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MODELLING SERVICE TIMES USING SOME BETA-BASED COMPOUND DISTRIBUTION

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ABSTRACT: *The design of the queueing model involves modelling the arrival and service processes of the system. Conventionally, the arrival process is assumed to follow Poisson while service times are assumed to be exponentially distributed. Other distributions such as Weibull, uniform, lognormal have been used to model service times however, generalized distributions have not been used in this regard. In recent times, attention have been shifted to generalised families of distributions including Beta generalized family of distributions which led to the development of Beta-based distributions. Distributions generated from a mixture of beta random variables are quite numerous in literature with little or no application to service times data. In this study, six Beta-based compound distributions - Beta-Log-logistic distribution (BLlogD), Beta-Weibull distribution (BWeiD), Beta-Lomax distribution (BLomD), Beta-exponential distribution (BExD), Beta-Gompertz distribution (BGomD) and Beta-log-normal distribution (BLnormD) - were compared with the classical service times model on four service times data sets. Maximum likelihood estimator was employed in estimating parameters of the selected models while Akaike Information Criteria (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan Quin information criterion (HQIC) statistics were employed to select the best model. CDFs, PDFs and PP-plots were used to fit the data of the suggested models. Results from the study shows that Beta-Exponential distribution (BExpD) performed better for the datasets I (AIC=640.3, CAIC=640.5,BIC=648.1 and HQIC=643.4), Beta-weibull distribution (BWeiD) performed better for the data sets II and III (AIC=204.2, 2142.4,CAIC=204.2, 2142.7,BIC=212.8, 2154.9 and HQIC=207.6, 2147.5) while Beta-log-logistic distribution (BLlogD) performed better for the data sets IV (AIC=2275.3,CAIC=2275.5, BIC=2289.3 and HQIC=2280.9). Findings from this study revealed some useful Beta-based compound distributions which performed better than the conventional service time model. From the findings of this study, we recommend researchers to dive deep into queueing theory.*

KEYWORDS: Service Times, Beta-based Compound distributions, Exponential distribution, Flexibility.

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INTRODUCTION

The queuing model involves modeling the arrival and service processes of a service system. Conventionally, the arrival process is assumed to be Poisson and service times are assumed to be exponentially distributed. But it has been suggested that this assumption may not hold for all service times data (Adedigba, 2005; Mandelbaum & Zeltyn, 2005). Literature revealed that the distribution of service times are heavily tailed in nature (Adedigba, 2005). It was also reported that log-normal distribution can be used to model the service times in call centers and that the assumption of exponential distribution was found to be irrelevant in this case (Mandelbaum, Sakov & Zeltyn, 2000; Ishay, 2003). Prado, Louzada, Rinaldi and Benze (2015) proposed a MAXGE distribution that describes a specific system with heavy traffic, a fast service and the service rate was said to depend on the state of the system. The parameter estimation was based on the maximum likelihood method. The model was applied to both simulated and real data and the proposed model was found to be better using AIC and BIC criteria for model selection.

Oladimeji and Ibidoja (2020) studied the distribution of service time of patients at the University of Ilorin Health Services Clinic; two distributions exponential and gamma were fitted and tested using Chi-squared goodness of fit test. Results showed that the arrival process follows a Poisson distribution with an arrival rate of 0.8 patient per 2 minutes while the service process follows a gamma.

Adegdiba (2005) studied statistical distributions for service times, in the review, the assumption of exponential service times was examined and a log-normal distribution was fitted to service times of the help desk. Results from the analysis carried out in the study verified the irrelevance of the assumption of exponential service times to this help desk hence log-normal distribution was said to have provided a reasonable fit to the data.

Akinsete, Famoye and Lee (2008) studied and proposed beta pareto distribution for heavy and long tail distributions. Eugene *et al*. (2002) defined the beta-normal (BN) distribution and stated that the beta-normal distribution generalizes the normal distribution and has flexible shapes such as unimodal and bimodal, thereby giving it greater applicability. Nadarajah and Kotz (2004) defined the beta-Gumbel distribution, and stated that it has greater tail flexibility and therefore, referred to it as type I extreme value distribution.

Nadarajah and Kotz (2006) studied the beta Exponential distribution, and stated that its hazard function can be increasing and decreasing thereby making it to be applied on skewed data. Famoye *et al.* (2005) defined the Beta-Weibull distribution and the application of the model to censored data was made by Lee *et al*. (2007) where it was said to have hazard function which can be either decreasing or increasing making it have greater applicability to skewed data. Similarly, Kong *et al.* (2007) proposed the Beta-gamma distribution, this distribution has a greater tail that can have wider application to skewed data sets.

The beta log-normal distribution was proposed by Castellares and Cordeiro (2011). The model was recommended for application to skewed data after determining the skewness and kurtosis for some values of parameters (a,b). Estimation of parameters was done using the maximum likelihood estimation method, and the model was applied to data which represented the fatigue life in hours of 10 bearings of a certain type. These data were used as an illustrative example for the three-parameter Weibull distribution by Cohen *et al.* (1984) and for the two-parameter Birnbaum-Saunders (BS) distribution by Ng *et al.* (2003).

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The Beta-Lomax distribution was proposed by Lemonte & Cordeiro, (2013), time plots for the distribution were obtained and it was recommended that beta lomax distribution can be used for modeling failure times in which initial failure probability is higher and it decreases during the aging process the author recommended application of this distribution in other areas and fields. The Beta-Gompertz (BG) distribution was introduced by Jafari, Tahmasebi and Alizadeh (2014) the distribution was said to include some well-known lifetime distributions such as Beta-exponential and generalized Gompertz distributions as special sub-models. The distribution was quite flexible and was recommended for use in modeling survival data and reliability problems. The method used for estimating parameters was maximum likelihood estimation. The distribution was recommended for further applications due to its flexibility.

