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ON THE NOVEL AUXILIARY LYAPUNOV FUNCTION AND UNIFORM ASYMPTOTIC PRACTICAL STABILITY OF NONLINEAR IMPULSIVE CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS VIA NEW MODELLED GENERALIZED DINI DERIVATIVE

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ABSTRACT: *In this paper, the uniform asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse is examined using an auxiliary Lyapunov functions which are analogues of vector Lyapunov functions. Together with comparison results, sufficient conditions for the uniform practical stability as well as the uniform asymptotic practical stability of the impulsive Caputo fractional order systems are established. An illustrative example is given to confirm the suitability of the obtained results.*

KEYWORDS: Uniform asymptotic practical stability, Caputo derivative, impulse, Lyapunov function.

INTRODUCTION

The systematic study of the concept of fractional calculus which is mainly concerned with the pure mathematical fields is traceable to the 19th century by Liouville, Riemann, Caputo, etc. [28], [38].

One of the trends in the stability theory of solutions of differential equations is the so-called practical stability [9], [28], [32], [33] and [34]. This aspect of stability was introduced by [25], and it is used in estimating the worst-case transient and steady-state responses together with verifying pointwise in time constraints imposed on the solution path or the trajectory curve. Fundamental results in this area were established in [9], [32], [41] for integer order derivative.

Rapidly revolving alongside the development of the theory of practical stability in recent years is the mathematical theory of impulsive differential equations which have experienced a massive research attention and development. The theory of impulsive differential equation is richer than the corresponding theory of differential equations [17] as they constitutes very important models for describing the true state of several real life processes and phenomena since many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are assumed to be subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. For instance, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems do exhibit impulsive effects [17].

Again, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [36], and one of the most versatile methods in the study of the stability properties of impulsive systems is the Lyapunov second method (see [5]).

The novelty of the Lyapunov's second method as observed in [5], [9] and [39] over other methods of examining stability properties of impulsive differential systems like the Razumikhin technique, the use of matrix inequality, etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential equation by seeking an appropriate continuously differentiable function (Lyapunov's function) that is positive definite and whose time derivative along the trajectory curve is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [12] and [31] and fundamental results have been obtained for its corresponding fractional order in [2], [5], [9] and [41]. [5] obtained fundamental results on the practical stability of impulsive Caputo fractional differential equations using the vector Lyapunov functions, stressing the importance of the method over the scalar Lyapunov function.

In this paper, the uniform asymptotic practical stability of impulsive Caputo fractional order systems is considered, and by means of the comparison principle, sufficient conditions for the uniform asymptotic practical stability of impulsive fractional order systems is established using a class of piecewise continuous functions. An illustrative example is given to confirm the suitability of the obtained results.

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Preliminary Notes and Definitions

The basic concept of calculus such as the derivative and integrals can be generalized to noninteger order using fractional calculus. This allows for more in-depth understanding of behavior of functions, particularly when they have complex or irregular behavior. There are multiple ways to define fractional derivatives and the integrals and the choice of definitions depends on the specific applications (see [15], [26], [27], [35] and [38]).

There are several definitions of fractional derivatives and fractional integrals.

General case. Let the number $n-1<\beta<n, \beta>0$ be given, where n is a natural number, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.1.

The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}^{RL}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t\geq t_0 $$

Definition 2.2.

The Caputo fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}^{C}_{t_0}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t\geq t_0 $$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3.

The Grunwald-Letnikov fractional derivative of order q of $\gamma(t)$ is given by (See [1])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0 $$

and

Definition 2.4. The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (See [1])

$$ {}^{GL}D_0^{\beta}\gamma(t)=\limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t\geq t_0 $$

where ${}^{\beta}C_r$ are the binomial coefficients and $\left[\frac{t-t_0}{h}\right]$ denotes the integer part o $\frac{t-t_0.}{h}$

Particular case. (when n=1). In most applications, the order of β is often less than 1, so that $\beta \in (0,1)$. For simplicity of notation, we will use ${}^{C}D^{\beta}$ instead of ${}^{C}_{t_0}D^{\beta}$ and the Caputo fractional derivative of order β of the function $\gamma(t)$ is

$$ {}^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{-\beta}\gamma'\,ds,\ \ t \ge t_0 \qquad (2.1)$$

Impulses in Fractional Differential Equations

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0<\beta<1$.

$$\begin{aligned} {}^{C}D^{\beta}\gamma(t) &= f(t,\gamma),\ t \ge t_0, \\ \gamma(t_0) &= \gamma_0, \end{aligned} \qquad (3.1)$$

where $\gamma \in R^N,\ f \in C[R_+ \times R^N, R^N],\ f(t,0) \equiv 0\ and\ (t_0, x_0) \in R_+ \times R^N$.

Some sufficient conditions for the existence of the global solutions to (3.1) are considered in [7, 11, 22, 23, 29, 35, 42].

