African Journal of Mathematics and Statistics Studies

ISSN: 2689-5323

Volume 7, Issue 4, 2024 (pp. 272-295)

www.abjournals.org

A NEW CLASSICAL TWO PARAMETER ASYMMETRIC PROBABILITY DISTRIBUTION: PROPERTIES AND APPLICATION

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Cite this article:

Nomuoja, A. J., Agu, C., Olawale, T. G., Adewole, C. J., Guobadia, E. K. (2024), A New Classical Two Parameter Asymmetric Probability Distribution: Properties and Application. African Journal of Mathematics and Statistics Studies 7(4), 272-295. DOI: 10.52589/AJMSS-ZXSCST58

Manuscript History

Received: 25 Mar 2023

Accepted: 10 Jul 2023

Published: 25 Nov 2024

ABSTRACT: *In this paper, the record is set straight on the technique for the development of classical distributions, where a new model called the Sky-Log distribution is proposed as an illustrative example of the methodical approach. The statistical properties of the proposed distribution were derived, and the very many known generating functions exist for the distribution. Lionel Messi's football record data were analyzed to validate the essence of the proposed model. Finally, it was discovered that the proposed distribution sub-model, termed Sky-X distribution, and the exponential distribution, are exact model fit alternatives.*

KEYWORDS: Model Theory, Probability distribution, Sky-Log distribution, Sky-X distribution, Lifetime data, Generating function.

Article DOI: 10.52589/AJMSS-ZXSCST58

DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

INTRODUCTION

In statistical modeling, probability distributions play a great role in providing best fit for the forecast of unprecedented events. This explains why some distributions are well known, suitable for the events they model. Humans witness unforeseen outcomes on a daily basis in different corners of the world, and so, the tireless search for better models that can appropriately project, with high accuracy, such unfolding events. In the past two to three decades, researchers have channeled much energy in the extensions of already existing distributions through compounding, convolution, mixture distribution, generalizations, etc. Sankaran (1970) introduced compounding in the Poison-Lindley distribution using the model:

$$p(x) = \int_0^\infty f(x|\theta)\, g(\theta)\, d\theta. \quad (1)$$

Jasiulewicz and Kordecki (2003) and Ghitany (2007) explored the convolution of independent Erlang distributions, and Lindley distributions respectively with:

$$p(z) = \int_{-\infty}^{\infty} f_x(z-y)\, f_y(y)\, dz \quad (2)$$

Lindsay (1995), in his theory compilation work, detailed the application of mixture model, which is given as:

$$p(x) = \sum_{i=1}^{k} v_i g_i(x)\,;\ \text{where}\ \sum_{i=1}^{k} v_i = 1,\ v_i > 0. \quad (3)$$

where v_i, as the weight, is the mixing probability (Friedman, 2009) and $g_i(x)$ is the distribution components. Originally, the equation in (3) was two dimensionally employed in the development of Lindley distribution (Lindley, 1958), before it received massive exploration of other k-dimensions. More so, generalizations are models designed such that other distributions can be incorporated directly into them without mathematical derivations. For instance, the Kumaraswamy generalization is given as:

$$f(x) = \alpha\beta g(x)[G(x)]^{\alpha-1}\{1 - [G(x)]^{\alpha}\}^{\beta-1} \quad (4)$$

where $g(x)\ and\ G(x)$ represent the probability and cumulative distribution respectively of a classical or baseline distribution, which combine to form a new compound of distribution. Cordeiro and de Castro (2011), Johnson (1995), Nadarajah and Kotz (2006), Oguntunde (2015), Rodrigues (2017), El-Nadi (2017), Obubu (2019), Ekhosuehi (2018), Opone (2020), and several authors have exhaustively explored the different generalizations. Another means through which distributions are developed is by parametric evaluation, termed as sub-models or special cases, as seen in the development of exponential, Erlang and chi-square distributions, which are gamma distribution offspring at different values of the parameters.

In the millennia, the aforementioned distribution techniques are predominant, whereas the development of classical types has remained a case of rarity. This is because these techniques and/or host models are already laid down, hence its ease of substitutive application. However, the methodologies for the developments of some baseline or classical distributions like Gamma distribution, Logistics distribution, Gumbel distribution, Weibull distribution, Pareto distribution, Normal distribution, etc. are not directly specified in literature (to the best of our knowledge). As observed in the different editions and volumes of the "continuous univariate

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DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

distribution compilations" by Johnson (1970 and 1995) and other relevant distribution materials, the derivation technique is rarely mentioned. The avalanche and stereotyped tilt in the developmental procedures of probability models in recent times informed the motivation for this research.

The aim of this paper, therefore, is to expose the model development theory which appears to be scarce in the literature, but had been used in developing some classical distributions in the 90s. Furthermore, we develop a novel classical probability model called Sky-Log probability distribution, which can forecast more accurately the lifetime success-events in football league championship development. The rest of the sections are arranged thus: the development of the new model, and its mathematical and statistical properties, where the data presentation and application follow suit as the last section.

METHODOLOGIES

Model Formulation

Proposition 1: A classical univariate continuous probability distribution $f(x)$ with random variable X, can be developed employing the mathematical technique:

$$D^{-1} \times v(x,\omega) = f(x,\omega) \quad (5)$$

Proof

Define any mathematical function as $v(x,\omega)$; where ω represent the vector of constant parameters. Again, define D as an integral operation with $v(x,\omega)$ in the similitude of the well-known continuous probability property $\int_{-\infty}^{\infty} p(x,\omega)dx = 1$, with an appropriate support $x \in R$, say {-∞, ∞} representing the minimum and maximum range; we have thus:

$$D = \int_{-\infty}^{\infty} v(x,\omega)dx \quad (6)$$

It is worthy of note that D may not be integrable. In other words, the chosen support may not converge for the integral solution of the function. However, the convergence of D proves the appropriateness of the variable support $x \in R$, and it is at this step that the software solution (or output) also indicates the parameter support. Now, the inverse of the following result D^{-1} is presumed to be a constant function of parameter(s). Furthermore, the product of the inverse of D and the defined mathematical function $v(x,\omega)$ is equal to a model $f(x,\omega)$.

$$D^{-1} \times v(x,\omega) = f(x,\omega) \quad (7)$$

Consequently, $f(x,\omega)$ as any mathematical function, could be a probability model on the premises of a defined parameter space. This space or domain houses the only parameter range or parameters combination range, in which a PDF yields probability outcome: $0 \leq p(x) \leq 1$

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Sky-Log Distribution Development

Proposition 2: A random variable X is said to follow a Sky-Log distribution with PDF and CDF if:

$$f_{sk-l}(x) = \lambda^x e^{\alpha x}(-\alpha - Log[\lambda]), \qquad (13)$$

$$x > 0, \ \ \alpha + Log[\lambda] < 0$$

$$F_{sk-l}(x) = 1 - e^{x\alpha}\lambda^x$$

Proof

The mathematical derivation of the Sky-Log distribution, combines the shape parameter and location parameter of both the logarithmic and exponential functions, constrained by the condition that $\int p(x)\, dx = 1$ in its combination; where $p(x)$ is a probability distribution.

α, λ, λ^x and $e^{\alpha x}$ are the shape parameter, location parameter, variable location component and variable exponential component respectively. Applying the proposed technique, with the assistance of mathematical software we have the following development.

