EXTENDING A LEVEL THREE ORDER POLYNOMIAL PROBABILITY MODEL

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ABSTRACT: *Inverse and power extensions were derived for a three order polynomial probability distribution and studied in this paper as post-development. The two extensions serve to make up for the limitations of their baseline model. Some relevant properties—shape of the PDF, moments, survival function, hazard function, quantile function, stress-strength reliability, order statistics and parameter estimations—were studied. The hazard shapes of the distributions were found to be inverted bathtub for the former and three different shapes namely: increasing function, decreasing function and bathtub, for the latter. This implies that the distributions can altogether model many varieties of datasets emanating from different life phenomena. This statement "if $S(\infty) = 0$, then $H(\infty) = \infty$" was examined and was discovered to apply for exponential distribution but not any of the extended distributions, and the baseline distribution too. Finally, the extensions showed to be competent over both the baseline distribution and their respective counterpart distributions, with respect to the datasets used. For the latter, the superiority was accountably hinged on the extra parameters, since the dataset has an outlier.*

KEYWORDS: Inverse transformation, power transformation, hazard shapes, baseline distribution, application.

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INTRODUCTION

Echebiri and Mbegbu (2022) developed a probability distribution as a combination of exponential distribution $f_1(x) = \theta e^{-\theta x}$ and three level polynomial components $f_2(x) = (1 + x + x^3)$:

$$p(x,\theta) = C_n f(x)$$

Where C_n is the normalization and given as $C_n = \left[\int_0^{\infty} f_i(x)dx\right]^{-1}$ and $f_i(x) = \theta e^{-\theta x}\,(1 + x + x^3)$

$$\rightarrow \qquad C_n = \frac{\theta^3}{6+\theta^2+\theta^3}$$

$$\rightarrow \qquad p(x,\theta) = \frac{\theta^3}{6+\theta^2+\theta^3}\left[\,\theta e^{-\theta x}\,(1 + x + x^3)\right]$$

$$= \frac{\theta^4(\,1+x+ x^3\,)}{e^{\theta x}(\theta^3+\theta^2+6)}\ ,\ x > 0,\ \theta > 0 \qquad (1)$$

Consequently, the cumulative distribution function (Cdf) from equation (1) is given as

$$P(x,\theta) = 1 - \left(\frac{1}{e^{\theta x}} + \frac{\theta x\,[\,\theta^2 + \theta^2 x^2 + 3\theta x + 6\,]}{e^{\theta x}(\theta^3 + \theta^2 + 6)}\right) \qquad (2)$$

The very many statistical features of the three order polynomial distribution (TOPD) were investigated, but remarkably, the comparative study on coefficient of variation showed that it is a flexible distribution. The distribution was applied in the modeling of online live-streaming and cancer data and it was found to be a better fit than some univariate one parameter distributions. TOPD exhibits a shape as increasing failure rate, hence limited in a sense to modeling phenomenon emanating from other hazard rate types, say, decreasing failure rate, bathtub and/or inverted bathtub shapes. However, some probability distributions with bathtub shape hazard function have been proposed for the analysis of lifetime data; notwithstanding, exhaustive work has not been done in the analysis of models with inverted bathtub shape hazard rate function. Very recently, Sharma (2015) discussed the use of inverted bathtub shape hazard rate function in real applications of the Inverse Lindley distribution. Keller and Kamath (1982) proposed the inverse exponential distribution, which can also model data sets with inverted bathtub shaped failure rate.

Furthermore, the limitations of inverse exponential distribution to properly model data sets that are highly skewed or that exhibit heavy tails was detailed in the development of Generalized Inverse Exponential distribution (Abouammoh & Alshingiti, 2009). Extra parameters which were added in the model made it feasible to model for heavily tailed and highly skewed datasets. It is usually a conventional deduction that any inclusion of an extra parameter(s) into an existing distribution improves its ability to model data with wild observations. In like manner, Power Lindley distribution (Ghitany, 2013) was seen to model data sets that are also highly skewed and that exhibit heavy tails. As studied, the hazard rate function of Power Akash distribution has different shapes which include monotonically increasing and decreasing failure rate.

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Consequently, ample study is owed to be done as regards other hazard shapes of distribution extensions like power distributions aside the increasing and decreasing failure rate.

However, generators in distributions are employed in the development of generalizations, which serve as models that meet the rising need of wide applications on data emanating from different real-life events. Some examples of these distribution family generators are the beta generalized family (Beta-G) Eugene (2002), Gamma-G (type 1) Zografos and Balakrishnan (2009), the Kumaraswamy-G Cordeiro and de Castro (2011), Gamma-G (type 2) Ristic (2012), Gamma-G (type 3) Torabi and Montazari (2012), Exponentiated-G (EG) Cordeiro (2013), Weibull-G Bourguignon (2014), Logistic-G Torabi and Montazari (2014), Lomax-G family by Cordeiro (2014), new Weibull-G family Tahir (2016), Lindley-G family Cakmakyapan and Ozel (2016), Gompertz-G family Alizadeh (2017), etc. It commonly can be deduced that the extra parameters these generated generalized distributions come with enhance greater flexibility, and hence could model data with wild observations. The development of generators from mixture model distributions is still a very rare adoption or development, which would improve the flexibility of distributions in this category.

The paper aims at introducing few extensions of TOPD probability distribution, which involve the Inverse and Power generalization without leaving out their properties. They primarily would suffice for the limitation of the baseline distribution to model datasets from systems with other forms of failure rate apart from the increasing failure rate, as shown in the hazard function of the baseline distribution.

FORMULATION OF TOPD EXTENSIONS

Inverse Distribution

Proposition 1: Let Y denote a non-negative continuous random variable such that $Y \sim TOPD(x, \theta)$, then the cdf and pdf of the Inverse TOPD distribution (ITOPD) are derived thus:

Using the transformation $Y = \frac{1}{X}$ of a random variable, the technique for deriving its cdf and pdf are respectively given by:

$$F(x) = 1 - G(\frac{1}{x}) \tag{7}$$

$$f(x) = \frac{1}{x^2}\, g\left(\frac{1}{x}\right) \tag{8}$$

Now, a new inverted form of TOPD distribution is obtained by making substitutions accordingly in Equations (1) and (2), following the cdf and pdf of TOPD distribution.