The IVP for FrDE (3.1) is equivalent to the following Volterra integral equation (See [2]),

$$\gamma(t) = \gamma_0 + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1} f(s,\gamma(s))ds,\ t \ge t_0 \qquad (3.2)$$

Consider the IVP for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0<\beta<1$,

$$\begin{aligned} {}^{C}D^{\beta}\gamma(t) &= f(t,\gamma),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,... \\ \Delta\gamma &= I_k(\gamma(t_k)),\ t = t_k,\ k \in N, \\ \gamma(t_0^+) &= \gamma_0, \end{aligned} \qquad (3.3)$$

where $\gamma, \gamma_0 \in R^N,\ f \in C[R_+ \times R^N, R^N],\ and\, t_0 \in R_+\ I_k : R^N \to R^N,\ k = 1,2,....$

under the following assumptions:

(i) $0 < t_1 < t_2 < ... < t_k < ...,\ and\ t_k \to \infty\ as\, k \to \infty;$

(ii) $f : R_+ \times R^N \to R^N$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $x \in R^N, k = 1,2,...,$ and $\lim_{(t,y)\to(t_k^+,\gamma)} f(t,y) = f(t_k^+,\gamma)$ exists;

(iii) $I_k \times R^N \to R^N$

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In this paper, we assume that $f(t,0) \equiv 0$, $I_k(0) \equiv 0$ for all k so that we have trivial solution for (3.3), and the points t_k, $k = 1,2,\ldots$ are fixed such that $t_1 < t_2 < \ldots.\ and\ \lim_{k\to\infty} t_k = \infty$. The system (3.3) with initial condition $\gamma(t_0) = \gamma_0$ is assumed to have a solution $\gamma(t;t_0,\gamma_0) \in PC^{\beta}([t_0,\infty), R^N)$.

Remark 3. 1. The second equation in (3.3) is called the impulsive condition, and the function $I_k(\gamma(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 3.1. Let $\Omega : R_+ \times R^N \to R^N$ Then Ω is said to belong to class χ if,

(i) Ω is continuous in $(t_{k-1}, t_k]$ and for each $\gamma \in R^N$ and $\lim_{(t,y)\to(t_k^+,\gamma)} \Omega(t,y) = \Omega(t_k^+,\gamma)$ exists;

(ii) Ω is locally Lipschitz with respect to its second argument x and $\Omega(t,0) \equiv 0$

Now, for any function $\Omega(t,\gamma) \in PC([t_0,\infty) \times \xi, R_+^N)$. we define the Caputo fractional Dini derivative as:

$$
{}^{C}D_{+}^{\beta}\Omega(t,\gamma) = \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \{\Omega(t,\gamma) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [\Omega(t - rh, \gamma - h^{\beta} f(t,\gamma) - \Omega(t_0,\gamma_0)]\} \tag{3.4}
$$

$t \geq t_0$ where $t \in [t_0,\infty)$, $\gamma, \gamma_0 \in \xi$, $\xi \in R^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T]$.

Definition 3.2. A function $g \in PC[R^n, R^n]$ is said to be quasimonotone nondecreasing in γ, if $\gamma \leq y$ and $\gamma_i = y_i$ for $1 \leq i \leq n$ implies $g_i(\gamma) = g_i(y)$.

Definition 3.3. The zero solution of (3.3) is said to be:

(*PS1*) practically stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \leq \delta$ implies $\gamma_0 \in R^N\ \|\gamma(t,t_0,\gamma_0)\| < \varepsilon$ for $t \geq t_0$;

(*PS2*) uniformly practically stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$, continuous in t_0 such that for any $\gamma_0 \in R^N$, $\|\gamma_0\| \leq \delta$ implies $\gamma_0 \in R^N\ \|\gamma(t,t_0,\gamma_0)\| < \varepsilon$ for $t \geq t_0$;

(*PS3*) asymptotically practically stable if it is practically stable and if for each $\varepsilon > 0\ and\ t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0,\varepsilon)$ such that for $t \geq t_0 + T$ and $\|\gamma_0\| \leq \delta$ implies $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon$;

(*PS4*) uniformly asymptotically practically stable if it is uniformly practically stable and $\delta_0 = \delta_0(\in)$ and $T = T(\varepsilon)$ such that for $t \geq t_0 + T$, the inequality $\|\gamma_0\| \leq \delta$ implies $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon.$

Definition 3.4. A function $a(r)$ is said to belong to the class K if $a \in PC([0,\psi), R_+)$, $a(0)=0$, and $a(r)$ is strictly monotone increasing in r.

In this paper, we define the following sets:

$$\overline{S}_{\psi} = \{\gamma \in R^N : \|\gamma\| \le \psi\}$$

$$S_{\psi} = \{\gamma \in R^N : \|\gamma\| < \psi\}$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$\begin{aligned} {}_{t_0}^{C}D^{\beta}u &= g(t,u),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,... \\ \Delta u &= \psi_k(u(t_k)),\ t = t_k,\ k \in N, \\ u(t_0^+) &= u_0, \end{aligned} \tag{3.5}$$

existing for $t \ge t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g : R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (3.5) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC^{\beta}([t_0, \infty), R^n)$.

Lemma 3.2. Assume $m \in PC([t_0,T] \times \overline{S}_{\psi}, R^N)$. and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*,\alpha_1) = m(t^*,\alpha_2)$ and $m(t,\alpha_1) < m(t,\alpha_2)$ for $t_0 \le t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) > 0$ holds.