Let

$$v(x, \alpha, \lambda) = \lambda^x e^{\alpha x}, \quad x > 0. \qquad (14)$$

and

$$D = \int_0^\infty v(x, \alpha, \lambda)\ dx = \int_0^\infty \lambda^x e^{\alpha x}\ dx \qquad (15)$$

$$= -\frac{1}{\alpha + Log[\lambda]}, \qquad \alpha + Log[\lambda] < 0$$

$$\rightarrow \qquad D^{-1} = -(\alpha + Log[\lambda]),$$

$$f(x) = D^{-1} * v(x, \alpha, \lambda) = (-\alpha - Log[\lambda])\, \lambda^x e^{\alpha x}$$

$$\therefore \qquad f_{sk-l}(x) = \begin{cases} \lambda^x (-\alpha - Log[\lambda])\, e^{\alpha x}, & x > 0, \ \ \alpha + Log[\lambda] < 0 \\ 0, & otherwise \end{cases} \qquad (16)$$

This parameter support $\alpha + Log[\lambda] < 0$, as obtained in Mathematica, forms the domain in which the distribution is defined as a probability function. In addition, the trigonometric equivalence of the PDF in Equation (16) can be represented thus:

$$f_{sk-l}(x) = \lambda^x(-\alpha - Log[\lambda])(Cosh[x\alpha] + Sinh[x\alpha]) \qquad (17)$$

where the corresponding Cumulative Distribution Function (CDF) is obtained as:

$$F_{sk-l}(x) = \int_0^x f_{sk-l}(t)dt \qquad (18)$$

$$F_{sk-l}(x) = 1 - e^{x\alpha}\lambda^x \qquad (19)$$

Hence, the Probability Distribution by nomenclature is called the ***Sky-Log Distribution***.

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Sub-Model

At $\lambda = 1$, the proposed distribution simplifies to a probability distribution with PDF $g(x)$ obtained from Equation (16) or (19) as:

$$g_{sk-l}(x) = -\alpha e^{\alpha x}, \quad x > 0, \ -\infty \leq \alpha \leq 0 \tag{20}$$

Or $$g_{sk-l}(x) = -\alpha \left(Cosh[x\alpha] + Sinh[x\alpha]\right) \tag{21}$$

and the corresponding CDF $G(x)$ is obtained from Equation (19) as:

$$\rightarrow \qquad G(x) = 1 - e^{\alpha x} \tag{22}$$

For nomenclature purposes, we term the sub-model, Sky-X distribution. Although the properties of the sub-model are not explored in this paper, its application will be carefully examined.

Remark: The sub-model of the Sky-Log distribution has a resemblance of the exponential distribution, a sub-model to gamma distribution. However, the functions are not mathematically equal:

$$-\alpha e^{\alpha x} \neq \alpha e^{-\alpha x} \quad : \quad \text{for the PDF} \tag{23}$$

$$1 - e^{\alpha x} \neq 1 - e^{-\alpha x} \quad : \quad \text{and the CDFs}$$

PROPERTIES OF SKY-LOG DISTRIBUTION

The characteristic features as exhibited by the distribution are presented in this section, where most of the models are mathematical and statistical derivations from the works of Sahoo (2006).

The PDF and CDF of the Sky-Log Distribution

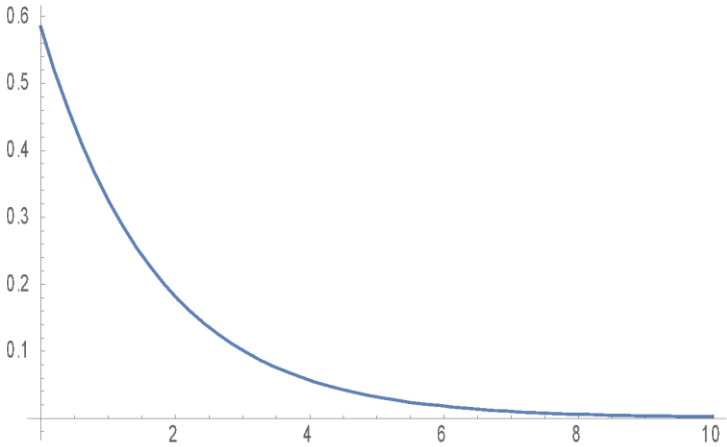

Figure 1: Plot of the PDF of Sky-log distribution at $\alpha = -1.5$, $\lambda = 2.5$

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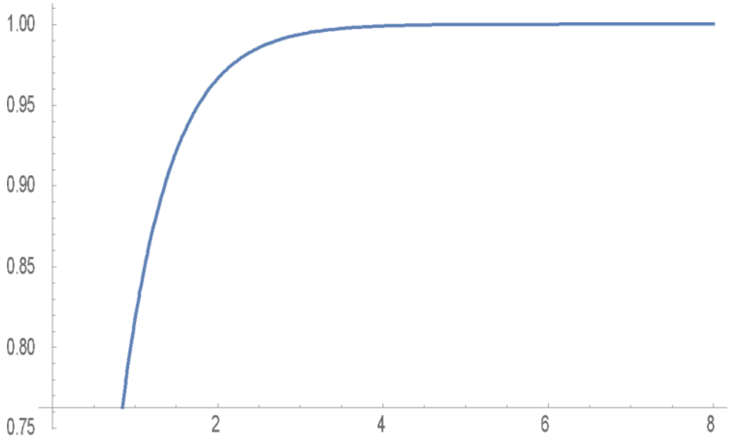

Figure 2: Plot of the CDF of Sky-log distribution at $\alpha = -1.5\,,\ \lambda = 1$

Remark: It is important to note that at other valid values of the parameters, the PDF is found to be positively skewed and unimodal as seen in Figure 1, while the CDF is an increasing function.

Mode

The Mode of Sky-Log distribution is obtained by differentiating the PDF in Equation (16), and equating to zero as:

$$\frac{d}{dx} f(x,\omega) = 0 \tag{24}$$

that is: $$\frac{d}{dx}\{-e^{x\alpha}\,\lambda^{x}\,(\alpha + Log[\lambda])\} = 0$$

$$-e^{x\alpha}\lambda^{x}(\alpha + Log[\lambda])^{2} = 0$$

$$\rightarrow \qquad e^{x\alpha}\,\lambda^{x} = 0 \tag{25}$$

Median

The Median of Sky-Log distribution is obtained by integrating Equation (16) within the range $(0 < x\ \ or\ \ x < \infty)$ and equating to 0.5.