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$$I_{jd}(x,\theta) = 1 - \left[1 - \left(1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right) e^{-\frac{\theta}{x}}\right]$$

$$= \left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right] e^{-\frac{\theta}{x}} \text{ and} \quad (9)$$

$$i_{jd}(x,\theta) = \frac{1}{x^2}\left[\frac{\theta^4}{\theta^3+\theta^2+6}\left(1 + \frac{1}{x} + \frac{1}{x^3}\right) e^{-\frac{\theta}{x}}\right]$$

$$i_{jd}(x,\theta) = \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\frac{\theta}{x}}, \quad x > 0, \ \theta > 0 \quad (10)$$

The Inverse TOPD distribution is a valid probability density function, that is, $\int f(x)dx = 1$.

Proof:

$$\int f(x)dx = \int_0^\infty i_{jd}(x,\theta)dx = \int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\frac{\theta}{x}} dx \quad (11)$$

let $m = \frac{1}{x} \to x = \frac{1}{m}, \frac{\partial x}{\partial m} = \frac{1}{m^2} \to \partial x = \frac{\partial m}{m^2}$

$$= \int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\left(m^2 + m^3 + m^5\right) e^{-\theta m} \frac{dm}{m^2}$$

$$= \frac{\theta^4}{\theta^3+\theta^2+6}\int_0^\infty \left(1 + m + m^3\right) e^{-\theta m}\, dm$$

but $\int_0^\infty x^C e^{-\theta x}\, \mathrm{dx} = \frac{\Gamma(C+1)}{\theta^{C+1}}$

Hence,
$$\int_0^\infty \frac{\theta^4}{\theta^3+\theta^2+6}\left(1 + m + m^3\right) e^{-\theta m}\, \mathrm{dm}$$
$$= \frac{\theta^4}{\theta^3+\theta^2+6} * \frac{\theta^3+\theta^2+6}{\theta^4} = 1 \quad (12)$$

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Properties of Inverse TOPD Distribution

Shape of Inverse TOPD Distribution

The first derivative of ITOPD-pdf at Equation (10) gives the mode of the distribution:

$$\frac{d[i_{jd}(x,\theta)]}{dx} = \frac{d\left[\frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5}\right)e^{-\frac{\theta}{x}}\right]}{dx}$$

$$\theta^5[x^{-5}+x^{-3}+x^{-2}] - \theta^4[5x^{-4}+3x^{-2}+2x^{-1}] = 0 \quad (13)$$

$$\frac{d[i_{jd}(x,\theta)]}{dx}]_{x=M_0} = 0$$

Moments of Inverse TOPD Distribution

The r^{th} moment of the Inverse TOPD distribution is obtained:

$$E(X^r) = \int_0^\infty x^r\, f(x)\; dx \quad (14)$$

$$= \int_0^\infty x^r\, \frac{\theta^4}{\theta^3+\theta^2+6}\left(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5}\right)e^{-\frac{\theta}{x}}\, dx$$

$$(X^r) = \frac{\theta^r\left[\theta^3-\theta^2(r-1)-(r-3)(r-2)(r-1)\right]\Gamma(1-r)}{(\theta^3+\theta^2+6)} \quad (15)$$

$E(X^r)$ is conditional, valid at $r < 1$. Hence, $E\left(X^r|_{r=1,2,3\ldots\infty}\right)$ does not converge. This implies that mean and variance and other measures of dispersion are indeterminate.

Reliability Analysis of Inverse TOPD Distribution

The survival function gives the description of the probability that a component will sustain after a given time, whereas hazard function is the likelihood that a system will fail after a given period of time or cycle. The survival $S_{ijd}(x,\theta)$ and hazard rate models $H_{ijd}(x,\theta)$ of Inverse TOPD distribution are given as:

$$S(x,\theta) = P(X \geq x) = \int_x^\infty f(t)\, dt \quad (16)$$

$$S_{ijd}(x,\theta) = 1 - I_{jd}(x,\theta)$$

$$= 1 - \left[1 + \frac{\frac{\theta^3}{x}+\frac{\theta^3}{x^3}+\frac{3\theta^2}{x^2}+\frac{6\theta}{x}}{\theta^3+\theta^2+6}\right]e^{-\frac{\theta}{x}} \quad (17)$$

$$H_{ijd}(x,\theta) = \frac{i_{jd}(x,\theta)}{S_{ijd}(x,\theta)} = \frac{\theta^4[\,\theta^3+\theta^2+1\,]e^{-\frac{\theta}{x}}}{x^5\left[(\theta^3+\theta^2+6)-\left\{(\theta^3+\theta^2+6)+\frac{\theta^3}{x}+\frac{\theta^3}{x^3}+\frac{3\theta^2}{x^2}+\frac{6\theta}{x}\right\}e^{-\frac{\theta}{x}}\right]} \quad (18)$$

In addition, Lawless (1982) derived the relationship between survival and hazard function as given by:

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$$S_{ijd}(x) = e^{-[H_{ijd}(x)]} \tag{19}$$

He also stated that: "if $S(\infty) = 0$, then $H(\infty) = \infty$". These would be verified in the empirical section alongside the behavior of $S_{ijd}(x)$ *and* $H_{ijd}(x,\theta)]_{\substack{x=0 \\ x=\infty}}$.

Parameter Estimation for Inverse TOPD Distribution

Let $X_i, i = 1,2,3, \dots, n$, be a random variable from Inverse TOPD distribution, the log-likelihood function $lnLf(x,\theta)$ is obtained as:

$$L(x,\theta) = \prod f(x,\theta) \tag{20}$$

$$L(x,\theta) = \left[\frac{\theta^4}{\theta^3+\theta^2+6}\right]^n \sum_i^n \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) e^{-\theta \sum_{i=1}^n \frac{1}{x_i}}$$

$$ln\mathrm{L}i_{jd}(x,\theta) = 4\mathrm{n}ln\theta - \mathrm{n}ln(\theta^3 + \theta^2 + 6) + \sum_{i=1}^n \ln\left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5}\right) - \theta \sum_{i=1}^n \frac{1}{x_i} \tag{21}$$

In estimation of MLE, the estimator is maximized at $\frac{\partial \, lnL}{\partial \theta} = 0$.