Proof. Let $\Omega(t,\gamma) = m(t,\alpha_1) - m(t,\alpha_2)$. Applying (3.4), we have

$$\begin{aligned} {}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \{ & [m(t^*,\alpha_1) - m(t^*,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ & - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)]\} \end{aligned}$$

when $\alpha_1 = \alpha_2$ we have

$$\begin{aligned} {}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h \to 0^+} \frac{1}{h^{\beta}} \{ & -[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\ & - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)]\} \end{aligned}$$

$$\begin{aligned} {}^{C}D_{+}^{\beta}(m(t^{*},\alpha_1)-m(t^{*},\alpha_2)) = &-\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}[m(t_0,\alpha_1)-m(t_0,\alpha_2)] \\ &+\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t^{*}-rh,\alpha_1)-m(t^{*}-rh,\alpha_2)] \\ &-\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)] \end{aligned}$$

$$\begin{aligned} {}^{C}D_{+}^{\beta}(m(t^{*},\alpha_1)-m(t^{*},\alpha_2)) = &-\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}[m(t_0,\alpha_1)-m(t_0,\alpha_2)] \\ &-\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)] \end{aligned}$$

$${}^{C}D_{+}^{\beta}(m(t^{*},\alpha_1)-m(t^{*},\alpha_2)) = -\limsup_{h\to 0^{+}}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r}\,{}^{\beta}C_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)]$$

Applying equation (3.8) in [1], we have

$${}^{C}D_{+}^{\beta}m(t^{*},\alpha_1) - {}^{C}D_{+}^{\beta}m(t^{*},\alpha_2) = -\frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)}\left[m(t^{*},\alpha_1)-m(t^{*},\alpha_2)\right]$$

By the lemma, we have that

$$m(t,\alpha_1)-m(t,\alpha_2)<0 \quad for\ t_0\le t\le t^{*}$$

And so, it follows that

$${}^{C}D_{+}^{\beta}m(t^{*},\alpha_1) - {}^{C}D_{+}^{\beta}m(t^{*},\alpha_2) > 0$$

Note that some existence results for (3.5) are given in [11], [13] and [14].

Remark 3.3. Lemma 3.2 extends Lemma 1 in [1], where the vectors $m(t,\alpha_1)\ and\ m(t,\alpha_2)$ are compared component-wise.

Lemma 3.4. Assume that:

(i) $\gamma(t,t_0,\gamma_0),\ \gamma^{*}\in PC([t_0,T],\ R^{N})$ is a solution of the IFrDE (3.5).

(ii) $\Omega\in PC[R_{+}\times R^{N},R_{+}{}^{N}]$ and $\Omega\in\chi\quad\gamma$ such that

$${}^{C}D_{+}^{\beta}\Omega(t,\gamma)\le -c(\|\gamma(t)\|),\ t\ne t_k,\ holds\ for\ all\ (t,\gamma)\in R_{+}\times S_{\psi},\ where\ \ c\in K.$$

There exists $\psi_0>0$ such that $\gamma_0\in S_{\psi}$ implies that $\gamma(t)+I_k(\gamma(t_k))\in S_{\psi}$ and

$\Omega(t_k,\gamma+I_k(\gamma(t_k)))\le\omega_k(\Omega(t,\gamma(t))),\ t=t_k,\gamma\in S_{\psi}$ and the function $\omega_k:R_{+}^{N}\to R_{+}^{N}$ is nondecreasing for $k=1,2,...$

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Then for $t_0 \in [t_0, T]$, the inequality

$$\Omega(t, \gamma^*(t)) \leq \Omega(t_0, \gamma_0) - \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} c(\|\gamma\|) ds$$

holds.

In the following, we shall establish the comparison result for the system (3.3).

Fractional Differential Inequalities and Comparison Results for Vector Fractional Differential Equations

In this section, again we assume that $0 < \beta < 1$.

Theorem 4.1 (Comparison Result).

Assume that:

(i) $g \in PC[R_+ \times R_+^n, R^n]$ and is continuous in $(t_{k-1}, t_k]$, $k = 1,2,...$ and $g(t,u)$ is quasimonotone

nondecreasing in u for each $u \in R^n$ and $\lim\limits_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+, u)$ exists;

(ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

${}^C D_+^{\beta} \Omega(t,\gamma) \leq g(t, \Omega(t,\gamma))$, $(t,\gamma) \in R_+ \times R^N$

and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \leq \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iii) $\vartheta(t) = \vartheta(t, t_0, u_0) \in PC^{\beta}([t_0, T], R^n)$ be the maximal solution of the IVP for the IFrDE system (3.5)

Then,

$$\Omega(t, \gamma(t)) \leq \vartheta(t), \ t \geq t_0, \tag{4.1}$$

where $\gamma(t) = \gamma(t, t_0, \gamma_0) \in PC^{\beta}([t_0, T], R^N)$ is any solution of (3.3) existing on $[t_0, \infty)$, provided that

$$\Omega(t_0^+, \gamma_0) \leq u_0 \tag{4.2}$$

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Proof:

Let $\eta \in S_\psi$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations,

$$\begin{aligned} {}^{C}D^{\beta}u &= g(t,u)+\eta, \quad for\, t\in[t_0,\infty) \\ u(t_0^+) &= u_0+\eta, \end{aligned} \tag{4.3}$$

for $t\in[t_0,\infty)$.