The Beta log-logistic distribution extends the log-logistic distribution this distribution was studied by Lemonte, (2014). The model was said to be quite flexible to analyze positive data. Estimation of the model parameters was performed using maximum likelihood. This model was applied to two real data sets. The model was hopeful to serve as an alternative model to other models available in the literature for modeling positive real data in many areas.

Many studies particularly those that used Beta based compound distributions have considered fitting these transformed distributions to the data sets related to rainfall, flood discharge, aluminum coupons, survival time of guinea pigs, annual flow of Nile river, and strength of glass fiber (Famoye *et al.,* 2005; Akinsete *et al.,* 2008; Alshwarbeh *et al.,* 2013; Alzaatreh *et al.,* 2013; Hanum *et al.,* 2015; Harini & Srinivasan, 2018).

Distributions generated from the logit of beta random variables are quite enormous and most of this family of beta generated distributions (118) have been enumerated by Selim (2020). Several suggestions had also been carried out by different authors, following the applications of these numerous Beta-based compound distributions. Despite the increasing number of Beta-based compound distributions, there has been scanty or no research to model service times data using Beta-based compound distributions.

This study therefore, sought to apply some Beta-based compound distributions (Beta-log-logistics distribution (BLlogD), Beta-Lomax distribution (BLomD), Beta-Gompertz distribution (BGomD), Beta-Weibull distribution (BWeiD), Beta-lognormal distribution (BLnormD) and Beta-exponential distribution to service times data, with the view of comparing the performances of the suggested Beta-based distributions to the classical service time model and to identify the best out of the fitted distributions on four service times data sets. These specific Beta-based compound distributions were primarily considered since earlier studies suggested that they can be applied to heavy-tailed datasets (Alzaatreh *et al*., 2016).

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METHODS

Method of Data Analysis

Six Beta based compound distributions - Beta-Log-logistic distribution (BLlogD), Beta-Weibull distribution (BWeiD), Beta-Lomax distribution (BLomD), Beta-exponential distribution (BExD), Beta-Gompertz distribution (BGomD) and Beta-log-normal distribution (BLnormD) - were compared with exponential distribution; the distributions were fitted and compared using, Akaike Information Criteria (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan Quin information criterion (HQIC) the statistics are described in detail in equations 7-10.

In general, the model with smaller values of AIC, CAIC, BIC and HQIC is chosen as the best model. The required computations were obtained by using the "library(MPS)" and "library(AdequacyModel)" in R-software.

Models

Beta-Weibull Distribution

According to Lee *et al.* (2004), Beta-weibull distribution is given by:

$$f(x;a,b,c,\beta)=\frac{\left(\frac{|c|}{\beta^c}\right)x^{c-1}e^{-\left(\frac{x}{\beta}\right)^c}\left[1-e^{-\left(\frac{x}{\beta}\right)^c}\right]^{a-1}\left[1-\left(1-e^{-\left(\frac{x}{\beta}\right)^c}\right)\right]^{b-1}}{B(a,b)}$$

$$=\frac{\left(\frac{|c|}{\beta^c}\right)x^{c-1}e^{-\left(\frac{x}{\beta}\right)^c}\left[1-e^{-\left(\frac{x}{\beta}\right)^c}\right]^{a-1}\left[e^{-\left(\frac{x}{\beta}\right)^c}\right]^{b-1}}{B(a,b)}$$

$$=\frac{\left(\frac{|c|}{\beta^c}\right)x^{c-1}e^{-b\left(\frac{x}{\beta}\right)^c}\left[1-e^{-\left(\frac{x}{\beta}\right)^c}\right]^{a-1}}{B(a,b)}, \quad a,b,c,\beta>0 \tag{1}$$

Where the pdf and cdf of weibull distribution is given as

$$g(x)=\left(\frac{|c|}{\beta^c}\right)x^{c-1}e^{-\left(\frac{x}{\beta}\right)^c}$$

$$G(x)=1-e^{-\left(\frac{x}{\beta}\right)^c}$$

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Beta-Exponential distribution

According to Nadarajah and Kotz (2006), the density of the Beta-exponential distribution is given by:

$$f(x;a,b,\lambda)=\frac{\lambda\exp(-\lambda x)\left[1-\exp(-\lambda x)\right]^{a-1}\left[1-(1-\exp(-\lambda x)\right]^{b-1}}{B(a,b)}$$

$$=\frac{\lambda\exp(-\lambda x)\left[1-\exp(-\lambda x)\right]^{a-1}}{B(a,b)} \quad a>0,\ b>0,\ \lambda>0,\ x>0 \qquad (2)$$

Where the pdf and cdf of exponential distribution is given as:

$$g(x)=\lambda\exp(-\lambda x)$$

$$G(x)=1-\exp(-\lambda x)$$

The Beta-Gompertz Distribution

According to Jafari, Tahmasebi and Alizadeh (2014), the Beta-Gompertz distribution is given as:

$$f(x)=\frac{\theta e^{\gamma x}e^{-\frac{\beta\theta}{\gamma}\left(e^{\gamma x}-1\right)}}{B\left(\alpha,\beta\right)}[1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}]^{\alpha-1} \qquad (3)$$