$$\int_0^x f(t)dt = \int_x^\infty f(t)dt = \frac{1}{2} \tag{26}$$

$$\int_0^x -e^{t\alpha}\,\lambda^{t}\,(\alpha + \mathrm{Log}[\lambda])\,dt = \frac{1}{2}$$

$$1 - e^{x\alpha}\lambda^{x} = \frac{1}{2}$$

$$e^{x\alpha}\lambda^{x} = \frac{1}{2}$$

Taking logs of both sides,

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$$log\ log\ [e^{x\alpha}\lambda^{x}]\ = log\ [\tfrac{1}{2}]$$

$$x\alpha + xlog[\lambda] = \ log\ [\tfrac{1}{2}]$$

$$x(\alpha + log[\lambda]) = \ log\ [\tfrac{1}{2}]$$

$$x = \frac{log\ [\frac{1}{2}]}{(\alpha + log[\lambda])} \tag{27}$$

Moment and Moment Generating Function

Let $X{\sim}SK - l$, then the rth moment about the origin $\mu_1^r$of the Sky-Log distribution and the moment generating function $M_x(t)$ are given respectively as:

$$\mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r\, f(x)dx \tag{28}$$

$$= \int_0^{\infty} x^r\, \lambda^x\, e^{\alpha x}(-\alpha - \mathrm{Log}[\lambda])dx$$

$$= \frac{(\alpha)^r\Gamma(r+1)}{(-\alpha)^r(\alpha+\mathrm{Log}[\lambda])^r}; \quad \alpha + \mathrm{Log}[\lambda] < 0, \quad r > -1 \tag{29}$$

and $$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}\, f(x)dx \tag{30}$$

$$= \int_0^{\infty} e^{tx}\, \lambda^x\, e^{\alpha x}(-\alpha - \mathrm{Log}[\lambda])dx$$

$$= \frac{\alpha+\mathrm{Log}[\lambda]}{t+\alpha+\mathrm{Log}[\lambda]}; \quad t + \alpha + \mathrm{Log}[\lambda] < 0 \tag{31}$$

Evaluating Equation (29) at $r = 1,2,3$ & 4, we obtain:

$$\mu'_1 = -\frac{1}{\alpha+Log[\lambda]} = \mu \qquad \mu'_2 = \frac{2}{(\alpha+Log[\lambda])^2} \tag{32}$$

$$\mu'_3 = -\frac{6}{(\alpha+Log[\lambda])^3} \qquad \mu'_4 = \frac{24}{(\alpha+Log[\lambda])^4} \tag{33}$$

The central moment about the mean of the Sky-Log distribution is:

$$\mu_n = E[\,(X - E[X])^n\,] = \sum_{j=0}^{n} \binom{n}{j}\, (-1)^{n-j}\, \mu'_j\mu^{n-j} \tag{34}$$

So that $$\mu_2 = \mu'_2 - \mu^2 = \frac{1}{(\alpha+Log[\lambda])^2} = \sigma^2 \tag{35}$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3 \tag{36}$$

$$= -\frac{2}{(\alpha+Log[\lambda])^3}$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4 \tag{37}$$

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$$= \frac{9}{(\alpha+Log[\lambda])^4}$$

Root Mean Square (RMS)

$$RMS = \sqrt{E(x^2)} = \sqrt{\mu_2'} \quad (38)$$

$$= \sqrt{2}\sqrt{\frac{1}{(\alpha+Log[\lambda])^2}}; \quad (\alpha + Log[\lambda]) < 0 \quad (39)$$

The Coefficient of Variation (CV), Coefficient of Skewness (CS), Coefficient of Kurtosis CK and the Variance to Mean Ratio (VMR) of Sky-log distribution are thus obtained as:

$$CV = \frac{\sigma}{\mu_1'} = -1 \quad (40)$$

$$CS = \frac{\mu_3}{\mu_2^{3/2}} = -2 \quad (41)$$

$$CK = \frac{\mu_4}{\mu_2^2} = 9 \quad (42)$$

$$VMR = \frac{\sigma^2}{\mu_1'} = -\frac{1}{\alpha+Log[\lambda]} = \mu \quad (43)$$

Cumulant Generating Function

Cumulants are derived following the relationship between cumulant generation function $K(t)$ and moment generating function $M_x(t)$. This is given as:

$$K(t) = Log\ M_x(t) \quad (44)$$

$$K(t) = Log\left[\frac{\alpha+Log[\lambda]}{t+\alpha+Log[\lambda]}\right], \quad t + \alpha + Log[\lambda] < 0$$

$$k_1 = K'(t) = -\frac{1}{t+\alpha+Log[\lambda]} \quad (45)$$

$$k_2 = K''(t) = \frac{1}{(t+\alpha+Log[\lambda])^2} \quad (46)$$

$$k_3 = K'''(t) = -\frac{2}{(t+\alpha+Log[\lambda])^3} \quad (47)$$

$$k_4 = K''''(t) = \frac{6}{(t+\alpha+Log[\lambda])^4} \quad (48)$$

when $t = 0,\ k_1 = \mu,\ k_2 = \mu_2 = \sigma^2, k_3 = \mu_3, k_4 = \mu_4 - 3\,\mu_2^2$, which also corresponds with the central moment relationship.

Factorial Moment Generating Function (FMGF)

The FMGF of a real valued random variable X $\aleph(t)$ is defined, if $M_x(t)$ exists in the neighborhood of $t = 1$; the generating function is obtained thus:

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$$\aleph(t) = E(t^x) = \int t^x f(x)dx \tag{49}$$

$$= \int_0^\infty t^x \, [\, \lambda^x \, e^{\alpha x}(-\alpha - \text{Log}[\lambda])]dx$$

$$= \frac{\alpha + Log[\lambda]}{\alpha + Log[t] + Log[\lambda]}, \quad \alpha + Log[t] + Log[\lambda] < 0 \tag{50}$$

Characteristic Function

The characteristic function, $\varphi_{it}(t)$ of Sky-Log distribution is:

$$\varphi_{it}(t) = E\left(e^{itx}\right) = \int e^{itx} f(x)dx \tag{51}$$

$$= \int_0^\infty e^{itx} \; [\, \lambda^x \, e^{\alpha x}(-\alpha - \text{Log}[\lambda])]dx \tag{52}$$

$$= \frac{\alpha + Log[\lambda]}{it + \alpha + Log[\lambda]}, \; it + \alpha + Log[\lambda] < 0 \tag{53}$$

where $i = \sqrt{-1}$ is the complex component.

Central Moment Generating Function

The central moment generating function $M_c(t)$ of the Sky-Log distribution is

$$M_c(t) = E[t(x-\mu)] = \int t(x-\mu) f(x)dx \tag{54}$$

$$= \int_0^\infty t\left(x - \left[-\frac{1}{\alpha + \text{Log}[\lambda]}\right]\right) e^{x\alpha} \lambda^x (-\alpha - \text{Log}[\lambda])dx \tag{55}$$

$$= 0$$

Remark: Any distribution whose moment generating function exists returns zero as its central moment generating function.

Table 2: Coefficient of Variation (CV) of different one-parameter distributions, valued at $\omega = 1$ and $\omega = 3$. Recall that ω is the vector of parameters

Distributions	$\left[CV = \frac{\sigma}{\mu}\right]_{\omega=1}$	$\left[CV = \frac{\sigma}{\mu}\right]_{\omega=3}$
Exponential	1	1
Fretchet	-	0.4889
Rayleigh	0.5223	0.5226
Logistics-	1.8145	1.8140
Sky-Log	-1	-1

Remark: In model development, distributions with $CV < 1$ are considered low-variance and $CV > 1$, high variance (Everitt, 1998). From Table 2, Sky-log distribution is found to be

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more precise compared to other models as indicated by the CVs. This is a clear support that the Sky-Log distribution has an edge as to the flexibility compared to other models considered.