The function $u_\eta(t,\alpha)$ is a solution of (4.3), where $\alpha>0$ the fractional differential equation (3.5) if and only if it satisfies the Volterra fractional integral equation,

$$u_\eta(t,\alpha)=u_0+\eta+\tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,u_\eta(s,\alpha))+\alpha)ds, \;\; t\in[t_0,\infty). \tag{4.4}$$

Let the function $m(t,\alpha)\in PC([t_0,T]\times S_\psi, R_+^N)$ be defined as $m(t,\alpha)=\Omega(t,\gamma^*(t))$

We now prove that

$$m(t,\alpha)<u_\eta(t,\alpha) \quad for \;\; t\in[t_0,\infty) \tag{4.5}$$

Observe that the inequality (4.5) holds whenever $t=t_0$ *i.e.*

$$m(t_0,\alpha)=\Omega(t_0,\gamma_0)\le u_0<u_\eta(t_0,\alpha)$$

Assume that the inequality (4.5) is not true, then there exists a point $t_1>t_0$ such that

$$m(t_1,\alpha)=u_\eta(t_1,\alpha) \;\; and \;\; m(t,\alpha)<u_\eta(t,\alpha) \text{ for } t\in[t_0,t_1).$$

It follows from Lemma 3.2 that

$${}^{C}D_+^{\beta}(m(t_1,\alpha)-u_\eta(t_1,\alpha))>0 \;\; i.e.$$

$${}^{C}D_+^{\beta}(m(t_1,\alpha))>{}^{C}D_+^{\beta}u_\eta(t_1,\alpha)$$

$${}^{C}D_+^{\beta}\Omega(t_1,\gamma(t_1))>{}^{C}D_+^{\beta}u_\eta(t_1,\alpha)$$

and using (4.3) we arrive at

$${}^{C}D_+^{\beta}\Omega(t_1,\gamma(t_1))>g(t_1,u(t_1,\alpha))+\eta>g(t_1,u(t_1,\alpha)$$

Therefore,

$${}^{C}D_+^{\beta}m(t_1,\alpha)>g(t_1,u(t_1,\alpha)) \tag{4.6}$$

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From Theorem 4.1, the function $\gamma^*(t)=\gamma(t,t_0,\gamma_0)$ satisfies the IVP (4.3) and the equality

$$\limsup_{h\to 0^+}\tfrac{1}{h^\beta}[\gamma^*(t)-\gamma_0-S(\gamma^*(t),h)]=f(t,\gamma^*(t)),\ \ holds \tag{4.7}$$

where $\gamma^*(t)=\gamma(t,t_0,\gamma_0)$ is any other solution of (3.5), and

$$S(\gamma^*(t),h)=\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\gamma^*(t-rh)-\gamma_0\right] \tag{4.8}$$

is the Grunwald Letnikov fractional derivative

Multiply equation (4.7) through by h^β

$$\limsup_{h\to 0^+}[\gamma^*(t)-\gamma_0-S(\gamma^*(t),h)]=h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t)-\gamma_0-[S(\gamma^*(t),h)+\rho(h^\beta)]=h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t)-h^\beta f(t,\gamma^*(t))=[S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)] \tag{4.9}$$

Then for $t\in[t_0,\infty),$ we have

$$\begin{aligned}
&m(t,\alpha)-m(t_0,\alpha)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[m(t-rh,\alpha)-m(t_0,\alpha)\right]=\\
&\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\Omega(t-rh,\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\}\\
&=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\}+\\
&\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\{[\Omega(t-rh),S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\Omega(t_0,\gamma_0)]-[\Omega(t-rh,\gamma^*(t-rh))-\Omega(t_0,\gamma_0)]\}
\end{aligned} \tag{4.10}$$

Since $\Omega(t,\gamma)$ is locally Lipschitzian with respect to the second variable, we have that,

$$\le L\left|(-1)^{r+1}\right|\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\gamma^*(t-rh)]\right\|$$

where $L>0$ is a Lipschitz constant.

$$\le L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[S(\gamma^*(t),h)+\rho(h^\beta))-(\gamma^*(t-rh)-\gamma_0)\right\| \tag{4.11}$$

Using equation (4.8), equation (4.11) becomes,

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$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)[(\gamma^*(t-rh)-\gamma_0)+\rho(h^{\beta}))-(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)(-1)^{r+1}[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r[(\gamma^*(t-rh)-\gamma_0)]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r\rho(h^{\beta})-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L(-1)^{r+1}\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^{\beta}C_r)[(\gamma^*(t-rh)-\gamma_0)]+[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^{\beta}C_r\rho(h^{\beta})\right\| \tag{4.12}$$

Substituting equation (4.12) into (4.10) we have

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$

$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$

$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

Dividing through by $h^{\beta}>0$ and taking the $\limsup\ as\ h\to 0^+$ we have,

$${}^{C}D_{+}^{\beta}m(t,\alpha)=\limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}\left\{\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^{\beta}f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]\right\}+$$

$$\limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^{\beta})\right\|$$

Recall that,

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)=-1\ and\ \ \limsup_{h\to 0^+}\tfrac{1}{h^{\beta}}\rho(h^{\beta})=0$$

From equations (3.6) and (3.7) in [1], we have that

$${}^{C}D_{+}^{\beta}m(t,\alpha)={}^{C}D_{+}^{\beta}\Omega(t,\gamma^*(t))+L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^*(t-rh)-\gamma_0)[-(-1)-1]+0\right\|$$

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Using condition (ii) of the Theorem 4.1, we obtain the estimate

$$ {}^{C}D_{+}^{\beta} m(t,\alpha) \le g(t,\Omega(t,\gamma^{*}(t)) = g(t,m(t,\alpha)), \tag{4.13} $$

Also,

$$ m(t_0^{+},\alpha) \le u_0 \;\; and \;\; m(t_k^{+},\alpha) = \Omega(t_k^{+},\gamma(t_k) + I_k(\gamma(t_k)) \le \rho_k(m(t_k)) \tag{4.14} $$

Now equation (4.14) with $t = t_1$ contradicts (4.6), hence (4.5) is true. □

For $t \in [t_0,T],$, we now establish that

$$ u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \quad \text{whenever } \eta_1 < \eta_2 \tag{4.15} $$