Where pdf and cdf of Gompertz distribution are given as,

$$g(x)=\theta e^{\gamma x}e^{-\frac{\beta\theta}{\gamma}\left(e^{\gamma x}-1\right)}$$

and

$$G(x)=1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)},x\geq 0,\beta>0,\gamma>0$$

The Beta Log-logistic

According to Lemonte (2014), the Beta log-logistic distribution is given as:

$$f(x)=\frac{\left(\beta/\alpha\right)}{B(a,b)}\frac{\left(x/\alpha\right)^{a\beta-1}}{[1+(x/\alpha)^{\beta}]^{a+b}},x>0 \qquad (4)$$

Where the pdf and cdf of log logistic distribution is given as

$$g(x)=\frac{\beta\left(x/\alpha\right)^{\beta-1}}{\alpha[1+(x/\alpha)^{\beta}]^{2}},x>0$$

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$$G(x)=\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}},x>0$$

The Beta Log-normal Distribution

According to Castellares and Cordeiro (2011), the Beta log-normal distribution is given as:

$$f(x)=\frac{\exp\left\{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^{2}\right\}}{x\sigma\sqrt{2\pi}B(a,b)}\Phi\left(\frac{\log x-\mu}{\sigma}\right)^{a-1}\left\{1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)\right\}^{b-1} \quad (5)$$

Where the pdf and cdf of Log-normal distribution is given as:

$$g(x)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^{2}\right\},x>0$$

$$G(x)=\Phi\left(\frac{\log x-\mu}{\sigma}\right)$$

Beta-Lomax Distribution

According to Lemonte & Cordeiro (2013), The Beta-lomax distribution CDF is given by:

$$f(x)=\frac{\alpha}{\lambda\beta(a,b)}\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]^{a-1}\left[1+\left(\frac{x-\mu}{\lambda}\right)\right]^{-(ab+1)} \quad (6)$$

Where the pdf and cdf of lomax distribution is given as:

$$f(x)=\frac{\alpha}{\lambda}\left[1+\left(\frac{x-\mu}{\lambda}\right)\right]^{-(a+1)} \quad x\geq\mu,\mu\geq0,\alpha>0,\lambda>0$$

$$F(x)=1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha};x\geq\mu,\mu\geq0,\alpha>0,\lambda>0$$

Model Selection Criteria

The AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan Quin information criterion) were used. The formulas for these statistics are given as follows:

$$AIC=-2ll+2k \quad (7)$$

$$BIC=-2ll+k\log(n), \quad (8)$$

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$$CAIC = -2ll + \frac{2kn}{(n-k-1)} \tag{9}$$

$$HQIC = -2ll + 2k \log\left[\log(n)\right] \tag{10}$$

Where ll denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

RESULTS

Dataset I

The dataset I represents the waiting times (in minutes) of 100 Bank customers, the data was examined and analyzed by Ghitany *et al*. (2013) for fitting the Lindley distribution. The data set is given as follows; 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5 . The descriptive statistics for the data, maximum likelihood parameter estimates, and the model selection criterion are provided in Tables 1, 2 and 3 while plots of the estimated densities and CDFs and the probability plots are presented in Figures 1 and 2.

Table 1: Summary Statistics for Dataset I

Parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Dataset I	100	0.80	4.675	8.10	13.02	9.877	38.500	52.37	1.4953	5.73

Table 2: Maximum Likelihood Parameter Estimates for Dataset I

Distribution	*Estimates*			
BLlogD ($\hat{a},\hat{b},\hat{\alpha},\hat{\beta}$)	0.9644696	2.0615885	1.8384059	13.4582742
BLomD ($\hat{\alpha},\hat{\lambda},\hat{a},\hat{b}$)	2.34699913	4.74248938	0.02168712	2.31901577
BWeiD ($\hat{\alpha},\hat{\beta},\hat{a},\hat{b}$)	3.1759960	2.5945249	0.7644417	10.5495448
BExD ($\hat{a},\hat{b},\hat{\lambda}$)	2.2578998	0.5016140	0.2765341	--
BGomD ($\hat{\theta},\hat{\gamma},\hat{\alpha},\hat{\beta}$)	2.555762239	0.282241297	0.004693969	0.440273845
BLnormD ($\hat{\lambda},\hat{\sigma},\hat{a},\hat{b}$)	0.8415086	13.6741978	4.4808788	1.3613294
ExpD($\hat{\lambda}$)	0.1013129	--	--	--

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Table 3: The Statistics AIC, CAIC, BIC and HQIC for Dataset I

Distribution	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
BLlogD	643.6422	644.0633	654.0629	647.8597	6th
BLomD	642.3136	642.7346	652.7342	646.531	4th
BWeiD	642.2874	642.7084	652.708	646.5048	3rd
BExD	640.2687	640.5187	648.0842	643.4318	1st
BGomD	642.2183	642.6393	652.6389	646.4357	2nd
BLnormD	642.4891	642.9102	652.9098	646.7065	5th
ExpD	660.0418	660.0826	662.647	661.0962	7th

