \tau + \mu_h) M_h^n \\
\frac{C_h^{n+1}-C_h^n}{h} &= \tau\, M_h^n - (\theta + \mu_h + \delta_h) C_h^n \\
\frac{R_h^{n+1}-R_h^n}{h} &= \varepsilon\, M_h^n + \theta C_h^n - (\psi + \mu_h) R_h^n \\
\frac{S_m^{n+1}-S_m^n}{h} &= \Lambda_m - \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n} + \mu_m\right) S_m^n \\
\frac{E_m^{n+1}-E_m^n}{h} &= \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n}\right) S_m^n - (\alpha_m + \mu_m) E_m^n \\
\frac{I_m^{n+1}-I_m^n}{h} &= \alpha_m E_m^n - \mu_m I_m^n
\end{aligned}\right\} \quad (6)$$

Rearranging equation (6), we have;

$$\left.\begin{aligned}
S_h^{n+1} &= S_h^n + h\left[\Lambda_h + \psi R_h^n - \left(\frac{b\beta_h I_m^n}{1+eI_m^n} + \mu_h\right) S_h^n\right] \\
E_h^{n+1} &= E_h^n + h\left[\left(\frac{b\beta_h I_m^n}{1+eI_m^n}\right) S_h^n - (\alpha_h + \mu_h) E_h^n\right] \\
M_h^{n+1} &= M_h^n + h[\alpha_h E_h^n - (\varepsilon + \tau + \mu_h) M_h^n] \\
C_h^{n+1} &= C_h^n + h[\,\tau\, M_h^n - (\theta + \mu_h + \delta_h) C_h^n] \\
R_h^{n+1} &= R_h^n + h[\,\varepsilon\, M_h^n + \theta C_h^n - (\psi + \mu_h) R_h^n] \\
S_m^{n+1} &= S_m^n + h\left[\Lambda_m - \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n} + \mu_m\right) S_m^n\right] \\
E_m^{n+1} &= E_m^n + h\left[\left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n}\right) S_m^n - (\alpha_m + \mu_m) E_m^n\right] \\
I_m^{n+1} &= I_m^n + h[\alpha_m E_m^n - \mu_m I_m^n]
\end{aligned}\right\} \quad (7)$$

We define, step I:

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$$\left.\begin{aligned}
S_{h1} &= h\left[\Lambda_h + \psi R_h^n - \left(\frac{b\beta_h I_m^n}{1+eI_m^n} + \mu_h\right) S_h^n\right] \\
E_{h1} &= h\left[\left(\frac{b\beta_h I_m^n}{1+eI_m^n}\right) S_h^n - (\alpha_h + \mu_h) E_h^n\right] \\
M_{h1} &= h[\alpha_h E_h^n - (\varepsilon + \tau + \mu_h) M_h^n] \\
C_{h1} &= h[\,\tau\, M_h^n - (\theta + \mu_h + \delta_h) C_h^n] \\
R_{h1} &= h[\,\varepsilon\, M_h^n + \theta C_h^n - (\psi + \mu_h) R_h^n] \\
S_{m1} &= h\left[\Lambda_m - \left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n} + \mu_m\right) S_m^n\right] \\
E_{m1} &= h\left[\left(\frac{b\beta_{1m} M_m^n}{1+eM_m^n} + \frac{b\beta_{2m} C_m^n}{1+eC_m^n}\right) S_m^n - (\alpha_m + \mu_m) E_m^n\right] \\
I_{m1} &= h[\alpha_m E_m^n - \mu_m I_m^n]
\end{aligned}\right\} \quad (8)$$

Step II:

