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A BLOCK HYBRID METHOD FOR THE DIRECT SOLUTION OF $y'' = f(x, y, y')$

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ABSTRACT: *This paper introduces a Block Hybrid Method for solving general second-order ordinary differential equations (ODEs) with initial value problems. The method is based on a continuous formulation of the second-order hybrid generalized Adams method, incorporating one off-grid point per step. Discrete schemes are derived from the continuous form and its first derivative, forming the block methods. Analysis shows the method is consistent, zero-stable, and convergent. Numerical results highlight its superiority over existing methods.*

KEYWORDS: Block Hybrid Method, Consistency, Zero-stability, Convergent, and Second Derivative.

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INTRODUCTION

Ordinary differential equations (ODEs) with second-order initial value problems (IVPs) are crucial for simulating dynamic systems in a variety of domains, including fluid flow, electrical circuits, and mechanical oscillations. These problems are mathematically formulated as:

$$y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y_1. \quad (1)$$

Most real-world systems are quite complex; finding analytical solutions to such equations is frequently impossible. Numerical approaches have become indispensable for approximating the solutions to these equations. Conventionally, second-order ordinary differential equations are resolved by transforming them into systems of first-order equations, which are subsequently tackled using numerical methods tailored for first-order systems.

However, this approach increases computational complexity and can introduce numerical instability (Awoyemi et al., 2015).

To overcome these challenges, recent advancements in numerical analysis have focused on direct methods that solve second-order ODEs without reduction. Familua and Omole (2017) introduced a five-point mono-hybrid point method to address the inefficiencies of traditional methods. Jator and Sennah (2018) proposed continuous collocation methods, which offer improved accuracy and computational efficiency. These methods use collocation and interpolation techniques to approximate solutions at multiple points within a block.

Block hybrid methods, such as those presented by Li, Wang, and Lu (2018), are particularly advantageous as they avoid the overlapping computations associated with predictor-corrector methods. The ability to solve for multiple points simultaneously reduces computational costs and enhances numerical stability. Olakiitan, Uwaheren and Obarhua (2017) demonstrated the effectiveness of Taylor series-based algorithms in achieving higher-order accuracy for second-order IVPs.

This study builds on these recent developments by presenting a new block hybrid method derived using power series functions as the basis. Tunde and Temilade (2019) showed that hybrid block methods could efficiently address second-order problems without requiring reduction, significantly improving solution accuracy. Similarly, Jator and Usoro (2020) highlighted the benefits of block methods in solving second-order ODEs directly, emphasizing their computational efficiency and accuracy.

Definition 1

A method (2) is said to be zero stable if no root of the first characteristics polynomial

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j \quad (2)$$

has modulus greater than one and every root with one is simple.

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METHODOLOGY

We first state the theorem that establishes the uniqueness of solutions of higher order ordinary differential equations.

Theorem 2.1

Given the general nth order initial value problem

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}) \qquad y^{(n)}(x_0) = m_k \qquad (3)$$

Let R be the region defined by the inequality $x_0 \le x \le x_0 + a, \ |s_t - c_t| \le b_t, \ t = 0,1,2,...,(n-1)$, where $c_t \ge 0$ *for* $k > 0$.

Suppose the function $f(x, s_0, s_1, ..., s_{n-1})$ *in (2.1) is*

- *non negative, continuous and non decreasing in each* x, $s_0, s_1, ..., s_{n-1}$ *in R;*
- $f(x, m_0, m_1, ..., m_{n-1}) > 0$ *for* $x_0 \le x \le x_0 + a$; *and*
- $m_k \ge 0, \ k = 0,1,2,...,n-1$ *then, the initial value problem (3) has a unique solution in R.*

Theorem 2.2

Let I be an identity matrix of dimension $(m+t)\times(m+t)$ and consider the matrices C and D. Then from the method of matrix inversion of Sirisena et al (1997), we have

(i) $DC = I$ (4)

(ii) $$\bar{y}(x) = \sum_{i=0}^{t+m-1} x^i \left(\sum_{j=0}^{t-1} C_{i+1,y+1} y_{n+j} + \sum_{j=0}^{m-1} C_{i+1,j+1} f_{n+j}\right) \qquad (5)$$

