DEGREE OF HOMOGENEITY OF FINITE TRANSITIVE EXTENSION PERMUTATION GROUPS WITH EMPHASIS ON SOCLE

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ABSTRACT: *Finite transitive groups could be extended through points, a line or a block; as such, the geometric constructions were motivated through the concept of the Steiner system.The transitive extension of the finite groups indicated a degree of homogeneity either by the partitioning of the groups or observing the orbits. We employed a different approach to the previous work carried out. The procedure for the determination of the degree of homogeneity considered here was based on the socle of the groups. The results indicated that the stabilizer of the transitively extended groups of finite primitive groups were mostly the actual groups before the extension. Further investigation revealed that the transitive extension of homogeneous groups were also homogeneous, and so it paved the way for the analytic approach in the determination of the degree of homogeneity much easier.*

KEYWORDS: Transitive extension, degree of homogeneity, socle of a permutation group.

INTRODUCTION

The development of the transitive extension of a group was as a result of the work of [10], as he was the first to mention the word transitive extension. It was further modified by using the geometry of Witt as described by [4]. This helped in the construction of the Mathieu groups as carried out by [7]. Thus, the transitive extension group has the stabilizer which is just the original group before the extension. The procedure for extending a permutation group followed from [3]. We begin by saying suppose H≤Sym(Ω) is a transitive group of rank r, then any fixed $\alpha \epsilon \Omega$; let us assume that $y_0 = 1, y_1, y_2, \ldots. y_r$ formed the sets of representatives for the double coset (H_α, H_α) in H. Let us choose a point $\mu \notin \Omega$ and set $\Omega^* = \Omega \cup \{\mu\}$. Then, it implies that x$\epsilon sym(\Omega^*)$ with sup(x). The group G=$\prec H, x \succ$ is a transitive extension of H and $x^2 \epsilon H$. Moreover, the double coset decomposition of the stabilizer is an invariant subgroup.

Further, we try to look at the degree of primitivity for finite transitive groups based on the sole of the groups as an object to determine the degree of homogeneity.

Preliminary Result

We begin the statement of basic definitions and theorems which are useful in the quest for obtaining our results.

Definition 2.1

A group G is sharply k-transitive if only the identity element fixes k distinct elements of G. A sharply 1-transitive G-set Ω is regular and G is a regular G-set for any group G.

Definition 2.2

Let | Ω |= n. The rank of G is the number of G_α −orbits in Ω.

Theorem 2.3

Let $G \leq sym(\Omega)$ be k-transitive of degree n. Then, for every k distinct elements, the order of G is n^k.

The next definition is useful in case a group is not primitive, then a block of imprimitivity exists.

Definition 2.4

A block is a subset $\beta \; of$ Ω such that for every $g \in G,$ either

$g\beta = \beta$ or $g\beta \cap \beta = \emptyset$.

Thus, in relation to the above definition, we can assert that a transitive permutation group could be trivial or non trivial, as to the definition that follows:

Definition 2.5

A transitive group is said to be primitive if it contains no nontrivial blocks; otherwise, it is imprimitive.

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The next result is a direct consequences of transitivity and primitivity, and so it follows that:

Theorem 2.6

If G is doubly transitive, then it is primitive. The next definition will aid in the classification of transitive extension groups without necessarily depending much on the usual geometric constructions of these groups.

Definition 2.7

Let G be a group acting on a set Ω and H a group acting on a set Δ. Then G acting on Ω is permutation isomorphic to H on Δ if there is isomorphism.

$\theta: G \rightarrow H$ and a bijection $\beta: \Omega \rightarrow \Delta$ such that for all $g \in G$ and $\alpha \in \Omega$, it follows that

$(\alpha^g)\beta = (\alpha)\beta^{(g)\theta}$. The pair (θ, β) is called a permutation isomorphism.

It follows analytically that stabilizer of these groups provides ease of determining the socle of the groups.