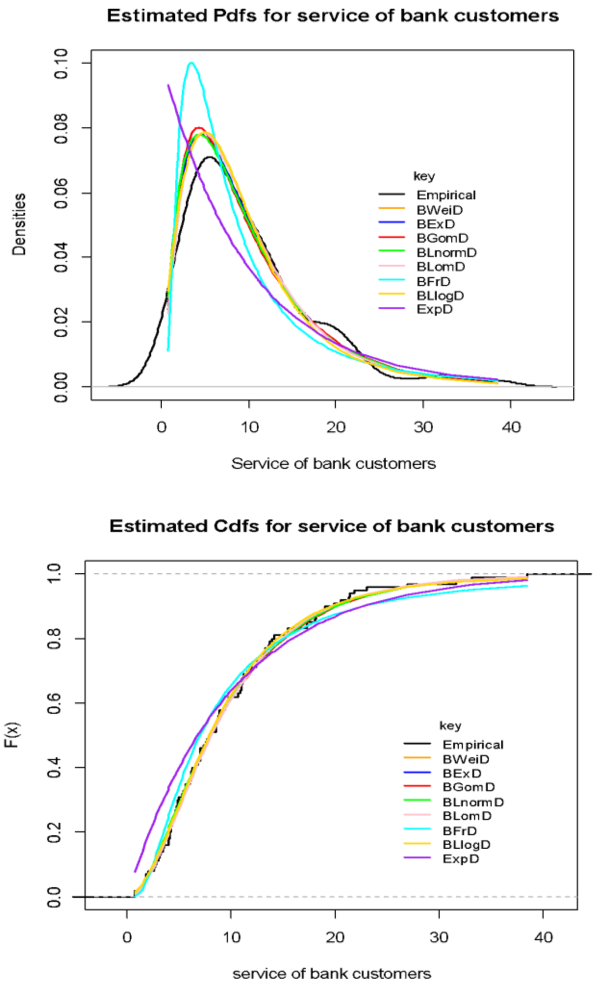

Figure 1: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset I.

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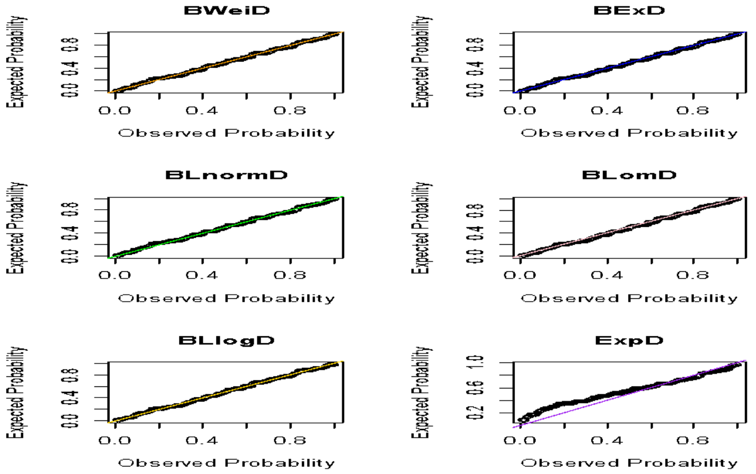

Figure 2: Probability plots for the fit of the models based on dataset I.

Dataset II

Data of service times for a particular model windshield given by Blischke and Murthy (2000). The data consist of 63 observations that is, service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h and the data is given as; 0.046, 0.140, 0.150, 0.248, 0.280, 0.313, 0.389, 0.487, 0.622, 0.900, 0.952, 0.996, 1.003, 1.010, 1.085, 1.092, 1.152, 1.183, 1.244, 1.249, 1.262, 1.436, 1.492, 1.580, 1.719, 1.794, 1.915, 1.920, 1.963, 1.978, 2.053, 2.065, 2.117, 2.137, 2.141, 2.163, 2.183, 2.240, 2.341, 2.435, 2.464, 2.543, 2.592, 2.600, 2.670, 2.717, 2.819, 2.820, 2.878, 2.950, 3.003, 3.102, 3.304, 3.483, 3.500, 3.622, 3.665, 3.695, 4.015, 4.628, 4.806, 4.881, 5.140. The descriptive statistics for the data, maximum likelihood parameter estimates, and the model selection criterion are provided in Tables 4, 5 and 6 while plots of the estimated densities and CDFs and the probability plots are presented in Figures 3 and 4.

Table 4: Descriptive Statistics for Dataset II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	0.05	1.122	2.06	2.820	2.09	5.14	1.55	0.43	-0.35

Table 5: Maximum Likelihood Parameter Estimates for Dataset II

Distribution	*Estimates*			
BLlogD ($\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}$)	0.259710	4.341365	4.099814	5.699922
BLomD ($\hat{\alpha}, \hat{\lambda}, \hat{a}, \hat{b}$)	1.84051295	4.98538174	0.04451189	3.74517449
BWeiD ($\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}$)	0.4545237	0.1439374	2.1927737	1.0998330

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BExD ($\hat{a}, \hat{b}, \hat{\lambda}$)	1.6935138	3.4711108	0.2101679	-
BGomD ($\hat{\theta}, \hat{\gamma}, \hat{\alpha}, \hat{\beta}$)	1.0699996	2.0781385	0.3893557	0.1235820
BLnormD ($\hat{\lambda}, \hat{\sigma}, \hat{a}, \hat{b}$)	0.1368542	6.7447406	2.2119220	0.5173722
ExpD($\hat{\lambda}$)	0.4789736			

Table 6: The Statistics AIC, CAIC, BIC and HQIC for Dataset II

Distribution	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
BLlogD	205.3109	206.0006	213.8835	208.6826	3rd
BLomD	216.0886	216.7782	224.6611	219.4602	6th
BWeiD	204.2003	204.8899	212.7728	207.5719	1st
BExD	212.3898	212.7966	218.8192	214.9186	4th
BGomD	204.9828	205.6724	213.5553	208.3544	2nd
BLnormD	213.4616	214.1513	222.0342	216.8333	5th
ExpD	220.6018	220.6673	222.7449	221.4447	7th