$$\left.\begin{aligned}
S_{h2} &= h\left[\Lambda_h + \psi\left(R_h^n + \frac{R_{h1}}{2}\right) - \left(\frac{b\beta_h\left(I_m^n + \frac{I_{m1}}{2}\right)}{1+e\left(I_m^n + \frac{I_{m1}}{2}\right)} + \mu_h\right)\left(S_h^n + \frac{S_{h1}}{2}\right)\right] \\
E_{h2} &= h\left[\left(\frac{b\beta_h\left(I_m^n + \frac{I_{m1}}{2}\right)\left(S_h^n + \frac{S_{h1}}{2}\right)}{1+e\left(I_m^n + \frac{I_{m1}}{2}\right)}\right) - (\alpha_h + \mu_h)\left(E_h^n + \frac{E_{h1}}{2}\right)\right] \\
M_{h2} &= h\left[\alpha_h\left(E_h^n + \frac{E_{h1}}{2}\right) - (\varepsilon + \tau + \mu_h)\left(M_h^n + \frac{M_{h1}}{2}\right)\right] \\
C_{h2} &= h\left[\,\tau\left(M_h^n + \frac{M_{h1}}{2}\right) - (\theta + \mu_h + \delta_h)\left(C_h^n + \frac{C_{h1}}{2}\right)\right] \\
R_{h2} &= h\left[\,\varepsilon\left(M_h^n + \frac{M_{h1}}{2}\right) + \theta\left(C_h^n + \frac{C_{h1}}{2}\right) - (\psi + \mu_h)\left(R_h^n + \frac{R_{h1}}{2}\right)\right] \\
S_{m2} &= h\left[\Lambda_m - \left(\frac{b\beta_{1m}\left(M_h^n + \frac{M_{h1}}{2}\right)}{1+e\left(M_h^n + \frac{M_{h1}}{2}\right)} + \frac{b\beta_{2m}\left(C_h^n + \frac{C_{h1}}{2}\right)}{1+e\left(C_h^n + \frac{C_{h1}}{2}\right)} + \mu_m\right)\left(S_m^n + \frac{S_{m1}}{2}\right)\right] \\
E_{m2} &= h\left[\left(\frac{b\beta_{1m}\left(M_h^n + \frac{M_{h1}}{2}\right)}{1+e\left(M_h^n + \frac{M_{h1}}{2}\right)} + \frac{b\beta_{2m}\left(C_h^n + \frac{C_{h1}}{2}\right)}{1+e\left(C_h^n + \frac{C_{h1}}{2}\right)}\right)\left(S_m^n + \frac{S_{m1}}{2}\right) - (\alpha_m + \mu_m)\left(E_m^n + \frac{E_{m1}}{2}\right)\right] \\
I_{m1} &= h\left[\alpha_m\left(E_m^n + \frac{E_{m1}}{2}\right) - \mu_m\left(I_m^n + \frac{I_{m1}}{2}\right)\right]
\end{aligned}\right\} \quad (9)$$

Step III:

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$$\left.\begin{aligned}
S_{h3} &= h\left[\Lambda_h + \psi\left(R_h^n + \frac{R_{h2}}{2}\right) - \left(\frac{b\beta_h\left(I_m^n + \frac{I_{m2}}{2}\right)}{1 + e\left(I_m^n + \frac{I_{m2}}{2}\right)} + \mu_h\right)\left(S_h^n + \frac{S_{h2}}{2}\right)\right] \\
E_{h3} &= h\left[\left(\frac{b\beta_h\left(I_m^n + \frac{I_{m2}}{2}\right)\left(S_h^n + \frac{S_{h2}}{2}\right)}{1 + e\left(I_m^n + \frac{I_{m2}}{2}\right)}\right) - (\alpha_h + \mu_h)\left(E_h^n + \frac{E_{h2}}{2}\right)\right] \\
M_{h3} &= h\left[\alpha_h\left(E_h^n + \frac{E_{h2}}{2}\right) - (\varepsilon + \tau + \mu_h)\left(M_h^n + \frac{M_{h2}}{2}\right)\right] \\
C_{h3} &= h\left[\tau\left(M_h^n + \frac{M_{h2}}{2}\right) - (\theta + \mu_h + \delta_h)\left(C_h^n + \frac{C_{h2}}{2}\right)\right] \\
R_{h3} &= h\left[\varepsilon\left(M_h^n + \frac{M_{h2}}{2}\right) + \theta\left(C_h^n + \frac{C_{h2}}{2}\right) - (\psi + \mu_h)\left(R_h^n + \frac{R_{h2}}{2}\right)\right] \\
S_{m3} &= h\left[\Lambda_m - \left(\frac{b\beta_{1m}\left(M_h^n + \frac{M_{h2}}{2}\right)}{1 + e\left(M_h^n + \frac{M_{h2}}{2}\right)} + \frac{b\beta_{2m}\left(C_h^n + \frac{C_{h2}}{2}\right)}{1 + e\left(C_h^n + \frac{C_{h2}}{2}\right)} + \mu_m\right)\left(S_m^n + \frac{S_{m2}}{2}\right)\right] \\
E_{m3} &= h\left[\left(\frac{b\beta_{1m}\left(M_h^n + \frac{M_{h2}}{2}\right)}{1 + e\left(M_h^n + \frac{M_{h2}}{2}\right)} + \frac{b\beta_{2m}\left(C_h^n + \frac{C_{h2}}{2}\right)}{1 + e\left(C_h^n + \frac{C_{h2}}{2}\right)}\right)\left(S_m^n + \frac{S_{m2}}{2}\right) - (\alpha_m + \mu_m)\left(E_m^n + \frac{E_{m2}}{2}\right)\right] \\
I_{m3} &= h\left[\alpha_m\left(E_m^n + \frac{E_{m2}}{2}\right) - \mu_m\left(I_m^n + \frac{I_{m2}}{2}\right)\right]
\end{aligned}\right\} \quad (10)$$