Therefore, $\frac{\partial \, ln\mathrm{L}i_{jd}(x,\theta)}{\partial \theta} = \frac{4n}{\theta} - \frac{n(3\theta^2 + 2\theta)}{\theta^3 + \theta^2 + 6} - \sum_{i=1}^n \frac{1}{x_i} = 0$

$$\rightarrow \quad (\theta^3 + 2\theta^2 + 24) - (\theta^4 + \theta^3 + 6\theta)\left(\frac{1}{\mathrm{n}} \sum_{i=1}^n \frac{1}{x_i}\right) = 0 \tag{22}$$

Quantile Function and Median of Inverse TOPD Distribution

The mathematical expression for quantile function is given as:

$$F(x) = u \rightarrow x = F^{-1}(u) \tag{23}$$

Hence, engaging the cdf of ITOPD, we obtain thus:

$$u = I_{jd}(x,\theta) = \left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right] e^{-\frac{\theta}{x}} \tag{24}$$

Taking the natural logarithm of both sides,

$$ln\,(u) = ln\left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right] - \frac{\theta}{x}$$

The quantile of inverse TOPD Distribution is not an explicit expression and is given by:

$$ln\left[1 + \frac{\frac{\theta^3}{x} + \frac{\theta^3}{x^3} + \frac{3\theta^2}{x^2} + \frac{6\theta}{x}}{\theta^3 + \theta^2 + 6}\right] - ln\, ln\,(u) - \frac{\theta}{x} = 0 \tag{25}$$

where $u \sim uniform(0,1)$

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The median of inverse TOPD distribution can be obtained by substituting for $u = 0.5$.

Stress-Strength Reliability of Inverse TOPD Distribution

The stress-strength reliability projects the description of a component's life, which has random strength X that is subjected to a random stress Y. When the stress applied to it overrides the strength, the system collapses at once, and it will function satisfactorily till $Y < X$. Therefore, $R = P(X > Y)$ is a measure of system reliability and in statistical literature, it is termed stress-strength parameter. It is widely applied in various fields of life especially in structuring engineering, deterioration of rocket motors, aging of concrete pressure vessels, etc.

Let X and Y be independent strength and stress random variables having Inverse TOPD distribution with parameter $i_{jd}(x, \theta_1) and\ i_{jd}(x, \theta_2)$ respectively, then the stress-strength reliability R of ITOPD can be obtained as:

$$R = P(X > Y) = \int_0^\infty P(X > Y | X = x) f(x) dx \tag{26}$$
$$= \int_0^\infty I_{jd}(x, \theta_2)\, i_{jd}(x, \theta_1)\, dx$$

$$= \int_0^\infty \left\{ \left[\left(1 + \frac{\frac{\theta_2{}^3}{x} + \frac{\theta_2{}^3}{x^3} + \frac{3\theta_2{}^2}{x^2} + \frac{6\theta_2}{x}}{\theta_2{}^3 + \theta_2{}^2 + 6} \right) e^{-\frac{\theta_2}{x}} \right] \left[\frac{\theta_1{}^4}{\theta_1{}^3 + \theta_1{}^2 + 6} \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5} \right) e^{-\frac{\theta_1}{x}} \right] \right\} dx$$

$$= \left\{ \frac{\theta_1{}^4}{(6 + \theta_1{}^2 + \theta_1{}^3)(\theta_1 + \theta_2)^7 (6 + \theta_2{}^2 + \theta_2{}^3)} \left[(6(\theta_1 + \theta_2)^3 (6 + (\theta_1 + \theta_2)^2 + (\theta_1 + \theta_2)^3) + 6b(\theta_1 + \theta_2)^2 (24 + 2(\theta_1 + \theta_2)^2 + (\theta_1 + \theta_2)^3) + \vartheta^2 (\theta_1 + \theta_2)(360 + 24(\theta_1 + \theta_2)^2 + 6(\theta_1 + \theta_2)^3 + (\theta_1 + \theta_2)^4 + (\theta_1 + \theta_2)^5) + \theta_2{}^3 (720 + 48(\theta_1 + \theta_2)^2 + 12(\theta_1 + \theta_2)^3 + 2(\theta_1 + \theta_2)^4 + 2(\theta_1 + \theta_2)^5 + (\theta_1 + \theta_2)^6)) \right] \right\},\ [\theta_1 + \theta_2] > 0 \tag{27}$$

Order Statistics (Inverse TOPD Distribution)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Inverse TOPD distribution. Let $X_1 < X_2 <, \ldots, < X_n$ denote the corresponding order statistics. The pdf of the kth order statistics say $Y = X_k$ is given by:

$$f_{i:n} = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} f(y) F^{k+l-1}(y) \tag{28}$$

Thus, the pdf of kth order statistics can be expressed from equation (28) given by:

$$f_{i:n} = \frac{n!\, \theta^4 (\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^5})\, e^{-\frac{\theta}{x}}}{(\theta^3 + \theta^2 + 6)(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left[1 - \left(1 + \frac{\theta x [\theta^2 + \theta^2 x^2 + 3\theta x + 6]}{\theta^3 + \theta^2 + 6} \right) e^{-\frac{\theta}{x(i+k+1)}} \right] \tag{29}$$

That implies that the pdf of minimum order statistics is obtained by substituting $j = k = 1$ in Equation (29) to have:

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$$f_{1:n} = \frac{\mathrm{n}[\theta^4(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5})\, e^{-\frac{\theta}{x}}]}{(\theta^3+\theta^2+6)} \sum_{l=0}^{n-1}\binom{n-1}{l}\ (-1)^l\ \left[1-\left(1+\frac{\theta x\,[\,\theta^2+\theta^2\,x^2+3\theta x+6\,]}{\theta^3+\theta^2+6}\right)e^{-\frac{\theta}{x(i+2)}}\right] \tag{30}$$

while the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = k = n$ in Equation (29):

$$f_{n:n} = \frac{n[\,\theta^4(\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^5})\, e^{-\frac{\theta}{x}}]}{(\theta^3+\theta^2+6)} \left[1-\left(1+\frac{\theta x\,[\,\theta^2+\theta^2\,x^2+3\theta x+6\,]}{\theta^3+\theta^2+6}\right)e^{-\frac{\theta}{x(i+n+1)}}\right] \tag{31}$$