Table 3: Variance to Mean Ratio (VMR) analysis

VMR	λ					
		1	**1.5**	**2**	**2.5**	**3**
	-1.5	0.6667	0.7553	0.8340	0.9074	0.9776
	-2.0	0.5	0.5483	0.5886	0.6242	0.6567
α	**-2.5**	0.4	0.4303	0.4548	0.4757	0.4943
	-5.0	0.2	0.2073	0.2128	0.2173	0.2211
	-10	0.1	0.1018	0.1031	0.1041	0.1050
	-100	0.01	0.0100	0.0100	0.0100	0.1050
	$-\infty$	0.000000…	0.000000…	0.000000…	0.000000…	0.000000…

Table 3 shows the results of the $VMR = \frac{\sigma^2}{\mu}$ at different parameter values. For $1 \leq \lambda \leq 3$ and $-\infty \leq \alpha \leq -1.5$, we can deduce that the Sky-Log distribution is under dispersed, an indication that the Sky-Log distribution exhibits lesser variation, which is a desirable feature for the flexibility of a model.

Reliability, Failure Rate and Cumulative Hazard Models

If X is a continuous random variable with Sky-Log PDF and CDF given in Equations (16) and (19) respectively, then the reliability, failure rate and cumulative hazard functions $\underline{F}\,(x)$, $h(x)\ and\ H(x)$ are expressed respectively as:

$$\bar{F}\,(x) = \int_x^\infty f(t)dt \tag{56}$$

$$= \int_x^\infty e^{t\alpha}\lambda^t(-\alpha - \text{Log}[\lambda])dt \tag{57}$$

$$= e^{x\alpha}\lambda^x \tag{58}$$

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{-e^{x\alpha}\lambda^x(\alpha+\text{Log}[\lambda])}{e^{x\alpha}\lambda^x} = -(\alpha + \text{Log}[\lambda]) \tag{59}$$

and

$$H(x) = \int_0^x h(t)dt \tag{60}$$

$$= \int_0^x -(\alpha + \text{Log}[\lambda])dt$$

$$= x(-\alpha - \text{Log}[\lambda]) \tag{61}$$

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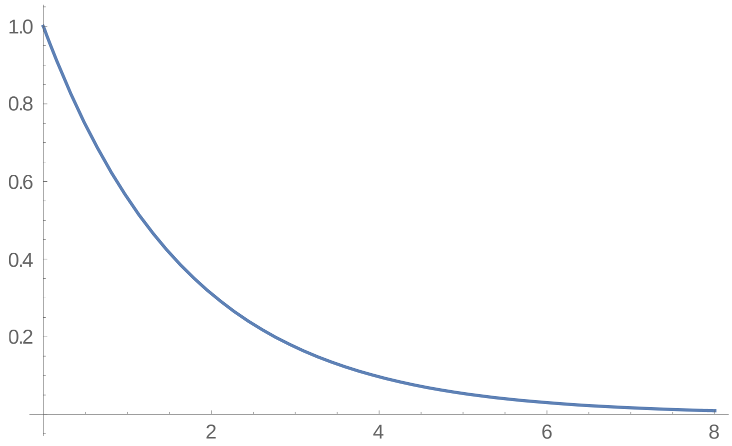
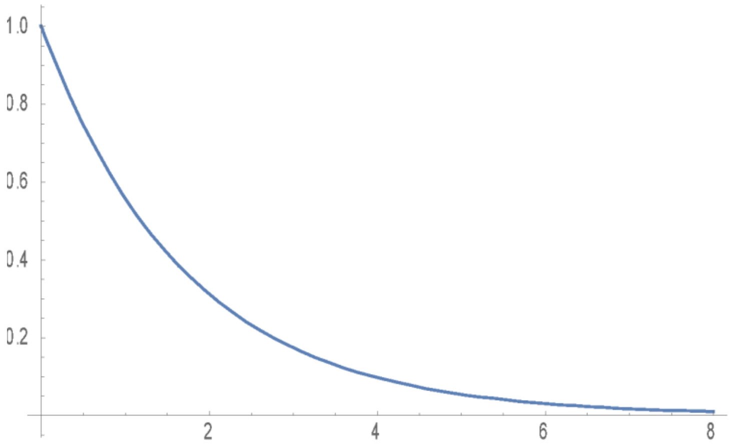


Figure 3. Plot of the Survival function of Sky-log distribution at $\alpha = -1.5$, $\lambda = 2$



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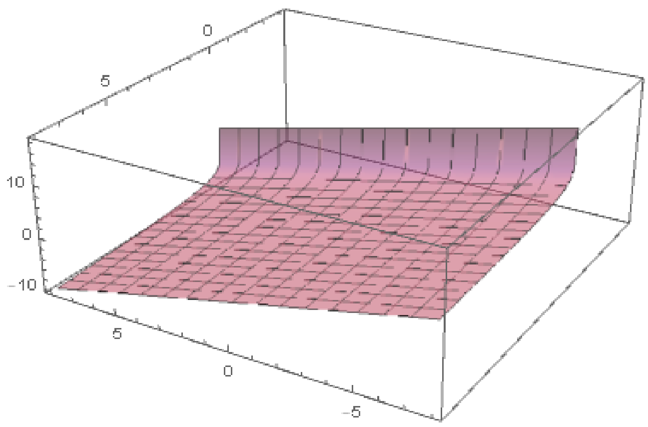

Figure 4: Plot of the hazard function of the Sky-log distribution at $\alpha = -2$, $\lambda = 2.8$

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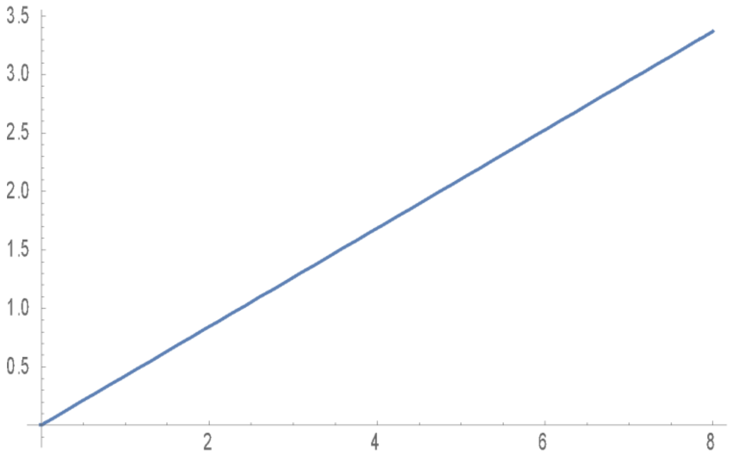

Figure 5: Plot of the cumulative hazard function of Sky-log distribution $\forall\ \alpha, \lambda$

Mean Time To Failure (MTTF) and Mean Residual Life (MRL)

Using the Laplace transform given as $l^*(t) = \int_0^\infty e^{-tx} u(x) dt$, we derive the MTTF of the Sky-Log distribution $\bar{F}^*(t)$ as

$$\bar{F}^*(t) = \int_0^\infty e^{-tx}\, \bar{F}(x)\, dx \tag{62}$$

where $\bar{F}(x)$ is survival function of the Sky-Log distribution

$$= \int_0^\infty e^{-tx}\, e^{x\alpha} \lambda^x\, dx$$

$$= -\frac{1}{t+\alpha+\text{Log}[\lambda]},\ t+\alpha+\text{Log}[\lambda] < 0,\ t+\alpha < 0\,, \tag{63}$$

When $t = 0$, we obtain

$$\bar{F}^*(0) = -\frac{1}{\alpha+\text{Log}[\lambda]} = \mu \tag{64}$$

The Mean Residual (remaining) Life of an item at age t is

$$\text{MRL(t)} = \frac{1}{\bar{F}(x)} \int_t^\infty \bar{F}(x)\, dx = \mu(t) \tag{65}$$

$$= \frac{1}{e^{x\alpha}\lambda^x} \int_t^\infty e^{x\alpha}\lambda^x\, dx$$

$$= -\frac{e^{t\alpha - x\alpha}\lambda^{t-x}}{\alpha+\text{Log}[\lambda]},\ \ \alpha+\text{Log}[\lambda] < 0\ \ \&\ \text{Log}[\lambda] < 0 \tag{66}$$

When $t = x$, $MRL = MTTF = \mu$.