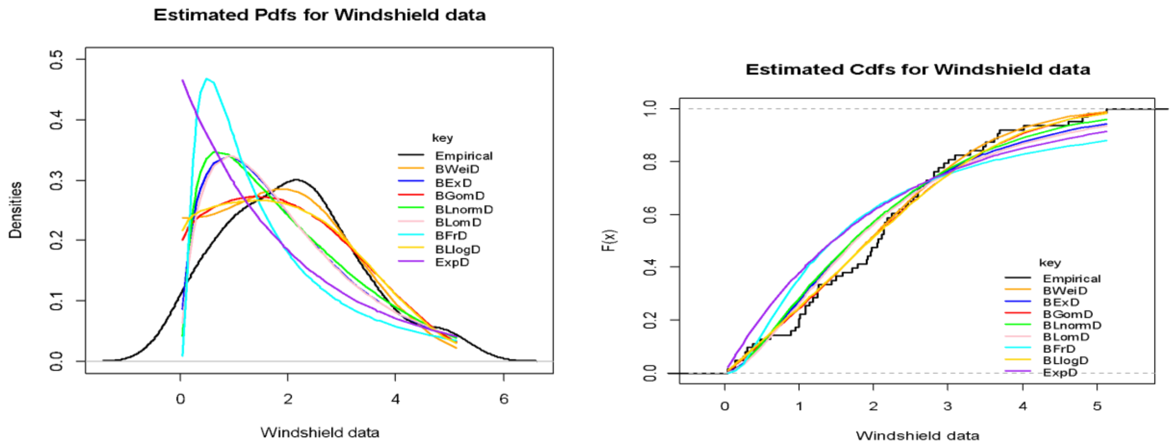

Figure 3: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset II.

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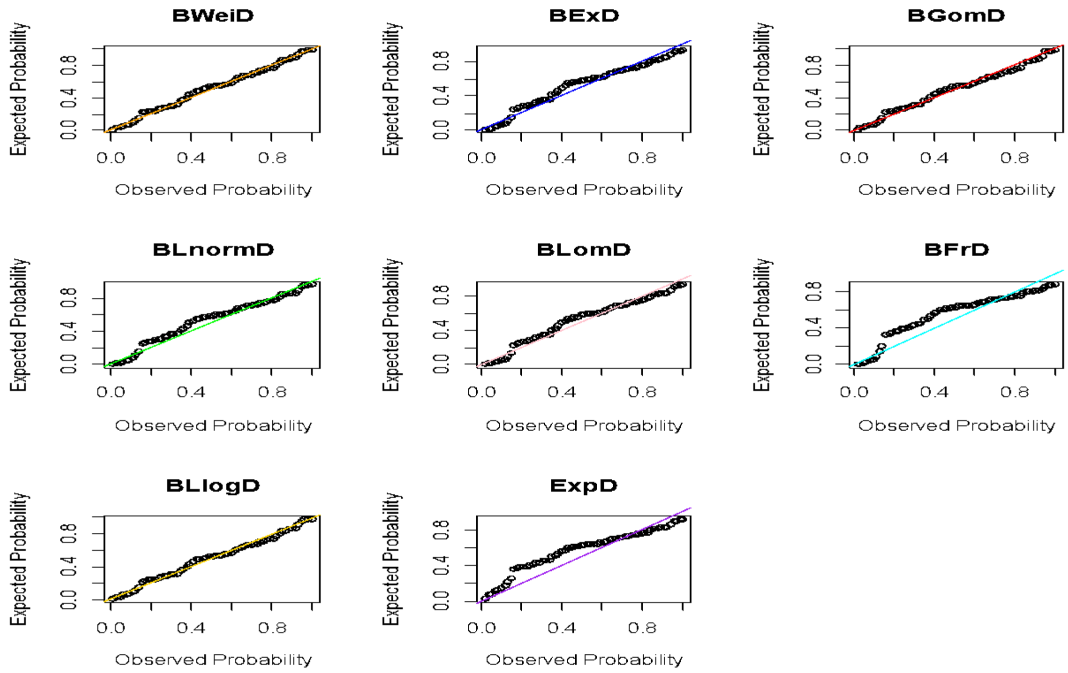

Figure 4: Probability plots for the fit of the models based on dataset II.

Dataset III

This data consist of service times of Modibbo Adama University Senate meetings (the time that Senate meeting begins and the time it ends); the data is measured in minutes and is given as;155,215,253,211,208,225,119,73,133,211,195,254,110,205,210,215,190,187,327,220,380,287,287,180,287,263,280,675,133,270,495,535,427,470,400,550,208,250,465,285,309,585,205,555,300,165,245,533,435,551,550,165,295,120,148,433,495,240,300,615,360,515,400,64,90,140,380,560,265,535,120,545,470,430,390,437,85,15,240,470,275,175,440,370,170,240,324,305,406,151,432,124,369,156,445,405,215,260,455,410,340,345,275,410,516,305,325,425,205,515,220,240,522,460,405,405,420,415,270,375,135,485,485,215,448,520,405,340,396,184,436,430,400,374,446,505,190,436,205,255,496,275,195,425,340,435,440,340,380,335,270,460,541,270,420,650,210,285,535,345,585,610,285,640,240,180,420,480. The descriptive statistics for the data, maximum likelihood parameter estimates, and the model selection criterion are provided in Tables 7, 8 and 9 while plots of the estimated densities and CDFs and the probability plots are presented in Figures 5 and 6.