Step IV:

$$\left.\begin{aligned}
S_{h4} &= h\left[\Lambda_h + \psi(R_h^n + R_{h3}) - \left(\frac{b\beta_h(I_m^n + I_{m3})}{1 + e(I_m^n + I_{m3})} + \mu_h\right)(S_h^n + S_{h3})\right] \\
E_{h4} &= h\left[\left(\frac{b\beta_h(I_m^n + I_{m2})(S_h^n + S_{h3})}{1 + e(I_m^n + I_{m3})}\right) - (\alpha_h + \mu_h)(E_h^n + E_{h3})\right] \\
M_{h4} &= h[\alpha_h(E_h^n + E_{h3}) - (\varepsilon + \tau + \mu_h)(M_h^n + M_{h3})] \\
C_{h4} &= h[\tau(M_h^n + M_{h3}) - (\theta + \mu_h + \delta_h)(C_h^n + C_{h3})] \\
R_{h4} &= h[\varepsilon(M_h^n + M_{h3}) + \theta(C_h^n + C_{h3}) - (\psi + \mu_h)(R_h^n + R_{h3})] \\
S_{m4} &= h\left[\Lambda_m - \left(\frac{b\beta_{1m}(M_h^n + M_{h3})}{1 + e(M_h^n + M_{h3})} + \frac{b\beta_{2m}(C_h^n + C_{h3})}{1 + e(C_h^n + C_{h3})} + \mu_m\right)(S_m^n + S_{m3})\right] \\
E_{m4} &= h\left[\left(\frac{b\beta_{1m}(M_h^n + M_{h3})}{1 + e(M_h^n + M_{h3})} + \frac{b\beta_{2m}(C_h^n + C_{h3})}{1 + e(C_h^n + C_{h3})}\right)(S_m^n + S_{m3}) - (\alpha_m + \mu_m)(E_m^n + E_{m3})\right] \\
I_{m4} &= h[\alpha_m(E_m^n + E_{m3}) - \mu_m(I_m^n + I_{m3})]
\end{aligned}\right\} \quad (11)$$

hence:

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$$\left.\begin{aligned}
S_h^{n+1} &= S_h^n + \frac{1}{6}[S_{h1} + 2S_{h2} + 2S_{h3} + S_{h4}] \\
E_h^{n+1} &= E_h^n + \frac{1}{6}[E_{h1} + 2E_{h2} + 2E_{h3} + E_{h4}] \\
M_h^{n+1} &= M_h^n + \frac{1}{6}[M_{h1} + 2M_{h2} + 2M_{h3} + M_{h4}] \\
C_h^{n+1} &= C_h^n + \frac{1}{6}[C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4}] \\
R_h^{n+1} &= R_h^n + \frac{1}{6}[R_{h1} + 2R_{h2} + 2R_{h3} + R_{h4}] \\
S_m^{n+1} &= S_m^n + \frac{1}{6}[S_{m1} + 2S_{m2} + 2S_{m3} + S_{m4}] \\
E_m^{n+1} &= E_m^n + \frac{1}{6}[E_{m1} + 2E_{m2} + 2E_{m3} + E_{m4}] \\
I_m^{n+1} &= I_m^n + \frac{1}{6}[I_{m1} + 2I_{m2} + 2I_{m3} + I_{m4}]
\end{aligned}\right\} \quad (12)$$

where $S_h^0 > 0,\ E_h^0 \geq 0,\ M_h^0 \geq 0,\ C_h^0 \geq 0,\ R_h^0 \geq 0,\ S_m^0 > 0,\ E_m^0 \geq 0$, and $I_m^0 \geq 0$.

Non-Standard Finite Difference (NSFD Scheme

Here, we developed a non-standard finite difference scheme that is dynamically consistent with the continuous system in (1). Following the idea of Mickens, (2000 & 2007), we discretize model (1) as follows:

Let $S_h^n, E_h^n, M_h^n, C_h^n, R_h^n, S_m^n, E_m^n$ and I_h^n respectively denote the numerical approximations of $S_h(t), E_h(t), M_h(t), C_h(t), R_h(t), S_m(t), E_m(t)$ and $I_m(t)$ at $t = nh$, $n = 0,1,2, \ldots$, where h is the time-step size. Then equation (1) becomes;