With matrix D and C defined as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & . & . & . & . & . & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & . & . & . & . & . & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & . & . & . & . & . & (t+m-1)(t+m-2)x_n^{t+m-1} \\ 0 & 0 & 2 & . & . & . & . & . & (t+m-1)(t+m-2)x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \qquad (6)$$

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$$C = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} & \cdots & \varphi_{t-1,1} & h^2\psi_{0,1} & . & . & h^2\psi_{m-1,1} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} & \cdots & \varphi_{t-2,1} & h^2\psi_{0,2} & . & . & h^2\psi_{m-2,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (7)$$

The columns of C which gives the continuous coefficients $\alpha_j(x)$, $j = 0,1,\ldots,t-1$ and $\beta_j(x)$, $j = 0,1,\ldots,m-1$ can be obtained from the corresponding columns of D^{-1}; thus, we have that (5) becomes

$$y(x) = (y_n,\ldots, y_{n+t-1}, f_n,\ldots, f_{n+m-1})C^T(1, x,\ldots, x^{t+m-1})^T \quad (8)$$

where T denotes Transpose

Modifying the matrix inversion approach as used by Sirisena et al. (1997) as illustrated in Theorem (2.2), we obtain the continuous form of the discrete methods as for $k = 3$

$$y(x) = \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_1(x)y_{n+1} + h^2\left(\beta_0(x)f_n + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}\right) \quad (9)$$

where $g(x) = y'(x), \alpha_v(x),\ \alpha_{v-1}(x)$, $\beta_j(x)$ are continuous coefficients of the methods to be determined for $k = 3$ using (6) and (9); we obtain D as

$$D = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & 12x^2_{n+2} & 20x^3_{n+2} & 30x^4_{n+2} \\ 0 & 0 & 2 & 6x_{n+3} & 12x^2_{n+3} & 20x^3_{n+3} & 30x^4_{n+3} \end{bmatrix}$$

Using Maple 18, the inverse of the D matrix $C = D^{-1}$ is computed and the entries of the inverse matrix are obtained.

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By using Maple 18, each column of the inverse matrix obtained above is being multiplied by the row matrix $\left[\ 1\ x\ x^2\ x^3\ x^4\ x^5\ x^6\ \right]$; we obtain the continuous coefficients of (9) and with some manipulation we obtain the continuous formulation of the form

$$y(x)=\left(2+\frac{2\zeta}{h}\right)y_{n+\frac{1}{2}}+\left(-1+\frac{2\zeta}{h}\right)y_{n+1}+\left(\frac{13}{640}h^2+\left(\frac{997}{5760}\zeta\right)h+\frac{1}{2}\zeta^2-\frac{23}{36}\frac{\zeta^3}{h}+\frac{7}{18}\frac{\zeta^4}{h^2}-\frac{13}{120}\frac{\zeta^5}{h^3}+\frac{1}{90}\frac{\zeta^6}{h^4}\right)f_n+$$
$$\left(\frac{21}{100}h^2+\left(\frac{7}{12}\zeta\right)h+\frac{16}{15}\frac{\zeta^3}{h}-\frac{44}{45}\frac{\zeta^4}{h^2}+\frac{8}{25}\frac{\zeta^5}{h^3}-\frac{8}{225}\frac{\zeta^6}{h^4}\right)f_{n+\frac{1}{2}}+\left(\frac{37}{1920}h^2+\left(\frac{72}{1920}\zeta\right)h-\frac{1}{2}\frac{\zeta^3}{h}+\frac{17}{24}\frac{\zeta^4}{h^2}-\frac{11}{40}\frac{\zeta^5}{h^3}+\frac{1}{30}\frac{\zeta^6}{h^4}\right)f_{n+1}+$$
$$\left(\frac{1}{1920}h^2+\left(\frac{17}{1920}\zeta\right)h+\frac{1}{12}\frac{\zeta^3}{h}-\frac{5}{36}\frac{\zeta^4}{h^2}+\frac{3}{40}\frac{\zeta^5}{h^3}-\frac{1}{90}\frac{\zeta^6}{h^4}\right)f_{n+2}+\left(-\frac{1}{9600}h^2+\left(\frac{7}{5760}\zeta\right)h-\frac{1}{90}\frac{\zeta^3}{h}+\frac{7}{360}\frac{\zeta^4}{h^2}-\frac{7}{600}\frac{\zeta^5}{h^3}+\frac{1}{450}\frac{\zeta^6}{h^4}\right)f_{n+3} \quad (10)$$