Table 7: Summary Statistics for the Dataset III

Parameters	**N**	**Minimum**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Values	168	15.0	215	337.5	437.8	337.47	675.0	20227.56	0.11	-0.85

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Table 8: Maximum Likelihood Parameter Estimates for Dataset III

Distribution	*Estimates*			
BLlogD ($\hat{a},\hat{b},\hat{\alpha},\hat{\beta}$)	0.6191815	0.7887228	4.4612437	336.5296425
BLomD ($\hat{\alpha},\hat{\lambda},\hat{a},\hat{b}$)	3.835803236	3.005416485	0.002166038	1.653534554
BWeiD ($\hat{\alpha},\hat{\beta},\hat{a},\hat{b}$)	0.5624346	0.5489026	3.5511960	372.5720661
BExD ($\hat{a},\hat{b},\hat{\lambda}$)	2.883771156	0.909051016	0.005627273	-
BGomD ($\hat{\theta},\hat{\gamma},\hat{\alpha},\hat{\beta}$)	0.9190899380	1.0471684763	0.0071839039	0.0003992496
BLnormD ($\hat{\lambda},\hat{\sigma},\hat{a},\hat{b}$)	0.09299415	11.31471689	6.77155257	0.24901815
ExpD($\hat{\lambda}$)	0.004447419			

Table 9: The Statistics AIC, CAIC, BIC and HQIC for Dataset III

Distribution	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
BLlogD	2171.639	2171.884	2184.135	2176.71	4th
BLomD	2205.46	2205.705	2217.956	2210.531	6th
BWeiD	2142.431	2142.676	2154.927	2147.502	1st
BExD	2188.364	2188.51	2197.736	2192.167	5th
BGomD	2153.905	2154.151	2166.401	2158.977	3rd
BLnormD	2145.927	2146.172	2158.423	2150.998	2nd
ExpD	2325.878	2325.902	2329.002	2327.146	7th

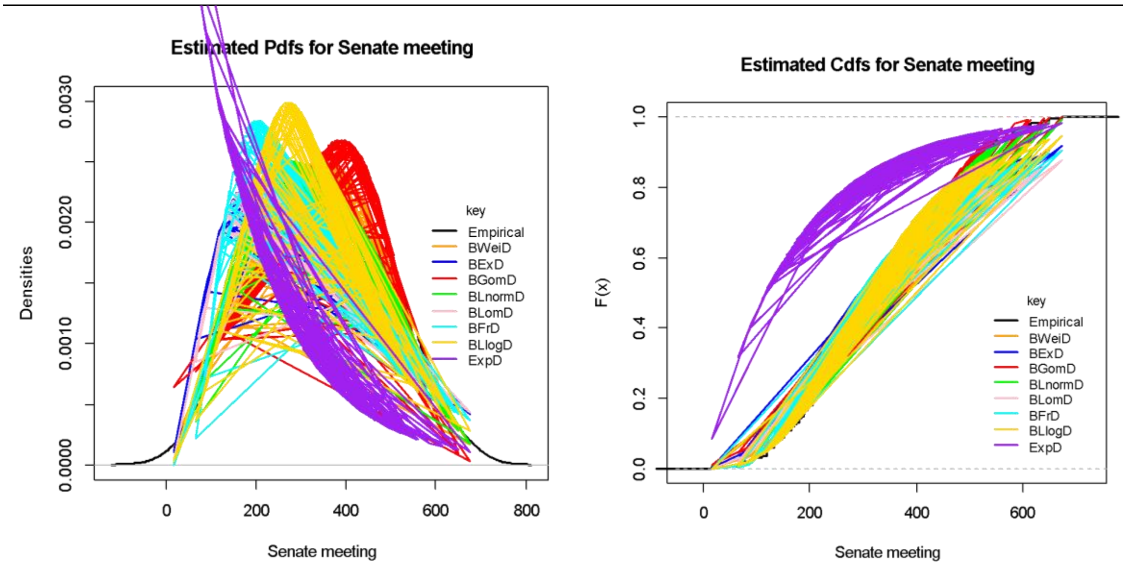

Figure 5: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset III.

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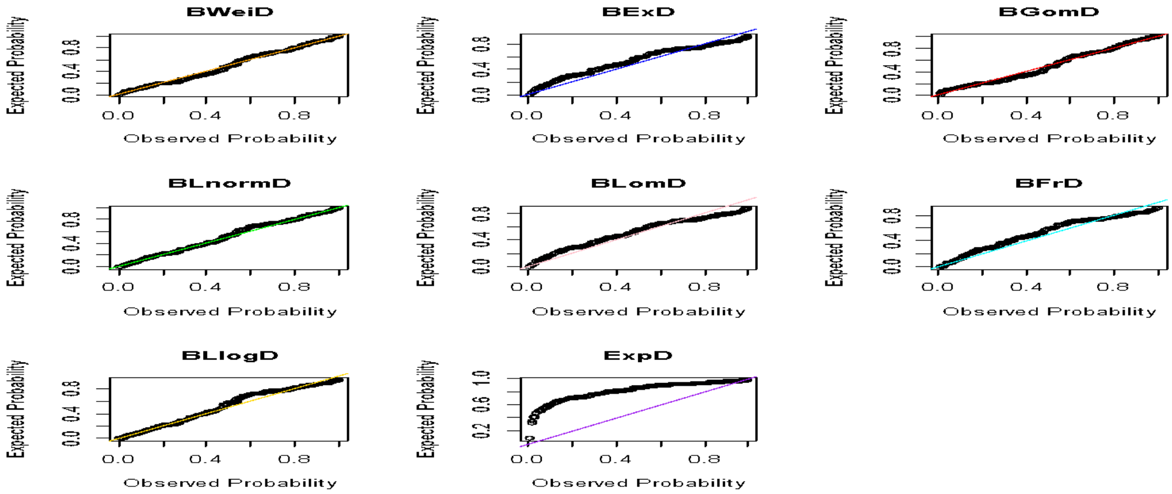

Figure 6: Probability plots for the fit of the models based on dataset III.