Power TOPD Distribution (PTOPD)

Proposition 2: Let Y denote a non-negative continuous random variable such that $Y \sim Juchez(y,\theta)$, then the cdf and pdf of the Power TOPD distribution are respectively obtained thus:

For the transformation $X = T^{\frac{1}{\varphi}} \rightarrow T = X^{\varphi}$ of a random variable, whose cdf and pdf is given by:

$$F(x) = G(x^{\varphi}) \tag{32}$$

$$f(x) = \frac{d}{dx}G(x^{\varphi}) = \varphi x^{\varphi-1}\, g(x^{\varphi})\,; \quad x > 0,\ \varphi > 0 \tag{33}$$

The pdf of Power TOPD distribution can be derived by direct substitution following Equations (32) and (33):

$$p_{jd}(x,\theta,\varphi) = \varphi x^{\varphi-1}\frac{\theta^4}{\theta^3+\theta^2+6}\,(1+x^{\varphi}+x^{3\varphi})\,e^{-\theta x^{\varphi}} \tag{34}$$

$$P_{jd}(x,\theta,\varphi) = 1-\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}} \tag{35}$$

Alternatively, it can be derived through mixture model as in Equation (1):

$$f(x) = d_1 g_1(x,\theta,\varphi,1) + d_2\, g_2(x,\theta,\varphi,2) + d_3\, g_3(x,\theta,\varphi,4) \tag{36}$$

where $$d_1 = \frac{\theta^3}{\theta^3+\theta^2+6},\ d_2 = \frac{\theta^2}{\theta^3+\theta^2+6} + d_3 = \frac{6}{\theta^3+\theta^2+6} \tag{37}$$

$$g(x;\theta,\alpha) = \{\frac{x^{\alpha-1}\theta^{\alpha}e^{-\theta x}}{\Gamma(\alpha)} \quad x>0 \quad 0, \quad x<0 \quad \rightarrow \quad g(x^{\varphi};\theta,\alpha) = \{\frac{x^{\varphi(\alpha-1)}\theta^{\alpha}e^{-\theta x^{\varphi}}}{\Gamma(\alpha)} \quad x>0 \quad 0, \quad x<0 \tag{38}$$

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But from Equation (33),

$$\varphi x^{\varphi-1}\times\ g_1(x^{\varphi};\theta,1\,)=\theta\varphi x^{\varphi-1}e^{-\theta x^{\varphi}}$$

$$\varphi x^{\varphi-1}\times g_2(x^{\varphi},\theta,2)=\theta^2\varphi x^{2\varphi-1}\,e^{-\theta x^{\varphi}} \tag{39}$$

$$\varphi x^{\varphi-1}\times g_3(x^{\varphi},\theta,4)=\frac{\theta^4\varphi x^{4\varphi-1}\,e^{-\theta x^{\varphi}}}{6}$$

$$\rightarrow \quad p_{jd}(x,\theta,\varphi)=\frac{\theta^3}{\theta^3+\theta^2+6}\left[\theta\varphi x^{\varphi-1}e^{-\theta x^{\varphi}}\right]+\frac{\theta^2}{\theta^3+\theta^2+6}\left[\theta^2\varphi x^{2\varphi-1}\,e^{-\theta x^{\varphi}}\right]+\frac{6}{\theta^3+\theta^2+6}\left[\frac{\theta^4\varphi x^{4\varphi-1}\,e^{-\theta x^{\varphi}}}{6}\right] \tag{40}$$

Simplifying the expression in Equation (40), we have the pdf of Power TOPD distribution and the corresponding cdf.

$$p_{jd}(x,\theta,\varphi)\ =\ \frac{\varphi\theta^4}{\theta^3+\theta^2+6}\,(\,x^{\varphi-1}\,+x^{2\varphi-1}\,+x^{4\varphi-1}\,)\,e^{-\theta x^{\varphi}}, x>0,\ \theta>\ 0,\varphi>0 \tag{41}$$

$$P_{jd}(x,\theta,\varphi)\ =\ 1-\left(1+\ \frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}} \tag{42}$$

Using software, Mathematica, the validity of PTOPD was tested for $\int_0^{\infty} j(x,\theta)dx=1$. It also suffices to state that: $\lim_{x\to\infty} F(x)=1$

$$P_{jd}(x,\theta,\varphi)=1-\left(1+\ \frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}$$

$$P_{jd}(x,\theta,\varphi)=1-(1+\infty)0$$

$$P_{jd}(x,\theta,\varphi)=1 \tag{43}$$

Properties of Power TOPD Distribution

Shape of Power TOPD Distribution

The first derivative of PTOPD-Pdf at Equation (41) gives the mode of the distribution:

$$\frac{d[p_{jd}(x,\theta,\varphi)]}{dx}=\frac{d\left[\frac{\varphi\theta^4}{\theta^3+\theta^2+6}\,(\,x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\,)\,e^{-\theta x^{\varphi}}\right]}{dx} \tag{44}$$

$$\{\theta^4\varphi[(\varphi-1)x^{\varphi-2}+(2\varphi-1)x^{2\varphi-2}+(4\varphi-1)x^{4\varphi-2}]\}-\{\theta^5\varphi^2x^{\varphi-1}[x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}]\}=0 \tag{45}$$

$$\frac{df(x)}{dx}\Big]_{x\to Mode}=0$$

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Moments of Power TOPD Distribution