$$\left.\begin{aligned}
\frac{S_h^{n+1} - S_h^n}{\phi_1(h)} &= \Lambda_h + \psi R_h^n + \left[\frac{b\beta_h I_m^n}{1 + eI_m^n}\right] S_h^{n+1} - \mu_h S_h^{n+1} \\
\frac{E_h^{n+1} - E_h^n}{\phi_1(h)} &= \left[\frac{b\beta_h I_m^n}{1 + eI_m^n}\right] S_h^{n+1} - (\alpha_h + \mu_h) E_h^{n+1} \\
\frac{M_h^{n+1} - M_h^n}{\phi_1(h)} &= \alpha_h E_h^{n+1} - (\varepsilon + \tau + \mu_h) M_h^{n+1} \\
\frac{C_h^{n+1} - C_h^n}{\phi_1(h)} &= \tau M_h^{n+1} - (\theta + \mu_h + \delta_h) C_h^{n+1} \\
\frac{R_h^{n+1} - R_h^n}{\phi_1(h)} &= \varepsilon M_h^{n+1} + \theta C_h^{n+1} - (\psi + \mu_h) R_h^{n+1} \\
\frac{S_m^{n+1} - S_m^n}{\phi_2(h)} &= \Lambda_m + \left[\frac{b\beta_{1m} M_m^n}{1 + eM_m^n} + \frac{b\beta_{2m} C_m^n}{1 + eC_m^n}\right] S_m^{n+1} - \mu_m S_m^{n+1} \\
\frac{E_m^{n+1} - E_m^n}{\phi_2(h)} &= \left[\frac{b\beta_{1m} M_m^n}{1 + eM_m^n} + \frac{b\beta_{2m} C_m^n}{1 + eC_m^n}\right] S_m^{n+1} - (\alpha_m + \mu_m) E_m^{n+1} \\
\frac{I_m^{n+1} - I_m^n}{\phi_2(h)} &= \alpha_m E_m^{n+1} - \mu_m I_m^{n+1}
\end{aligned}\right\} \quad (13)$$

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The initial values of the discrete model (13) are assumed to be positive: $S_h^0 > 0$, $E_h^0 \geq 0$, $M_h^0 \geq 0$, $C_h^0 \geq 0$, $R_h^0 \geq 0$, $S_m^0 > 0$, $E_m^0 \geq 0$, and $I_m^0 \geq 0$.

From the model (13), we have;

$$\frac{N_h^{n+1} - N_h^n}{\phi_1(h)} = \Lambda_h - \mu_h N_h^{n+1} - \delta_h C_h^{n+1} \tag{14}$$

where $N_h^n = S_h^n + E_h^n + M_h^n + C_h^n + R_h^n$

In the absence of disease, equation (14) simplifies to;

$$\frac{N_h^{n+1} - N_h^n}{\phi_1(h)} = \Lambda_h - \mu_h N_h^{n+1} \tag{15}$$

Solving equation (15) at $t = nh$ yields:

$$N_h^{n+1} = \frac{\Lambda_h}{\mu_h} + \left[N_h^n - \frac{\Lambda_h}{\mu_h}\right] e^{-\mu_h(nh)} \tag{16}$$

By comparing equation (15) and (16), we have;

$$\phi_1(h) = \frac{e^{\mu_h(h)} - 1}{\mu_h}$$

Similarly,

$$\phi_2(h) = \frac{e^{\mu_m(h)} - 1}{\mu_m}$$

The discrete model (13) can be rearranged to get explicit form as shown below:

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$$\left.\begin{aligned}
S_h^{n+1} &= \frac{S_h^n + \phi_1(h)[\Lambda_h + \psi R_h^n]}{1 + \phi_1(h)[\Phi(I_m^n) + \mu_h]} \\
E_h^{n+1} &= \frac{E_h^n + \phi_1(h)\Phi(I_m^n)S_h^{n+1}}{1 + \phi_1(h)[\alpha_h + \mu_h]} \\
M_h^{n+1} &= \frac{M_h^n + \phi_1(h)\alpha_h E_h^{n+1}}{1 + \phi_1(h)[\varepsilon + \tau + \mu_h]} \\
C_h^{n+1} &= \frac{C_h^n + \phi_1(h)\,\tau\, M_h^{n+1}}{1 + \phi_1(h)[\theta + \mu_h + \delta_h]} \\
R_h^{n+1} &= \frac{R_h^n + \phi_1(h)\,[\varepsilon\, M_h^{n+1} + \theta C_h^{n+1}]}{1 + \phi_1(h)[\psi + \mu_h]} \\
S_m^{n+1} &= \frac{S_m^n + \phi_2(h)\Lambda_m}{1 + \phi_2(h)[\Phi(M_h^n) + \Phi(C_h^n) + \mu_m]} \\
E_m^{n+1} &= \frac{E_m^n + \phi_2(h)[\Phi(M_h^n) + \Phi(C_h^n)]S_m^{n+1}}{1 + \phi_2(h)[\alpha_m + \mu_m]} \\
I_m^{n+1} &= \frac{I_h^n + \phi_2(h)\alpha_m E_m^{n+1}}{1 + \phi_2(h)\mu_m}
\end{aligned}\right\} \quad (17)$$

where

$$\Phi(I_m^n) = \frac{b\beta_h I_m^n}{1 + eI_m^n}$$

$$\Phi(M_h^n) = \frac{b\beta_{1m} M_m^n}{1 + eM_m^n}$$

$$\Phi(C_h^n) = \frac{b\beta_{2m} C_m^n}{1 + eC_m^n}$$

Hence, since all parameters in system (17) are positive, then the numerical solutions will also be positive.