Observe that the inequality (4.15) holds for $t = t_0$

Assume that (4.15) is not true. Then there exists a point t_1 such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_2,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha) \;\; for \; t \in [t_0,t_1).$

By Lemma 3.2, we have that

$$ {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) > 0 $$

However,

$$ \begin{aligned} {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) &= {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_2,\alpha)) \\ &= g(t_1,u(t_1,\alpha)) + \eta_1 - [g(t_1,u(t_1,\alpha)) + \eta_2] \\ &= \eta_1 - \eta_2 < \mathbf{0} \end{aligned} $$

which is a contradiction, and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_\eta(t,\alpha)\}, \, t \in [t_0,T]$ of (4.3) is uniformly bounded, i.e. there exists $\lambda > 0$ with $\left|u_\eta(t,\alpha)\right| \le \lambda$, with bound λ on $[t_0,T]$. We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0,T]$. Let $k = \sup\{| \, g(t,\gamma) \, | : (t,\gamma) \in [t_0,T] \times [-\lambda,\lambda]\},$, where λ is the bound on the family $\{u_\eta(t,\alpha)\}$. Fix a decreasing sequence $\{\eta_i\}_{i=0}^{\infty}(t)$, such that $\lim_{i\to\infty} \eta_i = 0$ and consider a sequence of functions $\{u_\eta(t,\alpha)\}$. Again, let $t_1, t_2 \in [t_0,T], \; with \; t_1 < t_2$, then we have the following estimates,

$$ \begin{aligned} \left\|u_\eta(t_2,\alpha) - u_\eta(t_1,\alpha)\right\| \le \| \, u_0 + \eta_i + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_2} (t_2 - s)^{\beta-1} (g(s,u(s,\eta_i)) + \eta_i) ds \\ - \left(u_0 + \eta_i + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t_2} (t_1 - s)^{\beta-1} (g(s,u(s,\eta_i)) + \eta_i) ds \right) \| \end{aligned} $$

$$\leq \frac{1}{\Gamma(\beta)}\left\| \left[\int_{t_0}^{t_2}(t_2-s)^{\beta-1} - \int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right](g(s,u(s,\eta_i))u(t_2,\eta_i)ds \right\|$$

$$\leq \frac{k}{\Gamma(\beta)}\left\| \int_{t_0}^{t_2}(t_2-s)^{\beta-1} - \int_{t_0}^{t_1}(t_1-s)^{q-1}\right\| ds$$

$$\leq \frac{k}{\Gamma(\beta)}\left\| \int_{t_0}^{t_1}(t_2-s)^{\beta-1} + \int_{t_1}^{t_2}(t_2-s)^{\beta-1} - \int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right\| ds$$

$$\leq \frac{k}{\Gamma(\beta)}\left\| \int_{t_0}^{t_1}(t_2-s)^{\beta-1} - \int_{t_0}^{t_1}(t_1-s)^{q-1} + \int_{t_1}^{t_2}(t_2-s)^{q-1}\right\| ds$$

$$\leq \frac{k}{\Gamma(\beta)}\left[\left\| \int_{t_0}^{t_1}(t_2-s)^{\beta-1} - \int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right\| + \left\| \int_{t_1}^{t_2}(t_2-s)^{\beta-1}\right\|\right]$$

$$\leq \frac{k}{\beta\Gamma(\beta)}\left[\left\|(t_2-t_0)^{\beta} - (t_1-t_0)^{\beta} - (t_2-t_1)^{\beta}\right\| + \left\|(t_2-t_1)^{\beta}\right\|\right]$$

$$\leq \frac{k}{\beta\Gamma(\beta)}\left[(t_2-t_1)^{\beta} + (t_2-t_1)^{\beta}\right] = \frac{2k}{\Gamma(\beta+1)}(t_2-t_1)^{\beta} < \varepsilon$$

provided $\|t_2-t_1\| < \delta(\varepsilon) = \left(\frac{\Gamma(\beta+1)\varepsilon}{2k}\right)^{\frac{1}{\beta}}$, proving that the family of solutions $\{u_{\eta}(t,\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t,\alpha)\}$ guarantees the existence of a subsequence $\{u_{\eta_{ij}}(t,\alpha)\}$ which converges uniformly to the function $\vartheta(t)$ on $[t_0,T]$. Then we show that $\vartheta(t)$ is a solution of (4.4). Thus, equation (4.4) becomes

$$u_{\eta_{i_j}}(t,\alpha) = u_{0i_j} + \eta_{i_j} + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g_{i_j}(s,u_{\eta_{i_j}}(s,\alpha)) + \eta_{i_j})ds,\ \ t\in[t_0,\infty).$$
(4.16)

Taking the $\lim\ as\ i_j \to \infty$ in (4.16) yields,

$$\vartheta(t) = u_0 + \eta_{i_j} + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,\vartheta(t))ds,\ \ t\in[t_0,\infty). \tag{4.17}$$

Hence, $\vartheta(t)$ is a solution of (3.5) on $[t_0,T]$. We claim that $\vartheta(t)$ is the maximal solution of (3.5). Then, from (4.5), we have that $m(t,\alpha) < u_{\eta}(t,\alpha) \leq \vartheta(t)\ on\ [t_0,T]$..

Suppose that in Theorem 4.1, $g(t,u)\equiv 0$, then we have the following results

Corollary 4.1.