The r^{th} moment of the Power TOPD distribution is obtained:

$$E(X^r) = \int_0^\infty x^r\, p_{jd}(x,\theta,\varphi)\ \ dx \tag{46}$$

$$E(X^r) = \frac{\theta^{\frac{-r}{\varphi}}\, r\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)\varphi^2 r+6\varphi r^2+r^3\right]\Gamma\left(\frac{r}{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{47}$$

$$\mu = \frac{\theta^{\frac{-1}{\varphi}}\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)\varphi^2+6\varphi+1\right]\Gamma\left(\frac{1}{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{48}$$

$$\mu_2' = \frac{\theta^{\frac{-2}{\varphi}}\, 2\left[(\theta^3+\theta^2+6)\varphi^3+(11+\theta^2)2\varphi^2+24\varphi+8\right]\Gamma\left(\frac{2}{\varphi}\right)}{(\theta^3+\theta^2+6)\varphi^4} \tag{49}$$

$$\rightarrow \qquad \mu_2 \;=\; \mu_2' - \mu^2 \;=\; \sigma^2 \tag{50}$$

Reliability Analysis of Power TOPD Distribution

The survival function $S_{pjd}(x,\theta,\varphi)$ and hazard rate function $H_{pjd}(x,\theta,\varphi)$ of PTOPD can be obtained thus:

$$S_{pjd}(x,\theta,\varphi) = 1 - P_{jd}(x,\theta,\varphi) = 1 - \left[1 - \left(1 + \frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right) e^{-\theta x^{\varphi}}\right] \tag{51}$$

$$= \left(1 + \frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right) e^{-\theta x^{\varphi}} \tag{52}$$

$$H_{pjd}(x,\theta,\varphi) = \frac{p_{jd}(x,\theta,\varphi)}{S_{pjd}(x,\theta,\varphi)} = \frac{\frac{\varphi\theta^4}{\theta^3+\theta^2+6}\left[x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1}\right]e^{-\theta x^{\varphi}}}{\left(1+\frac{\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}} \tag{53}$$

$$= \frac{\varphi\theta^4(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1})}{(\theta^3+\theta^2+6)+[\theta^3 x^{\varphi} + \theta^3 x^{3\varphi} + 3\theta^2 x^{2\varphi} + 6\theta x^{\varphi}]} \tag{54}$$

Order Statistics (Power TOPD Distribution)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Power TOPD distribution. Let $X_1 < X_2 <, \ldots, < X_n$ denote the corresponding order statistics. The pdf of the kth order statistics say $Y = X_k$ is given by:

$$f_{i:n} = \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l} f(y)F^{k+l-1}(y) \tag{55}$$

Thus the pdf of kth order statistics can be expressed from Equation (55) given by:

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$$f_{i:n} = \begin{Bmatrix} \frac{\mathrm{n}!\,\varphi\theta^4(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1})\,e^{-\theta x^{\varphi}}}{(\theta^3+\theta^2+6)(k-1)!(n-k)!} \sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l \\ \left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \end{Bmatrix} \tag{56}$$

That implies that the pdf of minimum order statistics is obtained by substituting $j = k = 1$ in Equation (56), to have:

$$f_{1:n} = \begin{Bmatrix} \frac{\mathrm{n}\left[\theta^4(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1})\,e^{-\theta x^{\varphi}}\right]}{(\theta^3+\theta^2+6)} \sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l \\ \left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \end{Bmatrix} \tag{57}$$

while the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = k = n$ in Equation (56):

$$f_{n:n} = \frac{n[\theta^4(x^{\varphi-1}+x^{2\varphi-1}+x^{4\varphi-1})\,e^{-\theta x^{\varphi}}]}{(\theta^3+\theta^2+6)}\left[\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}\right] \tag{58}$$

Quantile and Median of Power TOPD Distribution

The model in Equation (23) also applies here; hence

$$u = 1-\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}} \tag{59}$$

$$1-u = \left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)e^{-\theta x^{\varphi}}$$

$$ln\,ln\,(1-u) = ln\,ln\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)-\theta x^{\varphi}$$

Thus, the quantile for the simulation of samples from power TOPD distribution is:

$$ln\,ln\left(1+\frac{\theta^3x^{\varphi}+\theta^3x^{3\varphi}+3\theta^2x^{2\varphi}+6\theta x^{\varphi}}{\theta^3+\theta^2+6}\right)-ln\,ln\,(1-u) - \theta x^{\varphi} = 0 \tag{60}$$

where $u \sim uniform(0,1)$, and at $u = 0.5$, the quantile expression becomes the median of PTOPD.

Parameter Estimation for Power TOPD Distribution

Let $X_i, i = 1,2,3,\ldots,n$, be a random variable from Power TOPD distribution, then the log-likelihood function $lnLf(x,\theta,\varphi)$, is obtained as:

$$Lf(x,\theta) = \prod_{i=1}^{n} f(x,\theta) \tag{61}$$

$$ln\mathrm{L}p_{jd}(x,\theta,\varphi) = 4\mathrm{n}ln\theta + \mathrm{n}ln\varphi - \mathrm{n}ln(\theta^3+\theta^2+6) + \sum_{i=1}^{n}\ln({x_i}^{\varphi-1}+{x_i}^{2\varphi-1}+{x_i}^{4\varphi-1}) - \theta\sum_{i=1}^{n}{x_i}^{\varphi} \tag{62}$$

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The MLE's are maximized at $\frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta} = 0$ and $\frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\varphi} = 0$

$$\frac{\partial\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta} = \frac{4n}{\theta} - \frac{n(3\theta^2 + 2\theta)}{\theta^3 + \theta^2 + 6} - \sum_{i=1}^{n} x_i^{\varphi} \tag{63}$$

$$\frac{\partial\, ln\mathrm{L}f(x,\theta,\varphi)}{\partial\varphi} = \frac{n}{\varphi} + \sum_{i=1}^{n} \frac{x_i^{\varphi-1}\, lnx_i + x_i^{2\varphi-1} lnx_i + x_i^{4\varphi-1} lnx_i}{x_i^{\varphi-1} + x_i^{2\varphi-1} + x_i^{4\varphi-1}} - \theta \sum_{i=1}^{n} x_i^{\varphi} \ln x_i \tag{64}$$