Theorem 2.5

Let G and H be both transitive groups. Then they are permutation isomorphic if and only if there is an isomorphism $\theta: G \rightarrow H$ such that θ maps a point stabilizer of G onto some point stabilizer of H. In particular, if G and H are both subgroups of Sym (Ω) for some set Ω, then G and H are permutation isomorphic if and only if they are conjugate in Sym (Ω).

The next definition is one of the required conditions in the classification scheme for finite primitive groups based on socle type due to [9].

Definition 2.6

A group is said to be almost simple if it has a non abelian simple unique minimal normal subgroup.

It follows that if G has a socle other than H, then, by [8], the core of the subgroup H is subnormal in G. Therefore, we define the core of transitive subgroup as follows:

Definition 2.7

The core of a transitive subgroup H of G is the intersection of all G conjugates in H. This is the largest subgroup of G contained in H and it is denoted by $core(H)$.

Thus, if G is transitive, then H is also transitive.

The next two definitions are paramount to the statement of the first condition for the classification scheme for finite primitive groups based on the socle type due to [8].

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Definition 2.8

Let $g \in G$ *be* such that ρ_g, λ_g and ι_g are permutations of G defined by

$(x)\rho_g = x_g,\ x\lambda_g = g^{-1}x,\ (x)\iota_g = g^{-1}xg = (x)\lambda_g o \rho_g$ for $x \in G. Then$ the right regular representation of G is the subgroup of Sym (G) defined by

$G_R = \{\rho_g : g \in G\}$. Similarly, the left regular representation of G is the subgroup of Sym (Ω) defined by $G_L = \{\lambda_g : g \in G\}$.

Definition 2.9

The set Δ is said to be a minimal block for G if Δ is a block for G containing at least two points and no other block with at least two points is properly contained in Δ.

The idea of partitioning was used in the geometrical approach to the determination of the degree of homogeneity in finite primitive groups. In this situation [6], used the partition and it was further used in the $t -$ orbit homogeneity cited in the work of [2].

Definition 2.10

An ordered set of partition of Ω given as $P = (P_1, P_2, \dots, P_k)$ of pairwise disjoint non empty subsets of Ω whose union $\cup_i^k P_i = \Omega$,where $|P_i| \geq |P_{i+1}|$ for all $1 \leq i \leq k$, is called a partition for P.

In this view, Steiner system $s(t, k, V)$ was employed in the geometric constructions of the Mathieu groups based on the idea of affine and projective general linear groups.

Definition 2.11

A permutation group G on Ω is k-homogeneous for k≥ 1 if G acts k-transitively on k element subsets of Ω.

The next result is due to Livingstone and Wagner (1965) which gives condition for homogeneity in finite permutation groups.

Theorem 2.12

Suppose that the group G is k-homogeneous on a finite set Ω where $2 \leq k \leq |\Omega|/2$. Then G is (k-1)-transitive with the following exception:

i. $k = 2, ALS(1, q) \leq A\sum L(1, q), q \equiv 3(mod4)$.

ii. $k = 3, PSL(2, q) \leq P\sum L(1, q)$,then $q \equiv 3(mod4)$.

iii. $k = 3, G = AGL(1,8), A\Gamma L(1,32)$.

For $k = 4$, either $G = PGL(2,8)$,or$P\Gamma L(2,32)$.

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Definition 2.13

Let G be a permutation group, the kernel of G acting on Ω is the kernel of the permutation representation of the form $G_{(\Omega)} = \{\alpha^g$ where $\alpha \in \Omega\}$.

If the kernel of the action of a group G on a set Ω consists of only the identity element, then G is said to act faithfully on Ω.

Definition 2.14

Let G be a group. The minimal normal subgroup H of G is a normal subgroup such that $H \neq 1$ is a subgroup of G which does not properly contain any other nontrivial normal subgroup of G.

Definition 2.15

The socle of a group G is the subgroup generated by the set of all minimal normal subgroups of the group and it is denoted by soc(G) and called socle type.