Assume that Condition (i) of Theorem 4.1 holds and,

(i) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that

$$ {}^{C}D_{+}^{\beta}\Omega(t,\gamma) \le 0 $$

(4.18)

holds, and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

Then for $t \in [t_0, \infty),$ the inequality

$$ \Omega(t,\gamma(t)) \le \Omega(t_0^+, \gamma_0) \text{ holds} $$

MAIN RESULTS

In this section, we will obtain sufficient conditions for the uniform practical stability as well as uniform asymptotic practical stability of the system (3.3). Again we assume $0 < \beta < 1$.

Theorem 5.1. Assume that

(i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n$, $k = 1,2,...,$

and $\lim\limits_{(t,y)\to(t_k^+,u)} g(t,y) = g(t_k^+, u)$ exists, and $g(t,u)$ is quasimonotone nondecreasing in u

(ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ *and* $\Omega \in \chi$ such that

${}^{C}D_{+}^{\beta}\Omega(t,\gamma) \le g(t, \Omega(t,\gamma)), t \ne t_k, \textit{ holds for all } (t,\gamma) \in R_+ \times S_\Psi,$

there exists a $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iii) $b(\|\gamma\|) \le \Omega_0(t,\gamma) \le a(\|\gamma\|),$ where $a, b \in K$ and $\Omega_0(t,\gamma), = \sum_{i=1}^{N} \Omega_i(t,\gamma).$

Then the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $0 < \varepsilon < \psi$ and $t_0 \in R_+$ be given.

Assume that the solution (3.5) is uniformly practically stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{N} u_{i0} < \delta \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon),\, t \geq t_0 \tag{5.1}$$

where $u(t, t_0, u_0)$ is any solution of (3.5).

Choose $u_0 = \Omega(t_0^+, \gamma_0)$ and

$$\sum_{i=1}^{N} u_{i0} = a\left(t_0, \|\gamma_0\|\right)$$

Since $a(t, \mathrm{K})$ and $a \in C[R_+ \times R_+, R_+]$, we can find a positive function $\delta = \delta(t_o, \delta_1) > 0$ such that the inequalities

$$a\left(t_0, \|\gamma_0\|\right) < \delta_1 \textit{ and } \|\gamma_0\| < \delta \tag{5.2}$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta$ then $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Suppose that this claim is false, then there would exists a point $t_1 \in [t_0, t)$ and the solution $\gamma(t, t_0, \gamma_0)$ with $\|\gamma_0\| < \delta_1$ such that

$$\|\gamma(t_1)\| = \varepsilon \textit{ and } \|\gamma(t)\| < \varepsilon \text{ for } t \in [t_0, t_1) \tag{5.3}$$

This implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ for $t \in [t_0, t_1)$

So that using equations (4.1) and condition (iv) of Theorem 5.1 we have the estimate

$$b\left(\|\gamma(t_1)\|\right) \leq \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1)), \text{ implying}$$

$$b(\varepsilon) \leq \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1)) \leq \vartheta(t) \tag{5.4}$$

where $\vartheta(t) = \sum_{i=1}^{n} \vartheta_i(t, t_0, u_0)$ is the maximal solution of (3.5).

Then, using equations (5.4), (5.3) and condition (iii) of Theorem 5.1 we arrive at the estimate

$$b(\varepsilon) \leq \Omega_0(t_1, \gamma(t_1)) \leq \sum_{i=1}^{N} \vartheta_i(t, t_0, u_0) < b(\varepsilon)$$

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which leads to a contradiction.

Hence, the uniform practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform practical stability of the trivial solution $\gamma = 0$ of (3.3).

Theorem 5.2. Assume that:

(i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n, k = 1,2,...,$

and $\lim\limits_{(t,y)\to(t_k^+,u)} g(t,y) = g(t_k^+, u)$ exists, and $g(t,u)$ is quasimonotone nondecreasing in u

(ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi \quad \gamma$ such that

$^C D_+^\beta \Omega(t,\gamma) \le -c(\|\gamma(t)\|), t \ne t_k,$ *holds for all* $(t,\gamma) \in R_+ \times S_\psi$, *where* $c \in K.$

There exists $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iii) $b(\|\gamma\|) \le \Omega_0(t,\gamma) \le a(\|\gamma\|),$ for all $(t,\gamma) \in R_+ \times S_\Psi$, where $a, b \in K$ and

$\Omega_0(t,\gamma) = \sum\limits_{i=1}^N \Omega_i(t,\gamma).$

Then the uniform asymptotic practical stability of the trivial solution $u = 0$ of (3.5) implies the uniform asymptotic practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $\gamma^*(t)$ be any solution of (3.3). Then $\gamma^*(t)$ is uniformly asymptotically practically stable, if it is uniformly practically stable and uniformly attractive.

By Theorem 5.1, the trivial solution $\gamma = 0$ is uniformly practically stable. We now proceed to show that the solution $\gamma = 0$ is uniformly attractive, i.e. , for any $\mu > 0,$ there exists a $T = T(\mu) > 0$ such that for any $t_0 \in R_+, \gamma_0 \in R^n$ with $\|\gamma_0\| \le \delta,$ the inequality $\|\gamma^*(t, t_0, \gamma_0)\| < \mu$ holds for $t \ge t_0 + T.$

Let $\lambda \in (0, \delta)$ be a constant such that

$$a(\lambda) < b(\delta) \text{ with } \|\gamma_0\| < \lambda \tag{5.6}$$

Combining condition (iii) and equation (5.6) gives the estimate

$$b(\|\gamma_0\|) \le \Omega_0(t_0, \gamma_0) \le a(\|\gamma_0\|) < a(\lambda) < b(\delta). \tag{5.7}$$

Clearly, from (5.7), we can see that $\|\gamma_0\| < \delta$.