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ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the Sky-Log Distribution. Let $X_1 < X_2 < , \ldots, < X_n$ denote the corresponding order statistics. The pdf and the cdf of the kth order statistics say $Y = X_k$ is given respectively as:

$$f_Y(\mathrm{y}) = \frac{\mathrm{n}!}{(k-1)!(n-k)!} F^{k-1}(y)\{1-F(y)\}^{n-k} f(y) \tag{67}$$

$$= \frac{\mathrm{n}!}{(k-1)!(n-k)!} (1-e^{y\alpha}\lambda^{y})^{k-1}\{e^{y\alpha}\lambda^{y})\}^{n-k}\{e^{y\alpha}\lambda^{y}(-\alpha-\mathrm{Log}[\lambda])\}$$

$$= \frac{\mathrm{n}!}{(k-1)!(n-k)!} \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}\{e^{y\alpha}\lambda^{y})\}^{i}\{e^{y\alpha}\lambda^{y})\}^{n-k}\{e^{y\alpha}\lambda^{y}(-\alpha-\mathrm{Log}[\lambda])\}$$

$$= \frac{\mathrm{n}!(-\alpha-\mathrm{Log}[\lambda])}{(k-1)!(n-k)!} \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{i}\{e^{y\alpha}\lambda^{y})\}^{n-k+i+1} \tag{68}$$

and

$$F_Y(\mathrm{y}) = \sum_{j=k}^{n}\binom{n}{j} F^{j}(y)\{1-F(y)\}^{n-j} \tag{69}$$

$$= \sum_{j=k}^{n}\binom{n}{j}(1-e^{y\alpha}\lambda^{y})^{j}\{e^{y\alpha}\lambda^{y}\}^{n-j}$$

$$= \sum_{j=k}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(-1)^{l}(\{e^{y\alpha}\lambda^{y}\})^{j}\{e^{y\alpha}\lambda^{y}\}^{n-j}$$

$$= \sum_{j=k}^{n}\sum_{l=0}^{j}\binom{n}{j}\binom{j}{l}(-1)^{l}(\{e^{y\alpha}\lambda^{y}\})^{n} \tag{70}$$

respectively, for k = 1,2,3,…,n.

The PDF of the minimum order statistics is obtained by substituting $k = 1$ in Equation (68) to get:

$$f_{1:n} = n(-\alpha - Log[\lambda])\{e^{y\alpha}\lambda^{y})\}^{n} \tag{71}$$

while the corresponding PDF of maximum order statistics is obtained by making the substitution of $k = n$ in the same Equation (68).

$$f_{n:n} = \mathrm{n}(-\alpha-\mathrm{Log}[\lambda])\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{i}\{e^{y\alpha}\lambda^{y})\}^{i+1} \tag{72}$$

Stress-Strength Reliability

The stress-strength reliability gives an analogy about the life of a system, which has random strength X that is subjected to a random stress Y. When force is applied to override a restraint, the system collapses at once, and it will operate optimally till $Y < X$. Therefore, $R = P(X > Y)$ measures system reliability and in the literature, it is known as stress-strength parameter. This is mostly applied in diverse fields of life especially in structural engineering, deterioration of rocket motors, aging of concrete pressure vessels, etc.

Let X and Y be independent strength and stress random variables having Sky-Log distribution with parameter ω_1 *and* ω_2 respectively; then the stress-strength reliability R can be obtained as:

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

$$R = P(X > Y) = \int_0^\infty P(X > Y|X = x)f(x)dx \quad (73)$$

$$= \int_0^\infty F_{sk-l}(x, \omega_2)\, f_{sk-l}(x, \omega_1)\, dx$$

$$= \int_0^\infty \{F_{sk-l}(x, \alpha_2, \lambda_2)\, f_{sk-l}(x, \alpha_1, \lambda_1)\}]_{\lambda_1=\lambda_2=1}\, dx$$

$$= \int_0^\infty \{1 - e^{\alpha_2 x}\}\{ e^{\alpha_1 x}(-\alpha_1 - \text{Log}[1])\}\, dx$$

$$R_\alpha = \frac{\alpha_2}{\alpha_2+\alpha_1}; \quad \alpha_2 + \alpha_1 < 0 \ and \ \alpha_1 < 0 \quad (74)$$

More so,

$$R_\lambda = \int_0^\infty \{F_{sk-l}(x, \alpha_2, \lambda_2)\, f_{sk-l}(x, \alpha_1, \lambda_1)\}]_{\alpha_1=\alpha_2=1}\, dx \quad (75)$$

$$= \int_0^\infty \{1 - e^x \lambda_2^x\}\{ e^x \lambda_1^x(-1 - \text{Log}[\lambda_1])\}]_{\alpha_1=\alpha_2=1}\, dx$$

$$= \frac{Log[\lambda_2]}{Log[\lambda_2]+Log[\lambda_1]}; \ Log[\lambda_2],\ Log[\lambda_1] < 0 \quad (76)$$

Convolution of the Sky-Log Distribution

Let $X\ and\ Y$ be two random variables with density functions $f(x)\ and\ f(y)$ as given in Equation (16). Recall that both $f(x)\ and\ f(y)$ are defined for all real numbers. Then, the distribution of the sum of Independent random variables: $Z = X + Y$ is derived thus:

$$f_{X+Y}(z) = \int_{-\infty}^{z} f_y(z-y)f_y(y)dy \quad (77)$$

$$= \int_0^z \{\lambda^{z-y}\, e^{\alpha(z-y)}(-\alpha - \text{Log}[\lambda])\}\{\lambda^y\, e^{\alpha y}(-\alpha - \text{Log}[\lambda])\}dy$$

$$= ze^z\, \lambda^z\, (\alpha + \text{Log}[\lambda])^2 \quad (78)$$

The distribution of Ratios of Independent random variables: $Z = X/Y$ is given as

$$f_{X/Y}(z) = \int_{-\infty}^{\infty} y\, f_x\, (zy)\, f(y)dy \quad (79)$$

$$= \int_0^\infty \{\lambda^{zy}\, e^{\alpha(zy)}(-\alpha - \text{Log}[\lambda])\}\{\lambda^y\, e^{\alpha y}(-\alpha - \text{Log}[\lambda])\}dy$$

$$= \frac{1}{(1+z)^2} \quad (80)$$

Remark. Equations (78) and (80) are new probability models which satisfies the probability axiom $\int f(x)dx = 1$, where their respective CDFs are given as:

$$F_{X+Y}(z) = 1 + e^{z\alpha}\lambda^z(-1 + z(\alpha + \text{Log}[\lambda])) \quad (81)$$

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
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$$F_{X/Y}(z) = \frac{z}{1+z}, \quad [z] > -1 \tag{82}$$