Comparison of NSFD scheme with SFD (Forward Euler and RK-4 methods)

Here, we compare the NSFD method with Forward Euler scheme and RK-4 method at different values of the step-size, h and study the convergence of each scheme at disease-free equilibrium. The numerical experiments presented in this section were performed using MATLAB software with the following parameter values in table 3 compatible with malaria.

Table 3 Model Parameter Values for the convergence analysis at DFE

Parameters	Values	Parameters	Values
Λ_h:	0.000215	Λ_m:	0.07
μ_h:	0.0000548	μ_m:	$^1/_{15}$
δ_h;	0.001	b:	0.12

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β_h:	0.032	β_{1m}:	0.048
β_{2m}:	0.48	α_h:	$1/17$
α_m:	$1/18$	ψ:	$1/730$
e:	0.25	g:	0.25
ϕ:	0.05	τ:	0.4
θ:	0.05		

Sources: Olaniyi & Obabiyi, 2013; Otieno *et al*, 2016; Bala & Gimba, 2019; Aguilar & Gutierrez, 2020.

Figures 1a – 1d and figure 2a – 2d show that the three methods under study (Euler, RK-4 and NSFD) all converge to the disease-free equilibrium points for step-size, h = 1 and h = 2 respectively.

Case 1: $h = 1$

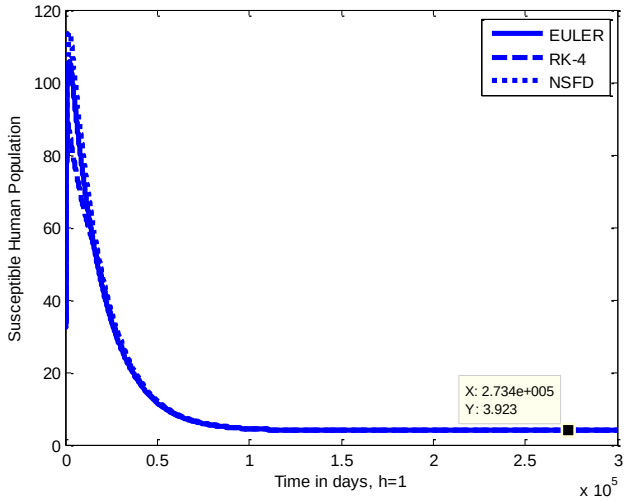

Figure 1a Graph of susceptible human population at $h = 1$ showing the convergence of Euler, Rk-4 and NSFD scheme

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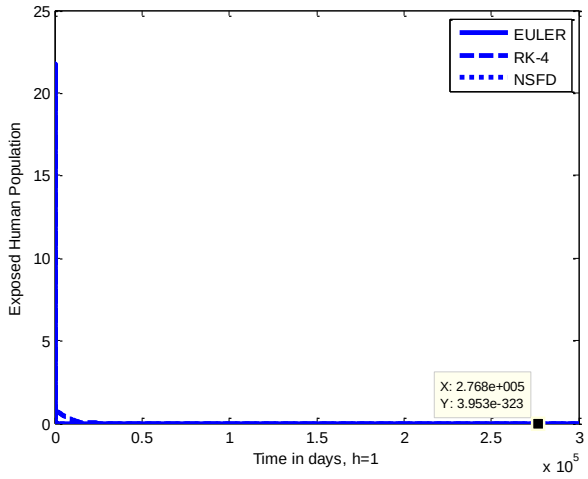

Figure 1b Graph of exposed human population at $h = 1$ showing the convergence of Euler, Rk-4 and NSFD scheme

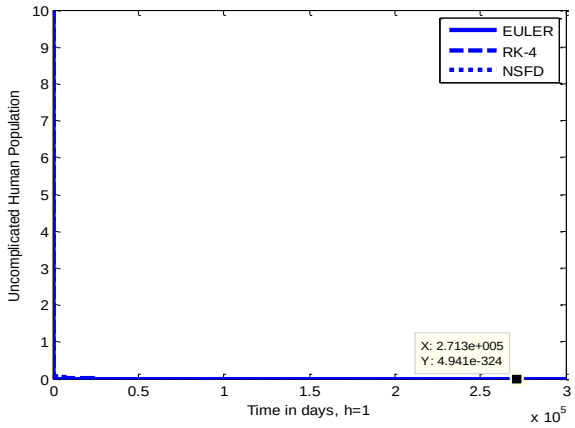

Figure 1c Graph of uncomplicated human population at $h = 1$ showing the convergence of Euler, Rk-4 and NSFD scheme