Let $\mu = \mu(\varepsilon),\ \mu \in (0, \varepsilon).$ Set $T(\mu) \geq \left[\left(\frac{\beta\Gamma(\beta)a(\delta)}{c(\beta)}\right)\right]^{1/\beta} > 0.$

We claim that

$$\|\gamma_0\| < \delta \text{ implies that } \|\gamma^*(t)\| < \mu,\ t \geq t_0 + T(\mu) \tag{5.8}$$

Assume that (5.8) is not true, then there existsatleast one $t \geq t_0 + T(\mu),$ such that

$$\|\gamma_0\| < \delta \text{ implies } \|\gamma^*(t)\| \geq \mu \tag{5.9}$$

Since $c \in K,$ condition (ii) of the Theorem can thus be written as

$${}^{C}D_{+}^{\beta}\Omega(t,\gamma) \leq -c(\mu) \tag{5.10}$$

Applying Lemma 3.4 to (5.10), we have that

$$\Omega(t,\gamma^*(t)) \leq \Omega(t_0,\gamma_0) - \frac{c(\mu)}{\Gamma(\beta)}\int_{t_0}^{t_0+T}(t-s)^{\beta-1}ds$$

This implies that

$$\Omega(t,\gamma^*(t)) \leq \Omega(t_0,\gamma_0) - \frac{c(\mu)}{\Gamma(\beta)}\frac{T^{\beta}}{\beta}$$

So that substituting for $T(\mu)$ we have that

$$\Omega(t,\gamma^*(t)) \leq \Omega(t_0,\gamma_0) - \frac{c(\mu)}{\beta\Gamma(\beta)}\left[\left(\frac{\beta\Gamma(\beta)a(\delta)}{c(\mu)}\right)^{\frac{1}{\beta}}\right]^{\beta}$$

$$\leq \Omega(t_0,\gamma_0) - a(\delta)$$

$$\leq a(\|x_0\|) - a(\delta)$$

$$\leq a(\delta) - a(\delta) = 0$$

which contradicts condition (iii) of the Theorem, and so (5.8) is established.

APPLICATION

Consider the system of fractional differential equations

$$\begin{aligned} {}^{C}D^{q}\gamma_1(t) &= -6\gamma_1 - \tfrac{\gamma_2^2\cos\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{\gamma_2^2 \tan\gamma_1}{\gamma_1},\ t \neq t_k \\ {}^{C}D^{q}\gamma_2(t) &= \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2\sin\gamma_1 - \gamma_2 \sec\gamma_1 - \gamma_1^2\cos\gamma_2,\ \ t \neq t_k \\ \Delta\gamma_1 &= \xi_k(\gamma(t_k)),\ t = t_k \\ \Delta\gamma_2 &= \tau_k(\gamma(t_k)),\ \ t = t_k\ ,\ k = 1,2,... \end{aligned} \tag{6.1}$$

with initial conditions

$$\gamma_1(t_0^+) = \gamma_{10}\ \ and\ \gamma_2(t_0^+) = \gamma_{20}$$

where $\gamma_1, \gamma_2 \in R^N$ are arbitrary functions.

Equation (6.1) is equivalent to (3.3) and $f = (f_1 f_2)$, where $f_1(t,\gamma_1) = {}^{C}D^{q}\gamma_1(t) = -6\gamma_1 - \tfrac{\gamma_2^2\cos\gamma_1}{2\gamma_1} + \gamma_1\sin\gamma_2 + \tfrac{\gamma_2^2\tan\gamma_1}{\gamma_1}$ and $f_2(t,\gamma_2) = {}^{C}D^{q}\gamma_2(t) = \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2\sin\gamma_1 - \gamma_2\sec\gamma_1 - \gamma_1^2\cos\gamma_2$

Consider a vector Lyapunov function of the form $\Omega = (\Omega_1, \Omega_2)^T$, where

$\Omega_1(t,\gamma_1,\gamma_2) = \gamma_1{}^2,\ \Omega_2(t,\gamma_1,\gamma_2) = \gamma_2{}^2$

So that $\Omega = (\Omega_1, \Omega_2)^T$ with $\gamma = (\gamma_1, \gamma_2) \in R^2$, so that $\|\gamma\| = \sqrt{\gamma^2 + y^2}$

$$\sum_{i=1}^{2} \Omega_i(t,\gamma_1,\gamma_2) = \gamma_1^2 + \gamma_2^2$$

The assumption,

$$b(\|\gamma\|) \leq \sum_{i=1}^{n} \Omega_i(\gamma, y) \leq a(t, \|\gamma\|) \text{ reduces to}$$

$$\sqrt{\gamma^2 + y^2} \leq \gamma^2 + y^2 \leq 2\left(\sqrt{\gamma^2 + y^2}\right)^2$$

with the proviso that $b(\wp) = \wp, and\ a(\wp) = 2\wp^2$.