5.4 Data set IV

This data represents the service time of 35 patients at the University of Ilorin Health Services Clinic observed on 7 days, the data is measured in minutes given by Oladimeji & Ibidoja (2020). The observations are given as follows; 95,55,92,49,53,61,48,115,54,89,53,56,66,51,136,30,100,46,51,70,48,125,62,88,50,62,65,40,93,65,70,41,52,179,40,133,52,52,39,53,64,45,132,49,92,38,55,74,43,92,133,86,44,80,76,38,106,57,109,43,60,59,36,105,88,103,51,59,69,49,84,83,108,41,58,55,48,98,85,97,30,60,62,59,95,89,101,119,74,79,56,68,52,67,40,82,56,36,75,147,94,37,67,64,52,75,51,98,32,172,56,48,82,83,87,41,61,63,37,81,56,78,32,64,54,46,81,62,75,45,49,65,34,72,60,91,40,17,57,43,98,59,41,27,43,70,52,96,58,82,36,46,168,34,97,62,83,36,33,185,46,36,76,70,82,42,55,59,55,78,74,91,37,51,68,54,71,80,70,33,39,56,16,65,76,83,40,45,68,51,58,81,82,67,33,53,29,56,66,79,20,42,50,24,82,66,69,28,32,61,42,114,65,76,29,30,66,38,72,71,84,33,40,49,73,79,79,73,47,46,49,75,94,63,61,65,49,111,73,12,61,74,48,61,83. The descriptive statistics for the data, maximum likelihood parameter estimates, and the model selection criterion are provided in Tables 10, 11 and 12 while plots of the estimated densities and CDFs and the probability plots are presented in Figures 7 and 8

Table 10: Summary Statistics for the dataset IV

Parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	245	12	48.0	61	80	65.53	185	765.758	1.35	3.08

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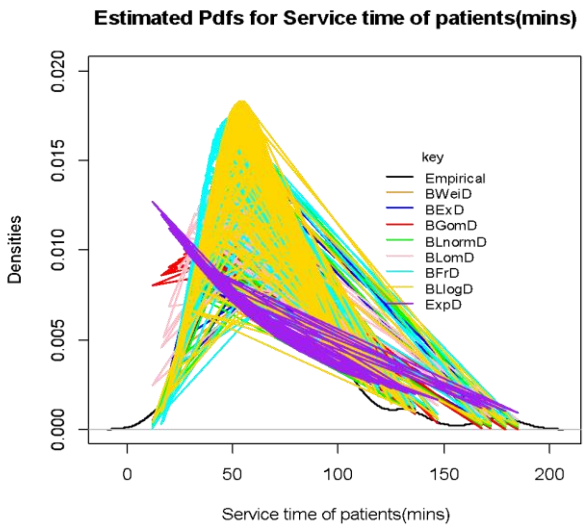
Table 11: Maximum Likelihood Parameter Estimates for Dataset IV

Distribution	*Estimates*			
BLlogD ($\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}$)	1.082735	1.122968	3.966269	61.346743
BLomD ($\hat{\alpha}, \hat{\lambda}, \hat{a}, \hat{b}$)	4.521293430	3.050107878	0.006578347	2.868076937
BWeiD ($\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}$)	9.1693109	4.9660673	0.8276144	56.5502103
BExD ($\hat{a}, \hat{b}, \hat{\lambda}$)	6.4167197	2.0546058	0.0247394	-
BGomD ($\hat{\theta}, \hat{\gamma}, \hat{\alpha}, \hat{\beta}$)	1.13327691	0.58631502	0.01765764	0.01435781
BLnormD ($\hat{\lambda}, \hat{\sigma}, \hat{a}, \hat{b}$)	1.2702600	4.8544297	4.7497232	0.6771358
ExpD($\hat{\lambda}$)	0.01783669			

Table 12: The Statistics AIC, CAIC, BIC and HQIC for Dataset IV

Distribution	*AIC*	*CAIC*	*BIC*	*HQIC*	**Ranks**
BLlogD	2275.304	2275.471	2289.309	2280.944	1st
BLomD	2328.419	2328.585	2342.424	2334.059	5th
BWeiD	2278.785	2278.952	2292.791	2284.425	4th
BExD	2278.05	2278.15	2288.554	2282.28	3rd
BGomD	2392.261	2392.428	2406.266	2397.901	6th
BLnormD	2277.978	2278.144	2291.983	2283.617	2nd
ExpD	2547.756	2547.772	2551.257	2549.166	7th



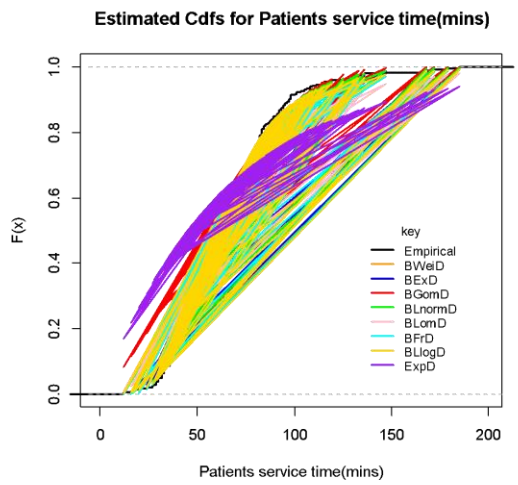


Figure 7: Histogram and plots of the estimated densities and cdfs of the fitted distributions to dataset IV.