Since $\frac{\partial(x^{\varphi})}{\partial\varphi} = x^{\varphi}\, lnx$

The likelihood equations in (63) and (64) can easily be solved iteratively using Fisher's scoring method, since at $\frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta} = 0$ and $\frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\varphi} = 0$ cannot be expressed as a closed form equation. We have thus:

$$\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\theta\, \partial\varphi} = \frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi\partial\theta} = \sum_{i=1}^{n} x_i^{\varphi} \ln x_i \tag{65}$$

$$\frac{\partial^2\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta^2} = -\frac{4n}{\theta^2} + \left\{-\frac{n(3\theta^2 + 2\theta)^2}{[\theta^3 + \theta^2 + 6]^2} + \frac{n(\,2+6\theta\,)^2}{\theta^3 + \theta^2 + 6}\right\} \tag{66}$$

$$\frac{\partial^2\, ln\mathrm{L}p_{jd}(x,\theta,\varphi)}{\partial\varphi^2} = \sum_{i=1}^{n} \left\{ \begin{matrix} \left[\frac{(x_i^{\varphi-1}\, lnx_i + x_i^{2\varphi-1} lnx_i + x_i^{4\varphi-1} lnx_i)(x_i^{\varphi-1}\, lnx_i + 2x_i^{2\varphi-1} lnx_i + 4x_i^{4\varphi-1} lnx_i)}{(x_i^{\varphi-1} + x_i^{2\varphi-1} + x_i^{4\varphi-1})^2}\right] + \\ \left[\frac{x_i^{\varphi-1}\, lnx_i^2 + x_i^{2\varphi-1} lnx_i^2 + x_i^{4\varphi-1} lnx_i^2}{x_i^{\varphi-1} + x_i^{2\varphi-1} + x_i^{4\varphi-1}}\right] \end{matrix} \right\} \tag{67}$$

Resolving the following matrix equations, the solutions of MLE $(\hat{\theta},\ \hat{\varphi})$ for $p_{jd}(x,\theta,\varphi)$ are obtained:

$$\left[\frac{\partial^2\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta^2}\ \frac{\partial^2\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta\,\partial\varphi}\ \frac{\partial^2\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta\,\partial\varphi}\ \frac{\partial^2\, lnLp_{jd}(x,\theta,\varphi)}{\partial\varphi^2}\right]\left[\hat{\theta} = \theta_0\ \ \hat{\varphi} = \varphi_0\right] = \left[\frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\theta}\ \frac{\partial\, lnLp_{jd}(x,\theta,\varphi)}{\partial\varphi}\right] \tag{68}$$

where θ_0 *and* φ_0 are initial values of θ *and* φ.

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EMPIRICAL INVESTIGATIONS

Having proposed two different models, it is however necessary to note that the PTOPD has the baseline distribution as a sub-model at parameter value $\varphi = 1$.

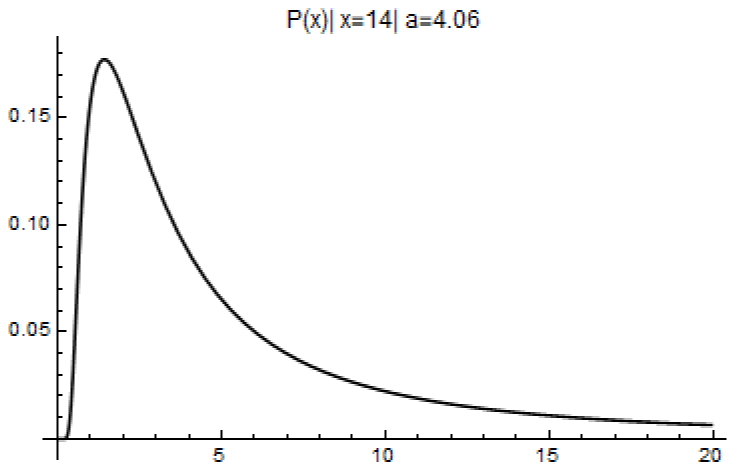

Figure 1: The PDF plot of the ITOPD

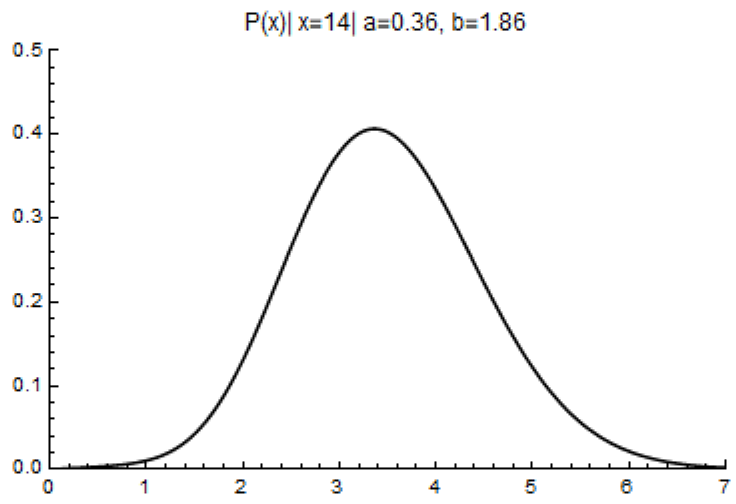

Figure 2: The PDF plot of the PTOPD

It could be deduced from Figure 1 and 2 that the shape of the pdfs are unimodal and positively skewed.

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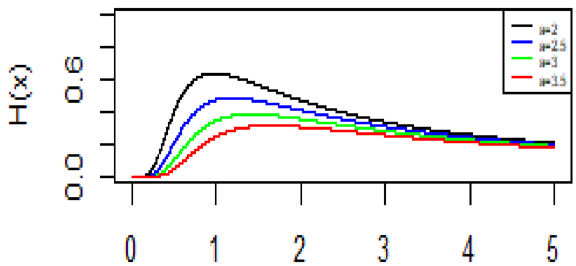

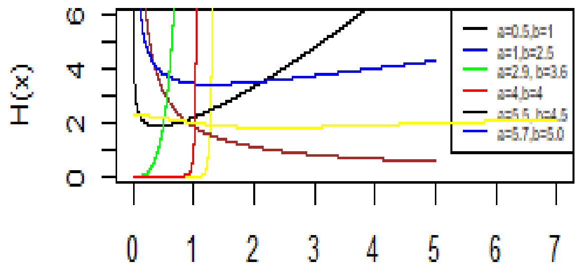

Figure 2: The Hazard plot for the ITOPD and PTOPD respectively

Figure 2 brings to light the hazard shapes of the ITOPD and PTOPD. The hazard function for ITOPD shows an inverted bathtub shape whereas PTOPD reveals three different shapes namely: bathtub or unimodal shape, increasing and decreasing failure rate shape.