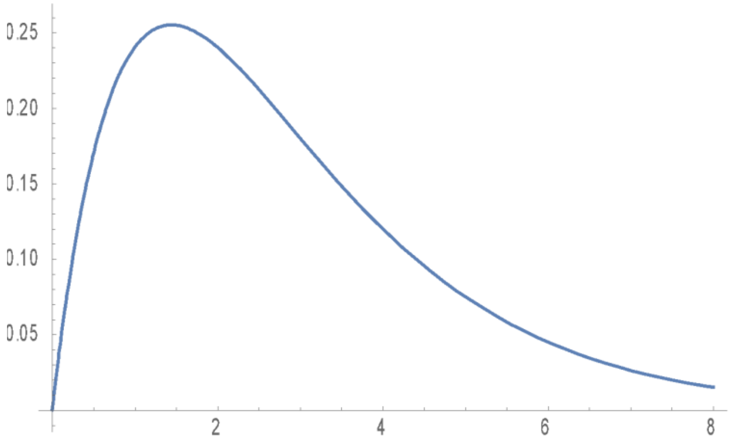

Figure 6: Plot of the PDF of the Convolution Sum

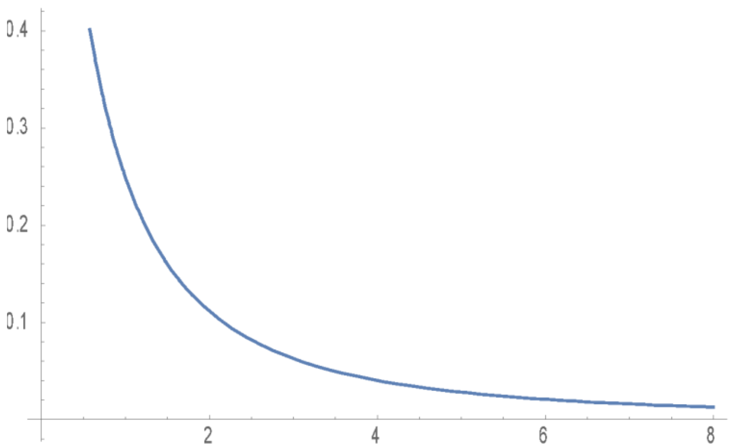

Figure 7: Plot of the PDF of the Convolution Ratio

Inverse Cumulative Distribution

The mathematical expression for the quantile function is derived thus:

$$F(x) = p; \qquad x = F^{-1}(p); \quad 0 \leq p \leq 1 \tag{83}$$

The Inverse Cumulative Function for the Sky-Log Distribution is thus obtained as:

$$F_{sk-l}(x) = p$$

$$p = 1 - e^{x\alpha}\lambda^{x}$$

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

$$x = \frac{Log(1-p)}{[\alpha+Log(\lambda)]} \tag{84}$$

This is also equal to Equation (27) at $p = 0.5$, as the median of the distribution.

Forecast of the Quantile of the Sky-Log Distribution

Forecasting the values of x in Equation (84) at $\alpha = -1.5, \lambda = 1$ for $n = 57$ gives:

0.26, 0.14, 0.61, 0.34, 0.44, 0.17, 0.20, 1.30, 0.11, 0.58, 0.54, 0.19, 3.46, 0.24, 0.04, 0.64, 0.42, 0.06, 0.08, 0.70, 0.35, 0.39, 0.25, 0.86, 2.64, 0.66, 1.47, 0.14, 0.78, 0.57, 2.74, 0.20, 1.09, 0.11, 0.23, 0.06, 3.73, 1.34, 1.45, 0.11, 0.62, 0.16, 2.02, 0.22, 1.32, 0.63, 1.23, 0.51, 0.15, 0.27, 0.08, 0.56, 0.62, 0.63, 0.43, 0.57, 0.70.

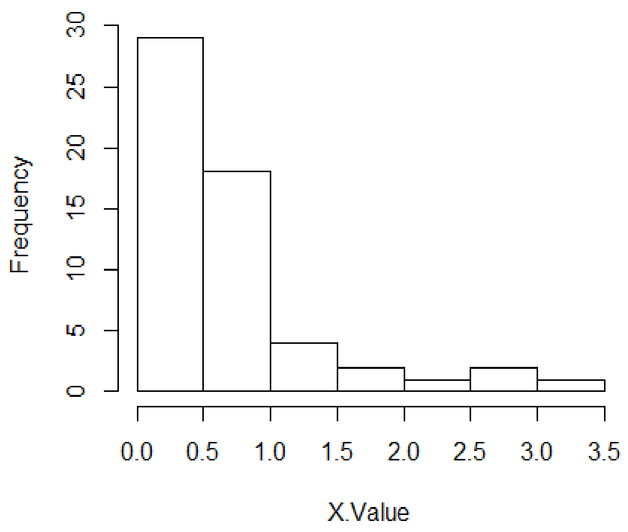

Figure 8: The Plot of the Histogram for Sky-Log quantile random variable X

Remark: The plot in Figure (8) is consistent with the plot of the PDF given in Figure (1).

Estimation

Let $X_i, i = 1,2,3, \dots, n$, be a random variable from the Sky-Log distribution; the Maximum Likelihood Estimator (MLE) of the parameters α *and* λ is obtained thus:

$$\mathrm{L}f(x,\omega) = (-\alpha - \mathrm{Log}[\lambda])^n \, \lambda^{\sum_{i=1}^{n} x_i} e^{\alpha \sum_{i=1}^{n} x_i} \tag{85}$$

$$ln\mathrm{L}f(x,\omega) = \mathrm{n}ln(-\alpha - \mathrm{Log}[\lambda]) + \sum_{i=1}^{n} x_i \, ln\lambda + \alpha \sum_{i=1}^{n} x_i$$

$$\frac{\partial \, ln\mathrm{L}f(x,\omega)}{\partial \omega} = 0$$

$$\rightarrow \quad \frac{\partial \, ln\mathrm{L}f(x,\omega)}{\partial \alpha} = \sum_{i=1}^{n} x_i - \frac{n}{(\alpha+\mathrm{Log}[\lambda])} = 0$$

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

$$\frac{\partial\, \ln \mathrm{L} f(x,\omega)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - \frac{n}{\lambda(\alpha+\mathrm{Log}[\lambda])} = 0$$

which gives $\underline{x} = \frac{1}{(\alpha+Log[\lambda])}$

$$\therefore \quad \hat{\alpha}_{mle} = \frac{1}{\underline{x}} - Log[\lambda] \tag{86}$$

$$\hat{\lambda}_{mle} = e^{[-\alpha+(\underline{x}^{-1})]} \tag{87}$$

Simulation Study

Employing the Monte Carlo simulation method, we examine the biasedness and consistency of the Sky-Log estimators for $n = 20, 50, 80, 100, 150\ and\ 200$ sample sizes, obtained for $M = 1000$, Monte Carlo times.

The estimator $\widehat{\omega}_{mle}$ of ω is biased if $E(\widehat{\omega}_{mle}) - \omega \neq 0$.