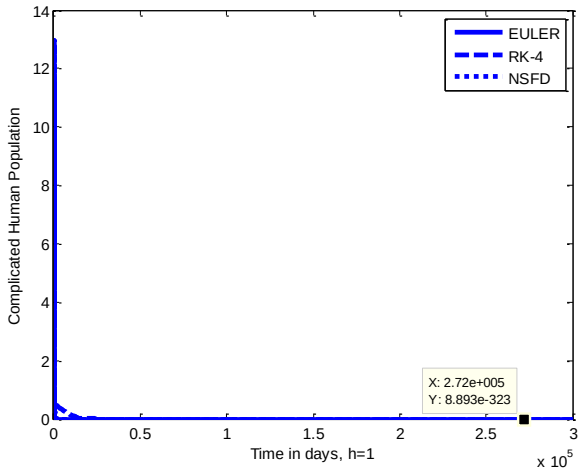

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Figure 1d Graph of complicated human population at $h = 1$ showing the convergence of Euler, Rk-4 and NSFD scheme

Case II: $h = 2$

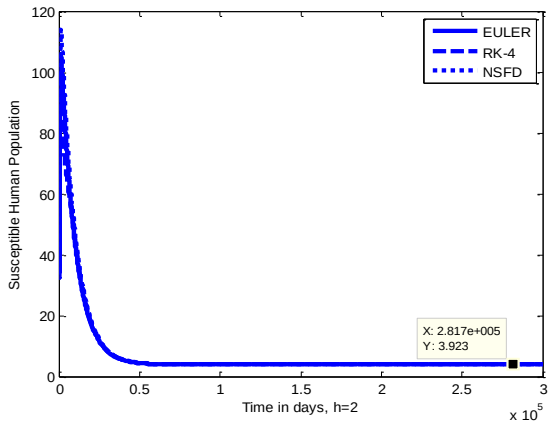

Figure 2a Graph of susceptible human population at $h = 2$ showing the convergence of Euler, Rk-4 and NSFD scheme

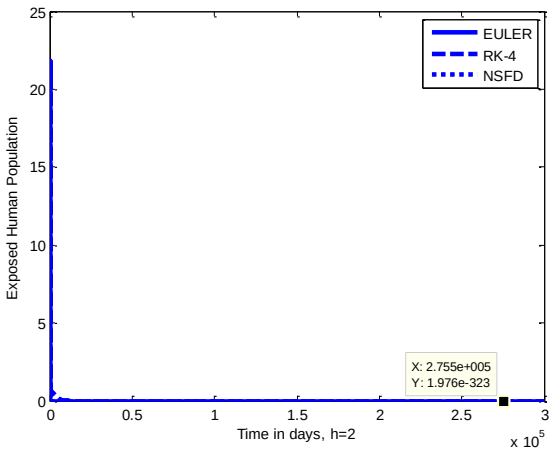

Figure 2b Graph of exposed human population at $h = 2$ showing the convergence of Euler, Rk-4 and NSFD scheme

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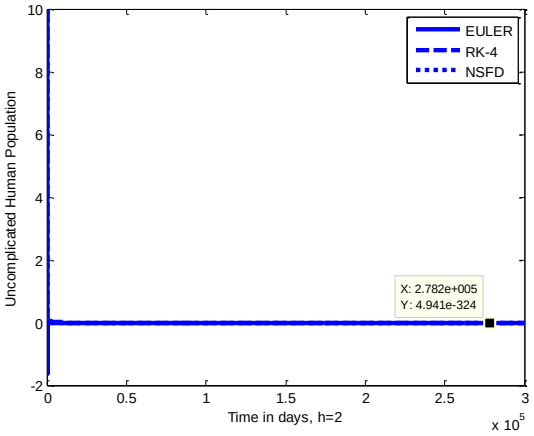

Figure 2c Graph of uncomplicated human population at $h = 2$ showing the convergence of Euler, Rk-4 and NSFD scheme

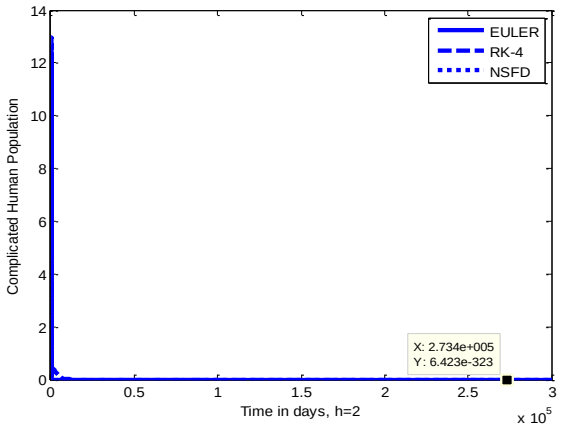

Figure 2d Graph of complicated human population at $h = 2$ showing the convergence of Euler, Rk-4 and NSFD scheme

Case: III: $h = 3$

However, figure 3a-3d show that for an increase in step-size to h = 3, the Euler method diverges and produces negative values while RK-4 and NSFD scheme converge to true equilibrium point of the continuous-time model.