Furthermore, we deduce that using equation (3.4) and $\Omega_1(t,\gamma_1,\gamma_2) = \gamma_1{}^2$

$${}^{C}D_{+}^{\beta}\Omega(t,\gamma_1,\gamma_2) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\Omega(t,\gamma_1,\gamma_2) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\,{}^{\beta}C_r\left[\Omega(t - rh, \gamma - h^\beta f_i(t,\gamma_1,\gamma_2)) - \Omega(t_0,\gamma_0)\right]\right\}, t \geq t_0$$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)\left[\Omega(t-rh, \gamma - h^\beta f_1(t,\gamma_1)) - \gamma_{10}^2\right]\right\}, t \geq t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) = \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r) \quad [(\gamma_1 - h^\beta f_1(t,\gamma_1))^2 - \gamma_{10}^2]\right\}, t \geq t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 - \gamma_{10}^2) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)[\gamma_1^2 - 2\gamma_1 h^\beta f_1(t,\gamma_1) + h^{2\beta} f_1{}^2(t,\gamma_1)) - \gamma_{10}^2]\right\}, t \geq t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\gamma_1{}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)\gamma_1^2 - \gamma_{10}^2 - + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)\gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r) h^\beta f_1(t,\gamma_1)\right\}, t \geq t_0 $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \limsup_{h\to 0^+} \tfrac{1}{h^\beta}\left\{\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)\gamma_1^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r)\gamma_{10}^2 - 2\gamma_1 \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r) h^\beta f_1(t,\gamma_1)\right\}, t \geq t_0 \qquad (6.2) $$

Recall from equations (3.7) and (3.8) in [1] that

$$ \limsup_{h\to 0^+} \tfrac{1}{h^\beta} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r) = \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} \quad and \quad \lim_{h\to 0^+} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r ({}^\beta C_r) = -1 \qquad (6.3) $$

and substituting (6.3) into (6.2), we have,

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} - \frac{\gamma_{10}^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1 f_1(t,\gamma_1) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1 f_1(t,\gamma_1) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} + 2\gamma_1(-6\gamma_1 - \tfrac{\gamma_2^2 \cos\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{\gamma_2^2 \tan\gamma_1}{\gamma_1}) $$

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \leq \frac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} - 12\gamma_1^2 - \gamma_2^2\cos\gamma_1 + 2\gamma_1^2 \sin\gamma_2 + \gamma_2^2 \tan\gamma_1 $$

$As\ t \to \infty,\ \dfrac{\gamma_1^2}{t^\beta \Gamma(1-\beta)} \to 0,$ so we now have that

$$ \begin{aligned} {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) &\leq -12\gamma_1^2 - \gamma_2^2\cos\gamma_1 + 2\gamma_1^2 \sin\gamma_2 + \gamma_2^2 \tan\gamma_1 \\ &= 2\gamma_1^2(-6 + \sin\gamma_2) + \gamma_2^2(-\cos\gamma_1 + 2\tan\gamma)_1 \\ &\leq 2\gamma_1^2(-6 + |\sin\gamma_2|) + \gamma_2^2(\tfrac{2|\sin\gamma_1|}{|\cos\gamma_1|} - |\cos\gamma_1|) \\ &\leq 2\gamma_1^2(-6+1) + \gamma_2^2(2-1) \\ &= \gamma_1^2(-10) + \gamma_2^2(1) \end{aligned} $$

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Therefore,

$$ {}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1) \le -10\Omega_1 + \Omega_2 \tag{6.4}$$

Also for $\gamma_0 \in S_{\psi}$, $for\ t = t_k,\ k = 1,2,...$ we have, $\Omega_1(t,\gamma(t)+\xi_k) = \left|\xi_k + \gamma(t)\right| \le \Omega_1(t,\gamma(t))$

Similarly, we compute for the Dini derivative for $\Omega_2(t,\gamma_2) = \gamma_2^2$ and follow through the same argument by substituting for $f_2(t,\gamma_2) = \frac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin\gamma_1 - \gamma_2 \sec\gamma_1 - \gamma_1^2 \cos\gamma_2$ in equation (3.4) to have that,

$$ {}^{C}D_{+}^{\beta}\Omega_2(t,\gamma_2) \le 4\Omega_1 - 4\Omega_2 \tag{6.5}$$

Also for $\gamma_0 \in S_{\psi}$, $for\ t = t_k,\ k = 1,2,...$ we have, $\Omega_2(t,\gamma(t)+\tau_k) = \left|\tau_k + \gamma(t)\right| \le \Omega_2(t,\gamma(t))$

By combining equations (6.4) and (6.5), we have

$$ {}^{C}D_{+}^{\beta}\Omega \le \begin{pmatrix} -10 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = g(t,\Omega) \tag{6.6}$$

Now, consider the comparison system ${}^{C}D^{\beta}u = g(t,\Omega) = Au$, where $A = \begin{pmatrix} -10 & 1 \\ 4 & -4 \end{pmatrix}$.

The vectorial inequality (6.6) and all other conditions of Theorem 5.2 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (6.1) is uniformly asymptotically practically stable. Therefore, the trivial solution $\gamma_0 = 0$ of the IFrDE (6.1) is uniform ly asymptotically practically stable.

CONCLUSION

In this paper, the asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations with fixed moment of impulse, using the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions is examined. By splitting the vector Lyapunov functions $V(t,x)$ into several components of the form V_1, V_2 ,…, each of the state variable or solution state , $x_1, x_2, ..., x_m$ can be inserted into each of the components rather than in one Lyapunov function, so that the arguments to obtained the desired practical stability result becomes less complicated, and the imposed conditions would be less restrictive. Together with the comparison results, sufficient conditions for the uniform asymptotic practical stability of the impulsive Caputo fractional order system (3.3) is established with an illustrative example.

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