The estimator $\widehat{\omega}_{mle}$ of ω is consistent if $\widehat{\omega}_{mle} p \to \omega\ as\ n \to \infty$.

$$\to \quad P(|\widehat{\omega}_{mle} - \omega| > \epsilon) = 0 \tag{88}$$

Table 4: Biasedness and Consistency of the Parameter Estimator

n	$\hat{\alpha}_{mle}\ at\ \alpha = -1.5$ $\hat{\lambda}_{mle}\ at\ \lambda = 2$	$\lvert\widehat{\omega}_{mle} - \omega\rvert$	$\hat{\alpha}_{mle}\ at\ \alpha = -5$ $\hat{\lambda}_{mle}\ at\ \lambda = 3$	$\lvert\widehat{\omega}_{mle} - \omega\rvert$
20	-1.59666 2.11271	0.09666 0.11271	-5.2659 3.3371	0.2659 0.3371
50	-1.57561 2.11040	0.07556 0.11040	-5.1729 3.2575	0.1729 0.2575
80	-1.56701 2.09968	0.06701 0.09968	-5.1416 3.2462	0.1416 0.2462
100	-1.55822 2.09729	0.05822 0.09729	-5.1034 3.2222	0.1034 0.2222
150	-1.54968 2.09357	0.04968 0.09357	-5.0864 3.2195	0.0864 0.2195
200	-1.54358 2.08794	0.04358 0.08794	-5.0734 3.1881	0.0734 0.1881

Table 4 reveals that MLEs are positively and negatively biased as $E(\widehat{\omega}_{mle}) - \omega \neq 0$. More so, $|\widehat{\omega}_{mle} - \omega|$ tends to zero as $n \to \infty$. That is, $\widehat{\omega}_{mle} p \to \omega\ as\ n \to \infty$, an indication that the estimator is consistent.

In what follows, we provide results for the average bias and mean square error for $M = 1000$ Monte Carlo simulations, over the selected values of (n, ω). The computational formulas for the Average Bias and the MSE are given by:

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$$Average\ Bias = \left[\frac{1}{M} \sum_{i=1}^{M} (\widehat{\omega}_i - \omega) \right] \ and\ MSE = \left[\frac{1}{M} \sum_{i=1}^{M} (\widehat{\omega}_i - \omega)^2 \right]$$

Table 5: Average Bias of the Estimators $\widehat{\omega}$

n	$\alpha = -1.5$	$\lambda = 1$	$\alpha = -5.5$	$\lambda = 1.5$	$\alpha = -10$	$\lambda = 3$
20	-0.2013	0.1247	-0.5394	0.3854	-0.6289	0.7190
50	-0.1269	0.1062	-0.2650	0.3302	-0.3761	0.5616
80	-0.1157	0.1028	-0.2198	0.2949	-0.2491	0.5271
100	-0.1076	0.1021	-0.2192	0.2790	-0.2346	0.4793
150	-0.0894	0.0967	-0.1809	0.2629	-0.2209	0.4782
200	-0.0994	0.0992	-0.1777	0.2702	-0.1789	0.4407

Table 6: Mean Square Error (MSE) of the Estimators $\widehat{\omega}$

n	$\alpha = -1.5$	$\lambda = 1$	$\alpha = -5.5$	$\lambda = 1.5$	$\alpha = -10$	$\lambda = 3$
20	0.2312	0.0256	1.9892	0.2715	5.3118	0.8186
50	0.0835	0.0183	0.7511	0.2553	2.0012	0.5627
80	0.0579	0.0170	0.4610	0.1800	1.1767	0.5170
100	0.0396	0.0164	0.2790	0.1766	0.9962	0.4845
150	0.0379	0.0158	0.2662	0.1669	0.6949	0.4460
200	0.0311	0.0153	0.2029	0.1658	0.4960	0.4024

Results from Tables 5 and 6 clearly show that the estimates of the average bias and the MSE decrease as the sample size n increases. These measurements by implication indicate the quality of the estimator, which could stand out in efficiency comparison with other models.

Applications of the Sky-Log Distribution and its Sub-model

In this section, we consider the application of the Sky-Log distribution to real lifetime dataset. The distribution is fitted to the dataset and compared with that of Rayleigh, Frechet, Logistic and Gumbel-Type 2 distributions. More so, the Sky-X distribution and the exponential distribution are also compared as sub-model case. Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the Log-Likelihood Component $[-2\ log\ log\ (L)\ ]$ are used as measures for the comparison.

Table 7: Lionel Messi's Goal Contribution Statistics

Season	Goal	Goal Assist
	Laliga	
2004-2005	1	-

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2005-2006	6	3
2006-2007	14	3
2007-2008	10	12
2008-2009	23	11
2009-2010	34	9
2010-2011	31	19
2011-2012	50	16
2012-2013	46	11
2013-2014	28	11
2014-2015	43	18
2015-2016	26	14
2016-2017	34	9
2017-2018	37	12
2018-2019	36	13
2019-2020	25	21
2020-2021	30	9
2021-2022	6	14
2022-2023	7	10
	Copa Del Rey	
2004-2005	-	-
2005-2006	1	1
2006-2007	2	1
2007-2008	-	-
2008-2009	6	1
2009-2010	3	-
2010-2011	10	2
2011-2012	6	6
2012-2013	6	1
2013-2014	5	3
2014-2015	5	4
2015-2016	6	6
2016-2017	6	6
2017-2018	5	4
2018-2019	3	5
2019-2020	3	1
2020-2021	3	1
2021-2022	-	-
2022-2023	1	-
	Champions League	
2004-2005	-	-
2005-2006	1	2
2006-2007	1	-
2007-2008	6	2
2008-2009	9	5

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 7, Issue 4, 2024 (pp. 272-295)
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2009-2010	8	-
2010-2011	12	4
2011-2012	14	5
2012-2013	8	3
2013-2014	8	-
2014-2015	10	6
2015-2016	6	1
2016-2017	11	2
2017-2018	6	2
2018-2019	12	3
2019-2020	3	4
2020-2021	5	2
2021-2022	5	-
2022-2023	4	4
	UEFA Super Cup	
2006-2007	-	-
2009-2010	-	1
2011-2012	1	1
2015-2016	2	1
	Club World Cup	
2009-2010	2	-
2011-2012	2	1
2015-2016	1	-

Data Set: The detailed data set in Table (7) as summarized here depicts Lionel Messi's club *goals* and *goal assists* respectively, prior to the landmark 2022 world cup championship (Goal.com, 2023).

Data 1 (Goals)

1 6 14 10 23 34 31 50 46 28 43 26 37 34 36 25 30 6 7 1 2
6 10 6 6 5 5 6 6 5 3 3 3 1 1 1 6 9 8 12 14 8
8 10 6 11 6 12 3 5 5 4 1 2 2 2 1

Data 2 (Goal Assists)

3 3 12 11 9 19 16 11 11 18 14 9 12 13 21 9 14 10 1 1 1 2
6 1 3 4 6 6 4 5 1 1 2 2 5 3 6 1 2 2 3 4 2 4
1 1 1

Table 8: **Application of the Sky-Log distribution to Data 1 (Goal Scored)**

Model	**Parameter**	**Std. Error**	$-2\ log\ log\ (L)$	**AIC**	**BIC**
Rayleigh	$a = -12.53$	0.8298	-235.230	472.461	474.50
Fretchet	$a = 3.9270$	0.5869	-200.595	405.191	409.277