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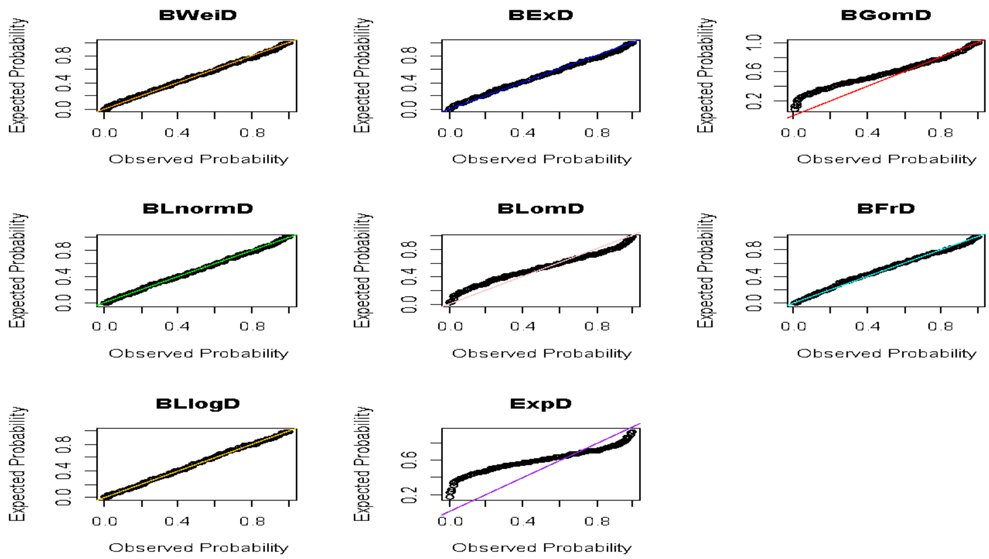

Figure 8: Probability plots for the fit of the models based on dataset IV.

DISCUSSION OF RESULTS

From the results in Tables 3, 6, 9 and 12, it was observed that the Beta-Exponential distribution (BExpD) performed much better for the datasets I, compared to the other fitted distributions including the classical service time modelBeta-Gompertz distribution (BGomD), Beta-Lomax distribution (BLomD), Beta-Weibull distribution (BWeiD), Beta-lognormal distribution (BLnormD), Beta-log-logistic distribution (BLlogD) and Exponential distribution. It was also observed that Beta-Weibull distribution (BWeiD) performed much better for the data sets II and III, compared to other fitted distributions. Beta-log-logistic distribution (BLlogD) performed much better compared to other fitted distributions on data sets IV.

Based on these results, it can be deduced that Beta-based compound distributions are flexible when applied to service times datasets irrespective of the nature of the datasets compared to the conventional service time model and this could be attributed to the fact that most of these Beta-based compound distributions have additional shape parameter(s) which vary the tail weight of the resulting distributions thereby inducing it with skewness to accommodate different data sets.

In this study, the Beta-Exponential distribution performed much better compared to other fitted distributions for data sets I. This finding agrees with a recent study carried out by Das (2020), who studied the well-established and widely used exponential and other generalized distributions and found out that the generalized models perform better than the exponential model. This also agrees with the study initialized by Nadarajah and Kotz (2006) and Adedigba, (2005) whose results from the analysis showed the irrelevance of the assumption of exponential distribution. For the data sets II and III, results from this study shows that Beta-weibull distribution (BWeiD) performed much better compared than other fitted distributions including exponential distributions. This study equally agrees with the studies conducted by Das (2020), Adedigba, (2005) whose application on service times showed that the assumption of exponential distribution for service times was irrelevant.

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The findings from this study further revealed that Beta-log-logistic distribution (BLlogD) performed much better compared to other fitted distributions including the conventional service time model for data sets IV. This could be attributed to the fact that Beta-log-logistic model is quite flexible in modeling positive data such as the data set IV. This also agrees with Lemonte, (2014) who stated that the model can serve as an alternative model to other models when modeling positive real data in many areas

CONCLUSION

In this study, a review of some Beta-based distributions has been done and a comparison of seven beta-based compound distributions with the classical service time model was carried out on four real life datasets (service times) of different nature. From the results based on all the model selection measures, it was observed that the Beta-Weibull distribution (BWeiD), Beta-log-logistics distribution (BLlogD) and Beta-Exponential distribution (BExpD) performed much better compared to the other four fitted distributions including the exponential distribution which is the least in all cases irrespective of the nature of the data sets and their sample sizes. This indicates that the BLlogD, BweiD and BExpD are more flexible than other beta-based distributions including the conventional service times model. The selected models will be useful in modeling different types of data.

RECOMMENDATIONS

Our recommendation from the findings of this research is that the useful Beta-based compound distributions identified in this study should be considered for modeling service times data in different areas such as engineering, health sciences, banks, call centers, education and economics.

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