Theorem 2.16

Let G be a group which acts primitively and faithfully on Ω with $|\Omega| = n$. Let $H = soc(G)$ and $\alpha \in \Omega$. Then, H is homogeneous of type T and exactly one of the following cases holds:

1. Affine, T is abelian of order p, where $n = p^m$ and G_α is a complement to it which acts on it and is simple.

2. Almost simple, we have $A_\alpha \cap {S_n}^3 \approx {S_1}^{\times 5}$ and $soc(G) = soc(A)$.

3. Diagonal action and either:

(i) m=2 and G acts intransitively on $\{T_{1,} T_2\}$ or

(ii) $m \geq 2$ and G acts primitively on $\{T_{1,} \dots T_m\}$.

In (1) T_1 and T_2, both act regularly. Moreover, the point stabilizer K_α of K is the form $diag(Aut(T)^{\times m}) \cdot S_m \cong Aut(T) \times S_m$ and thus $H_\alpha = diag(T^{\times m})$

4. Product type $m = rs$ with s > 1, we have $G \leq A = S_n w_r S_m$ and the wreath product acts in product action with A acting primitively but not regularly on d points and S_m acting transitively on 5 points. Thus, $n = d^5$. The group S_n is primitive of either:

(i) Type (1) with socle T^2($i.e$ $r = 2,$ $s < m$) or

(ii) Type 3(2) with socle T^2($i.e$ $r > 1,$ $s < m$).

5. Twisted wreath type. It acts regularly and $n = |T|^m$.

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Definition 2.17

A group G acting on set Ω is said to be a Jordan set and its complement $\Delta = \Omega \backslash \Gamma$ iff $|\Gamma| > 1$, and $G_{(\Delta)}$ acts transitively on Γ.

Definition 2.18

A group G acting transitively on the set Ω is a J-flag or Jordan flag if and only if Ω has a finite Jordan complement, $\emptyset = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k$ with the property that whenever $\triangle$ is a complement for Ω such that Δ_i for any $\Delta_{i-1} \subseteq \triangle \subseteq \Delta_i$, $implying$ $\triangle = \Delta_i$.

Theorem 2.19

Let G$\leq Sym(\Omega)$ be a sharply 2-transitive group, and $K = \{x \in G | x = 1$ or $fix(x) = \emptyset\}$. Then,

(i) G has a unique conjugacy class T of elements of order 2, and each point stabilizer G_α contains at most one element of T.

(ii) For some α the elements of K lie in distinct right cosets of G_α, and K is a regular normal abelian subgroup of G.

Proof

(i) Let $\alpha, \beta \in \Omega$. Then there exists an element of order 2 which interchanges α and β. So we have $(\alpha, \beta)^t = (\beta, \alpha)$ which fixes both α and β . Therefore, T has order 2 as required. Suppose $s \in G$such that β is interchanged by s. If we choose $z \in G$ such that $(\alpha, \beta)^Z = (\gamma, \delta)$ and $s^{-1}z^{-1}tz$ fixes both γ and δ, then
$s^{-1}z^{-1}tz = 1$ and so $s = z^{-1}tz \in T$. This shows that T is the unique conjugacy class of elements of order 2 in G.

Now suppose that $s, t \in T$ both lie in G_α and pick $\alpha^t \neq \alpha$. Then $\gamma = \beta^s \neq \beta$ and $\delta = \beta^t \neq B$. If we choose $z \in G$ such that $(\beta, \delta)^Z = (\beta, \gamma)$, then it shows that $s = z^{-1}tz$ since $fix(\delta) = fix(t) = \{\alpha)$. This show that z fixes α as well as β and so $z = 1$. Thus, s=t and G_α does not contain more than one element in T.