Empirically, from Equations (17), (18), (52) and (54), it can be deduced that:

$$S_{ijd}(x,\theta) = \{0,\ x = \infty\ ■,\ x = 0 \text{ and } H_{ijd}(x,\theta) = \{0,\ x = \infty\ ■,\ x = 0\ . \quad (74)$$

More so, $S_{pjd}(x,\theta) = \{1,\ x = 0\ ■,\ x = \infty$ and $H_{pjd}(x,\theta) = \{0,\ x = 0\ ■,\ x = 0$. (75)

By careful observation, the statement: “if $S(\infty) = 0$, then $H(\infty) = \infty$” by Lawless (1982) applies for exponential distribution but does not apply for these distributions, likewise Lindley and TOPD distributions. However, Equations (74) and (75) are also consistent with the relationship between hazard and survival function: $S(x) = e^{-[H(x)]}$, as given in Equation (19).

Hence,
$$S_{ijd}(x) = \{0,\ x = \infty\ ■,\ x = 0 \quad (76)$$

This refers to the time or settings where an individual or a system, fails. Following the same relationship,

$$S_{pjd}(x) = \{1,\ x = 0\ ■,\ x = \infty \quad (77)$$

where ■ implies “indeterminate.”

Table 1: Probability Analysis for the PDF of ITOPD and PTOPD Distribution ($\theta = 0.1$ to $\theta = 0.5$)

		ITOPD				**PTOPD**	**[$\varphi = 0.5$]**	
X	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.35$	$\theta = 0.5$	$\theta = 0.1$	$\theta = 0.25$	$\theta = 0.35$	$\theta = 0.5$
1	0.000045	0.001501	0.005146	0.01784	0.000022	0.000750	0.002573	0.008920
2	0.000008	0.000284	0.001021	0.00382	0.000027	0.000836	0.002750	0.008960
3	0.000003	0.000109	0.000401	0.00154	0.000032	0.000954	0.003038	0.009438

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4	0.000002	0.000057	0.000209	0.00081	0.000037	0.001072	0.003324	0.009918
5	0.0000009	0.000034	0.000127	0.00050	0.000043	0.001184	0.003587	0.010332
6	0.0000006	0.000023	0.000085	0.00033	0.000048	0.001290	0.003825	0.010671
7	0.0000004	0.000016	0.000061	0.00024	0.000053	0.001389	0.004038	0.010939
8	0.0000003	0.000012	0.000046	0.00018	0.000059	0.001482	0.004230	0.011147
9	0.0000002	0.000009	0.000035	0.00014	0.000064	0.001568	0.004400	0.011302
10	0.0000002	0.000007	0.000028	0.00011	0.000069	0.001649	0.004553	0.011411
11	0.0000002	0.000006	0.000023	0.00009	0.000073	0.001725	0.004689	0.011485
12	0.0000001	0.000005	0.000019	0.00008	0.000078	0.001796	0.004811	0.011525
13	0.0000001	0.000004	0.000016	0.00006	0.000083	0.001862	0.004919	0.011537
14	0.00000009	0.000004	0.000014	0.00006	0.000087	0.001925	0.005015	0.011524
15	0.00000008	0.000003	0.000012	0.00005	0.000092	0.001984	0.005101	0.011491
16	0.00000007	0.000003	0.000010	0.00004	0.000096	0.002039	0.005177	0.011444
17	0.00000006	0.000002	0.0000092	0.00004	0.00010	0.002091	0.005244	0.011380
18	0.00000006	0.000002	0.0000082	0.00003	0.00010	0.002140	0.005303	0.011303
19	0.00000005	0.000002	0.0000073	0.00003	0.00011	0.002186	0.005354	0.011216
20	0.00000005	0.000002	0.0000066	0.00003	0.00011	0.002229	0.005399	0.011119
21	0.00000004	0.000002	0.0000059	0.00002	0.00012	0.002270	0.005437	0.011015
22	0.00000004	0.000001	0.0000054	0.00002	0.00012	0.002309	0.005471	0.010904
23	0.00000003	0.000001	0.0000049	0.00002	0.00012	0.002345	0.005499	0.010787
24	0.00000003	0.000001	0.0000045	0.00002	0.00013	0.002379	0.005522	0.010667
25	0.00000002	0.000001	0.0000041	0.00002	0.00013	0.002412	0.005541	0.010542

Carefully studying the patterns in Table 1, across the parameters, down the x-values, we deduce that the distributions are unimodal and positively skewed. These explicitly agree with the shape of their pdfs. More so, the probability outcomes at different values of x is consistent with the probability axiom: $0 < P(x) < 1$.

The performance comparisons involve the baseline distribution and the extended distributions with other counterpart distributions. It is noteworthy to express that the distribution which corresponds to the lowest AIC, BIC and or highest log-likelihood is considered the best fit.

Data Set I: The waiting time (in minutes) of one hundred (100) bank customers before service is being rendered (Ghitany, 2008).

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5

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Table 2: Descriptive Statistics for Dataset 1

n	$\underline{x}$	σ^2	σ	Outlier Threshold $(3 \times \sigma) \pm \underline{x}$	Minimum	Maximum
100	9.877	52.37	7.24	(11.83, 31.59)	0.8	38.5

Data Set II: The tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge length of 20 mm reported in Ghitany (2013).

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861,1.865, 1.944,1.958,1.966,1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426,2.434, 2.435,2.478, 2.490, 2.511,2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 4.594.