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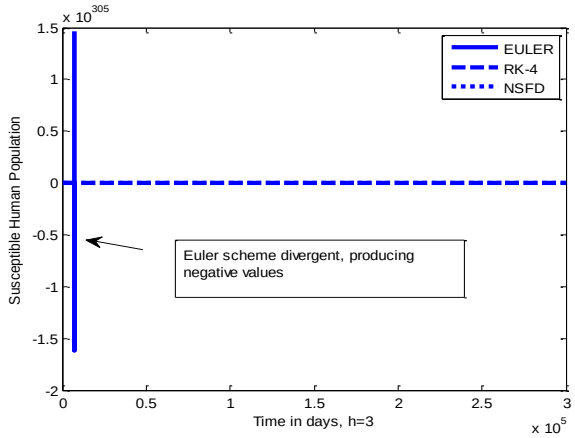

Figure 3a Graph of susceptible human population at $h = 3$ showing the convergence of Euler, Rk-4 and NSFD scheme

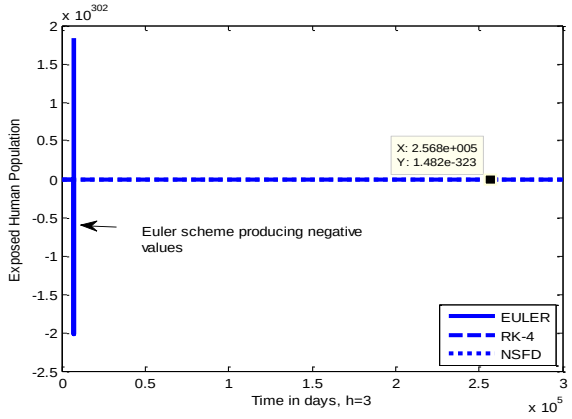

Figure 3b Graph of exposed human population at $h = 3$ showing the convergence of Euler, Rk-4 and NSFD scheme

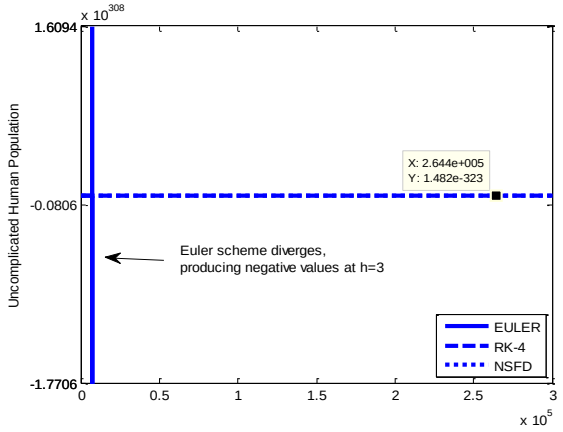

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Figure 3c Graph of uncomplicated human population at $h = 3$ showing the convergence of Euler, Rk-4 and NSFD scheme

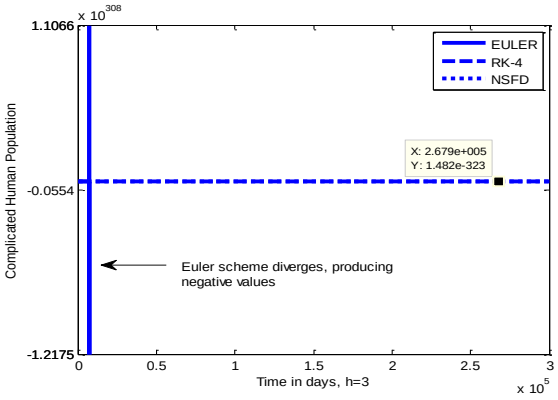

Figure 3d Graph of complicated human population at $h = 3$ showing the convergence of Euler, Rk-4 and NSFD scheme

Case IV: $h = 4$

Further increase in step-size to h = 4 as shown in figure 4a – 4d revealed that RK-4 scheme diverges and produces negative solutions while NSFD scheme remains convergent.