(ii) We must show that either K contains an element of T or else $TT \subseteq K$. If K contains no element from T, then K is the unique conjugacy class of T. It implies T has a unique fixed point $\alpha \neq \beta$. Thus, for $s, t \in T$, then $st \in K$. Also, let us assume to the contrary that K have a fixed point γ. Then $s^{-1}(st)s = ts = (st)^{-1}$ and so

$\{\gamma\} = fix(st) = fix((st)^{-1}) = \{\gamma^s\}$.

Thus s fixes γ so $\beta = \gamma$. Hence, $s_1 t \in G_\gamma$ and so s=t by conjugacy $st \notin K$; therefore, for any T, we have that $TT \subseteq K$. In particular, K=1.

Next, suppose that K is a nontrivial subgroup of G, then $K \lhd G$. Since G is k- primitive, then it is necessarily k-transitive and the point stabilizer of G is trivial.

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Finally, if K is abelian and since T has a conjugacy class, it implies that $T \subseteq K$. However, the 2-transitivity of G shows that G_α acts transitively by conjugation on the set K^* of non-identity elements of K. This shows that $K^* = T$.

It remains in a case when $TT \subseteq K$. Then for each $t \in T$, and so $K \subseteq TT$. Suppose that α is a unique fixed point of T for each $x \neq 1$ in K. Let us choose $s \in T$ such that α and α^x are interchanged by s. Then $tsx^{-1} \in K$ and fixes α. So $tsx^{-1} = 1$ and $x = ts \in TT$. Thus, $TT \subseteq K \subseteq TT$, and so $K = TT$, as claimed.

Now fix $t \in T$ and consider the conjugation by t on K. For each $x \in K$ we have $s \in T$ such that $x = ts$ and $t^{-1}xt = t^{-1}tst = st = s^{-1}t^{-1} = x$. Hence, conjugation by t inverts the element of K. In particular, for all $x, y \in K$ we have $xy = t^{-1}(xy)^{-1}t = (t^{-1}y^{-1}t)(t^{-1}x^{-1}t)yx$. Thus, K is abelian as well.

Lemma 2.20

G is sharply n-homogeneous on Ω, for some $n \geq 1$.

Proof

First observe that G cannot be k-homogeneous for all $k \in N$ since, otherwise, there would be an infinite descending chain. $G, G_{x_1}, G_{x_1 x_2}, \ldots$ of subgroups. To prove that H is homogeneous, it suffices to show that if for some k the group G is k-homogeneous but not sharply k-homogeneous on Ω, then G is (k+1)-homogeneous. We prove this by induction on k.

Suppose that G is k-homogeneous but not sharply k-homogeneous. We claim that (G_x, Γ) is not sharply (k-1)-homogeneous. Suppose otherwise and pick $x_1 < \ldots < x_{k-1}$ in Γ. Then $G_{xx,\ldots,x-1}$ fixes $\{2: y < z\}$, so $G_{yx_1 \ldots x_{k-1}} < G_{xx_1 \ldots x_{k-1}}$. Since these groups are conjugates, this contradicts the descending chain condition on the subgroups. It follows by induction that (G_x, Γ) is k-homogeneous and hence (G, Ω) is (k+1)-homogeneous as required.

Lemma 2.21

Suppose that G is a t-transitive Jordan group acting on a set Ω and that it is a transitive extension of H acting on the set $\Omega^* = \Omega \cup \{\omega\}$. Suppose further that H has a proper Jordan complement Δ of size k, where any set of t -points of Ω is contained in exactly λ complements of this size. Let $V = |\Omega|\}$ and $\Gamma = \Delta \cup \{\omega\}$, then:

(i) the set Γ is a Jordan complement for G on Ω^*;

(ii) the group induced on Γ by $G_{\{\Gamma\}}$ is a transitive extension of the group induced on Δ by $H_{\{\Delta\}}$;

(iii) any t+1 points of Ω^* are in exactly λ Jordan complement for G of size k+1;

(iv) the number b of Jordan complements of size k+1 for G is
$b = (v+1)(v)(v+1)\ldots(v-t+1)(k+1)(k)(k-1)\ldots(k-t+1)$.