Article DOI: 10.52589/AJMSS-ZXSCST58
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	$b = 0.9399$	0.0931			
Gumbel Type2	$a = 0.3409$ $b = 0.0823$	0.0451 0.0109	-427.550	859.100	863.19
Logistic	$a = 9.6146$ $b = 6.7346$	1.5299 0.7868	-224.714	453.427	457.513
Sky-Log	$a = -0.693$ $b = 1.8427$	3.2837 6.0516	-199.303	402.605	406.691
Sky-X	$a = -0.0824$	0.01091	-199.303	400.605	402.65

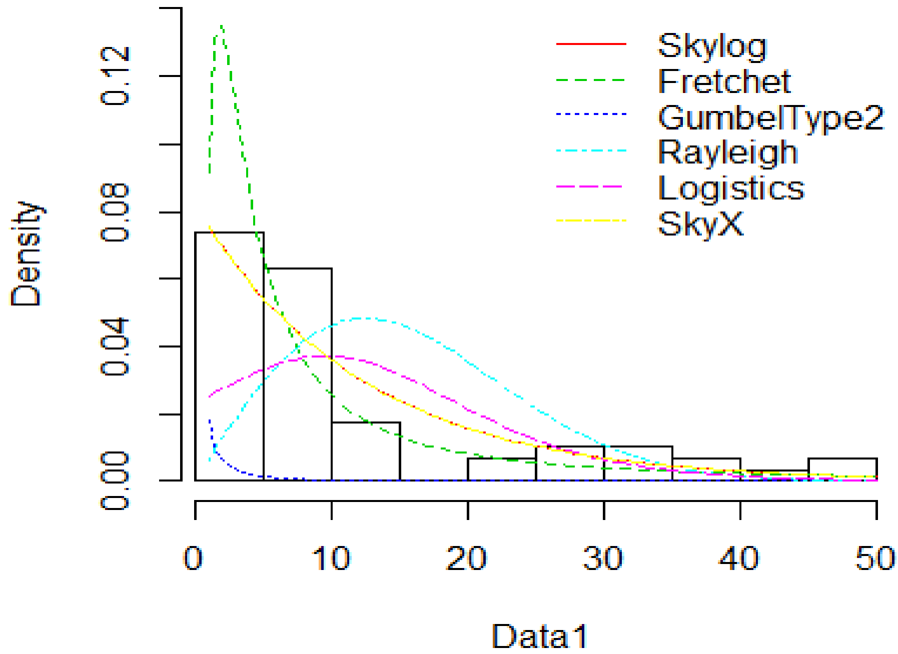

Figure 8: The plot of the histogram and densities of the distribution using Data 1

Table 9: Application of the Sky-Log distribution to Data 2 (Goal Assists)

Model	**Parameter**	**Std. Error**	-2 $log\ log\ (L)$	**AIC**	**BIC**
Rayleigh	$a = 5.9017$	0.4304	-147.73	297.473	299.323
Fretchet	$a = 2.4954$ $b = 1.1151$	0.3458 0.1263	-133.83	271.673	275.37
Gumbel Type2	$a = 0.4153$ $b = 0.1589$	0.0606 0.0232	-287.913	579.825	583.53
Logistics	$a = 5.5721$ $b = 3.1187$	0.8025 0.3782	-146.987	297.975	301.675
Sky-Log	$a = -0.8096$ $b = 1.9171$	4.7553 9.1166	-133.49	270.97	274.68
Sky-X	$a = -0.1588$	0.02316	-133.49	268.98	270.83

Article DOI: 10.52589/AJMSS-ZXSCST58
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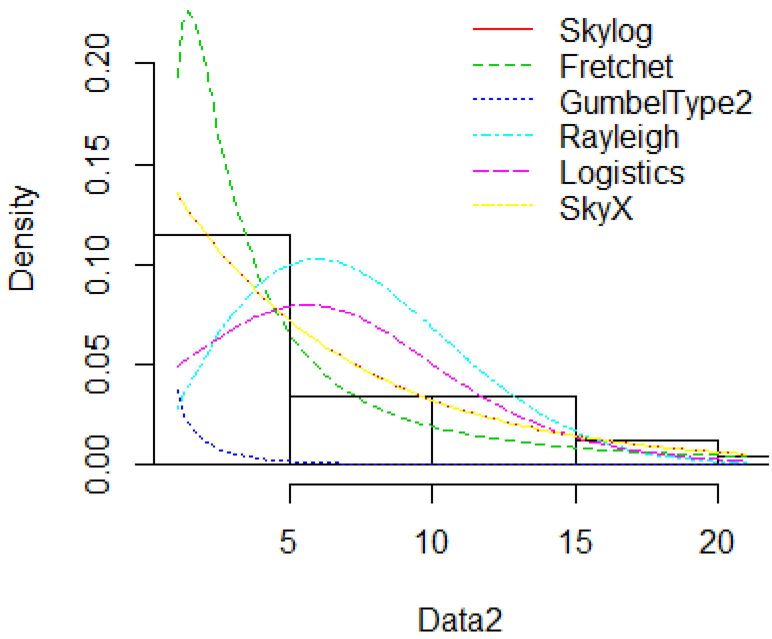

Figure 9: The plot of the histogram and densities of the distribution using Data 2

Tables 8 and 9 clearly reveals that the Sky-Log distribution and its sub-model have the least value of the Log-likelihood (-2logL), AIC and BIC Statistics, an indication that they are more superior in the modeling and fitting of the datasets considered when compared to the other distributions investigated. This claim is further buttressed by the plots in Figures 8 and 9.

Table 10: Summary Statistics for the Sky-X and Exponential distribution

Model	Parameter	Std. Error	$-2 \log\log(L)$	AIC	BIC
			Data 1		
Exp	$a = 0.0824$	0.01091	-199.30	400.61	402.65
Sky-X	$a = -0.0824$	0.01091	-199.30	400.61	402.65
			Data 2		
Exp	$a = 0.1588$	0.02316	-133.49	268.98	270.83
Sky-X	$a = -0.1588$	0.02316	-133.49	268.98	270.83

Remark: The results in Table 10 clearly show the very close relationship between the Sky-X distribution and the exponential distribution in fitting a real lifetime dataset. This is an indication that the Sky-X distribution can be used as an alternative to the exponential distribution in modeling and fitting lifetime datasets.

Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58

DISCUSSIONS OF RESULTS

The record is set straight for the theoretical method used in the derivation of most classical distributions, which is scarce in the literature, and a new distribution has been developed employing the technique as well. More so, the properties of the proposed distribution have been derived which include the mean, mode, median, moments, survival, hazard and cumulative hazard functions, mean time to failure, mean residual function, parameter estimation, stress strength reliability, order statistics, convolution of sums and ratio, inverse cumulative function, and simulation study. Unlike many other distributions, the different generating functions like moment generating function, characteristic function, factorial moment generating function and central moment generating function exist for the new distribution. The flexibility of the distribution was not just seen in its tractability, but in lack of over-dispersion and least variance, as it was comparatively examined using the VMR and CV.

CONCLUSION

Unprecedentedly, Lionel Messi's football record data is fitted to the proposed distribution and its sub-model, in comparison with other classical distributions, and they showed to be comparatively better. However, this model fitness is only in respect of the data employed in this study. Finally, the two sub-models which share structural resemblance also showed to be exact alternatives for data modeling.

CONFLICT OF INTEREST

None

FUNDING

None

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Article DOI: 10.52589/AJMSS-ZXSCST58
DOI URL: https://doi.org/10.52589/AJMSS-ZXSCST58