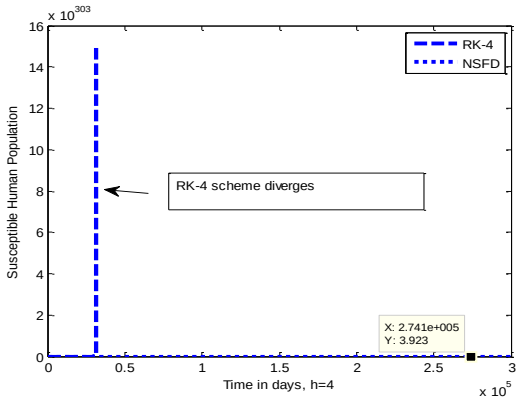

Figure 4a Graph of susceptible human population at $h = 4$ showing the convergence of Rk-4 and NSFD scheme

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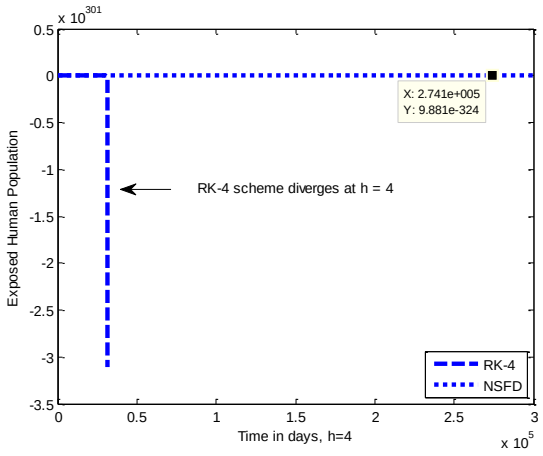

Figure 4b Graph of exposed human population at $h = 4$ showing the convergence of Rk-4 and NSFD scheme

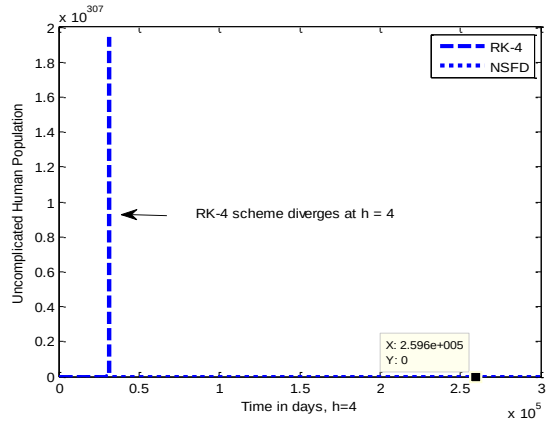

Figure 4c Graph of uncomplicated human population at $h = 4$ showing the convergence of

Rk-4 and NSFD scheme

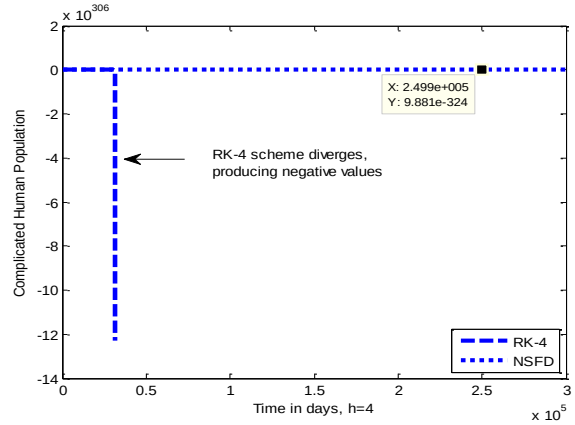

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Figure 4d Graph of complicated human population at $h = 4$ showing the convergence of Rk-4 and NSFD scheme

Case V: Effect of step-size on NSFD scheme

Figure 5 affirms that the proposed NSFD scheme preserves positivity property and converges to the true equilibrium point of the continuous-time model for large value of the step-size. It should be noted that in modeling infectious diseases for a long period of time, large step-size reduces time and cost of computations, hence the need for a numerical method that convergences at large step-size.

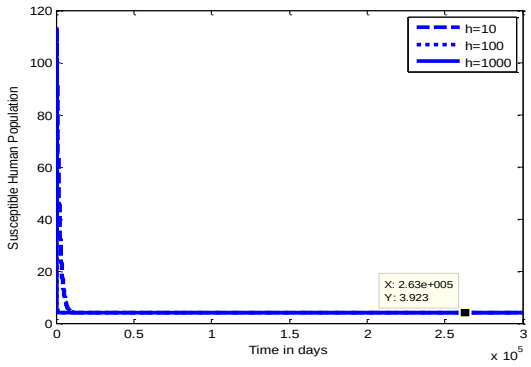

Figure 5: Graph of susceptible human population showing the effects of step-size on the convergence NSFD scheme at $h = 10$, $h = 100$ and $h = 1000$

CONCLUSION AND RECOMMENDATIONS

The continuous-time model was discretized using the Non-standard Finite difference (NSFD) method. The numerical experiments on the proposed NSFD scheme showed that NSFD preserves the properties of the continuous-time model, converges unconditionally and compares favorably with other standard finite difference methods. We therefore recommend the use of non-standard finite difference scheme for discretization of continuous-time models as it converges unconditionally and preserves the essential properties of the model.

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