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Next we show that the socle of primitive $2-$ transitive groups were the stabilizers of the complements

Theorem 2.22

Let G be a group acting primitively on Ω, such that soc(G)=$G_{(\Delta)}$. If G has a finite nonempty Jordan complement, then G is 2-transitive and hence G is 2- homogeneous.

RESULTS

Theorem 3.1

Let H be a permutation group and G a transitive extension of H, then G is k-homogeneous.

Proof

Let $H \leq G$ be a subgroup of Ω that is k-transitive on $\Omega = \Omega' \cup (\Psi\}$ such that $\Omega' \neq \Omega$. Suppose that H = soc(G) and H = G_α. We see that the stabilizer of H is less than or equal to G_α. Thus, $H \leq sym(\Omega)$ and H is k-transitive by Theorem 2.16. If G is homogeneous, then for any $y \in G$, the double coset of $G = G_\alpha y, G_\alpha \cup ... \ G_\alpha y_{r-1} G_\alpha$ has rank r with r orbits. Hence, for any $x \in sym(\Omega')$ which does not fix a point in Ψ, we have that $G =< H, x >$ is k-homogeneous.

Theorem 3.2

Suppose that G is $\lambda -$transitive acting on a Jordan set Ω, and H is a transitive extension of G of size k, then H is homogeneous if and only if G_Δ is a subgroup of $H_{\Delta'}$.

Proof

Since G is $\lambda -$transitive and Ω is the Jordan set then G_Δ is $\lambda - 1$-transitive which also follows 2.12. Suppose that H is a subgroup of G, then the transitive extension of the set Ω is the set $\Omega^* \cup \{\alpha\}$, where |H|= k. Since $H_{\Delta'}$ is λ -1 –transitive by Theorem 2.22 and the condition Theorem 2.16 implies that G is homogeneous of degree at most λ and so $H_{\Delta'} \geq G_\Delta$. Thus, we have that H is k-homogeneous.

Next, let G_Δ be a subgroup of $H_{\Delta'}$. In that case any x, y$\in G_\Delta$ implies that $xy^{-1} \epsilon G_\Delta$. Therefore $\lambda - 1$ –transitivity of H imply $\lambda - 1$-homogeneity.

From the results above, it shows that every Jordan group is homogeneous since the necessary and sufficient condition is that the stabilizer of the group be transitive.

Theorem 3.3

Let G be a non-trivial extension of H$\leq Sym(\Omega)$. If |H|=k, such that H is k-homogenous, then G is at least k-homogeneous.

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Proof

If H is a subgroup of G and H is k-transitive, then it follows that from Theorem 2.12 and Theorem 2.12, G is k-transitive and G_α is k-1 transitive. By Theorem 2.16, G is k+1 transitive.

Therefore, we can investigate the level of homogeneity via an extension of a Jordan groups. Thus it follows

Theorem 3.4

Let H $\leq Sym(\Omega)$ be a Jordan group with $H \leq G$. Then G is also a Jordan group and is homogeneous.

Proof

Let H be a Jordan group and $\Delta = \Omega \backslash \Gamma$ be its complement. Then it follows from Theorem 2.14 that G is also a Jordan group. If |G|=k and k$\geq$ 1, then by Lemma 2.16, G is k= homogeneous and faithful. Hence, it implies that G is sharply k-homogeneous.

Thus, we can assert that the extension of a Jordan group give rise to another Jordan group. In further investigation, it shows that the order of G is very useful in obtaining the level and degree of homogeneity.

CONCLUSION

The notion of transitive extension of transitive groups gave rise to the suitable procedure for the construction of Mathieu groups. This led to the emergence of sporadic groups. Most of the groups fall under the classification scheme due to Jordan. It followed that these groups were homogeneous. The degree of homogeneity of the transitive extension was determined based on the socle of the groups. The stabilizer of the groups under extension were just the groups before extension due to the fact that these groups considered were finite and primitive.

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