Table 3: Descriptive Statistics for Dataset 2

N	$\underline{x}$	σ^2	σ	Outlier Threshold $(3 \times \sigma) \pm \underline{x}$	Minimum	Maximum
69	2.4660	0.2936	0.5418	(+4.0941, - 0.8405)	1.312	4.594

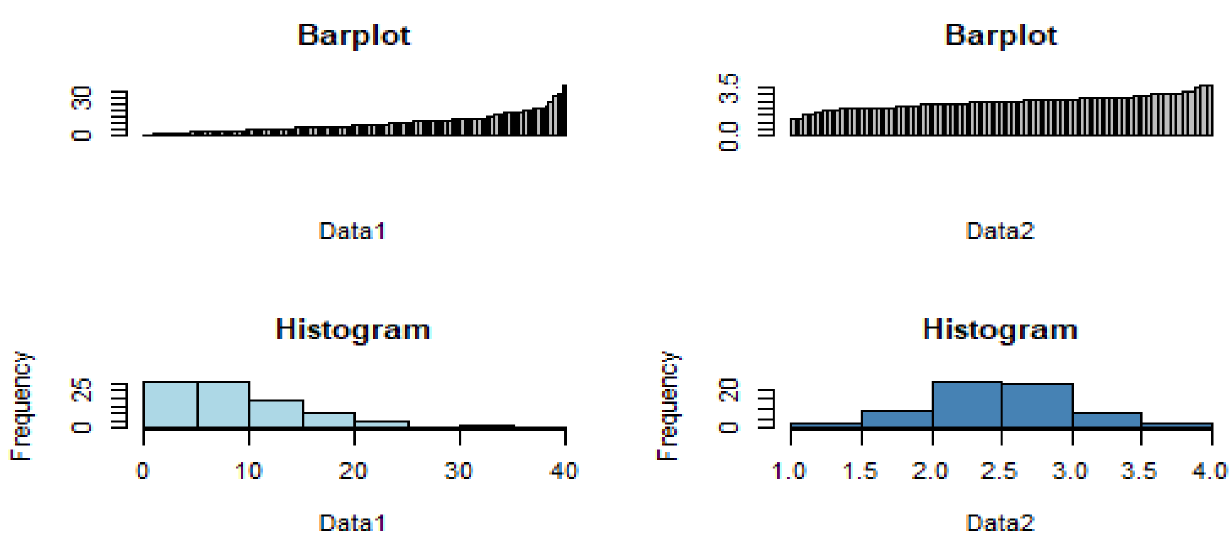

Figure 3: A graphical summary of the datasets 1 & 2

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The bar-plot and the histogram used here as graphical summary, gives an insight that the datasets are heavily skewed to the right or positively skewed. This is suggestive that the two extension distribution could sustain the capacity to model them.

Table 4: Performance rating for the ITOPD and the baseline distribution

			Data Set 1		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
TOPD	$\theta = 0.3968$	-329.59	661.18	663.79	2
ITOPD	$\theta = 6.3736$	-325.94	653.80	656.50	1

Table 5: Performance rating for the PTOPD and the baseline distribution

				Data Set 2	
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
TOPD	$\theta = 1.2429$	-108.04	218.08	220.31	2
PTOPD	$\theta = 0.4016$ $\varphi = 2.4335$	-55.037	114.07	118.54	1

For Datasets 1 and 2, as shown in Tables 4 and 5, the extended distributions showed to have better fit than the baseline distribution.

Table 6: Performance rating for the ITOPD and counterpart distributions

			Data Set 1		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
ITOPD	$\theta = 6.3736$	-325.94	653.80	656.50	1
ILD	$\theta = 6.1007$	-336.62	675.24	677.85	3
IED	$\theta = 5.3474$	-336.56	675.12	677.72	2
IAD	$\theta = 2.3698$	-399.45	800.91	803.51	4

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The distributions compared alongside Inverse TOPD distribution (ITOPD) are Inverse Lindley distribution (ILD), Inverse exponential distribution (IED) and Inverse Aishenawy distribution (IAD). Table 6 shows that the ITOPD exhibits better fit with respect to the applied data.

Table 7: The performance rating for the PTOPD and counterpart distributions

			Data Set 2		
Model	**Parameter Estimate**	**lnL**	**AIC**	**BIC**	**Rank**
PTOPD	$\theta = 0.4016$ $\varphi = 2.4335$	-55.037	114.07	118.54	1
PLD	$\theta = 0.0739$ $\varphi = 3.4041$	-56.557	117.11	121.58	3
PAD	$\theta = 0.2171$ $\varphi = 2.7661$	-55.406	114.81	119.28	2
PED	$\theta = 0.0116$ $\varphi = 4.5229$	-58.997	121.99	126.46	4

In this performance comparison, the baseline distribution and other counterpart power distributions were taken into consideration alongside PTOPD. These distributions are power Lindley distribution, power akash distribution and power exponential distribution (which is also known as Weibull distribution). In Table 7, it is clearly shown that PTOPD is a better fit compared to some power distributions and owing to the data used.

DISCUSSION AND CONCLUSION

The development of extensions of TOPD distribution was considered in this paper. The inverse and the power transformation approach were adopted to form new distributions termed “Inverse TOPD Distribution and Power TOPD Distribution.” These distributions were proven both mathematically and empirically to be valid distributions, where the baseline distribution is a special case of the power TOPD distribution at $\varphi = 1$. Properties like distribution mode, moment, mean and variance, survival and hazard functions, parameter estimation, quantile function, stress-strength reliability and order statistics were derived. The hazard shape for ITOPD is that of an inverted or upside-down bathtub shape; this implies that ITOPD can model outcomes that improve with time or use. PTOPD on the other hand exhibits three different shapes at various parameter values: bathtub shape, decreasing and increasing failure rate shape. This means that PTOPD is one such a flexible distribution that can model different sets of data. Inclusive are those that wear out with time and vice versa, and finally, those that exhibit early failure. More so, the pdf of the PTOPD is highly skewed and this is suggestive that it can model effectively data from wild

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observations. Methodically, a generalized family generator called "TOPD-G" is proposed and could be used to generate other flexible distributions. Highlights on the performance comparison—the ITOPD and PTOPD—show to be better fit than both the baseline distribution and or their counterpart distributions. PTOPD having extra parameters suffices for the limitation of the baseline in modeling data with outliers, as observed in Dataset 2 and Tables 3 and 5.

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