where $\zeta = x - x_n$

Evaluating (10) at the following points $\zeta = 0, 2h, 3h$, and also differentiating (10) to obtain

$$g(x)=y'(x)=\left(\zeta-\frac{23}{12}\frac{\zeta^2}{h}+\frac{14}{9}\frac{\zeta^3}{h^2}-\frac{13}{24}\frac{\zeta^4}{h^3}+\frac{1}{5}\frac{\zeta^5}{h^4}\right)f_n+\left(\frac{16}{5}\frac{\zeta^2}{h}-\frac{176}{45}\frac{\zeta^3}{h^2}+\frac{8}{5}\frac{\zeta^4}{h^3}-\frac{16}{75}\frac{\zeta^5}{h^4}\right)f_{n+\frac{1}{2}}+$$
$$\left(-\frac{3}{2}\frac{\zeta^2}{h}+\frac{17}{6}\frac{\zeta^3}{h^2}-\frac{11}{8}\frac{\zeta^4}{h^3}+\frac{1}{5}\frac{\zeta^5}{h^4}\right)f_{n+1}+\left(\frac{1}{4}\frac{\zeta^2}{h}-\frac{5}{9}\frac{\zeta^3}{h^2}+\frac{3}{8}\frac{\zeta^4}{h^3}-\frac{1}{15}\frac{\zeta^5}{h^4}\right)f_{n+2}+$$
$$\left(-\frac{1}{30}\frac{\zeta^2}{h}+\frac{7}{90}\frac{\zeta^3}{h^2}-\frac{7}{120}\frac{\zeta^4}{h^3}+\frac{1}{75}\frac{\zeta^5}{h^4}\right)f_{n+3} \quad (11)$$

and evaluating (11) also at the following points $\zeta = 0, \frac{1}{2}h, h, 2h, 3h$ and manipulating the schemes obtained from evaluating (10) and (11), we obtain the following discrete schemes:

$$g_{n+\frac{1}{2}} = g_n + \frac{1057}{5760}hf_n + \frac{91}{225}hf_{n+\frac{1}{2}} - \frac{193}{1920}hf_{n+1} + \frac{83}{5760}hf_{n+2} - \frac{53}{28800}hf_{n+3}$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hg_n + \frac{763}{11520}h^2f_n + \frac{49}{600}h^2f_{n+\frac{1}{2}} - \frac{101}{3840}h^2f_{n+1} + \frac{1}{256}h^2f_{n+2} - \frac{29}{57600}h^2f_{n+3}$$

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$$g_{n+1} = g_n + \frac{59}{360} hf_n + \frac{152}{225} hf_{n+\frac{1}{2}} + \frac{19}{120} hf_{n+1} + \frac{1}{360} hf_{n+2} - \frac{1}{1800} hf_{n+3} \quad \textbf{(12)}$$

$$y_{n+1} = hg_n + y_n + \frac{11}{72} h^2 f_n + \frac{28}{75} h^2 f_{n+\frac{1}{2}} - \frac{1}{30} h^2 f_{n+1} + \frac{1}{120} h^2 f_{n+2} - \frac{1}{900} h^2 f_{n+3}$$

$$g_{n+2} = g_n + \frac{11}{45} hf_n + \frac{64}{225} hf_{n+\frac{1}{2}} + \frac{16}{15} hf_{n+1} + \frac{19}{45} hf_{n+2} - \frac{4}{225} hf_{n+3}$$

$$y_{n+2} = 2hg_n + y_n + \frac{16}{45} h^2 f_n + \frac{64}{75} h^2 f_{n+\frac{1}{2}} + \frac{2}{3} h^2 f_{n+1} + \frac{2}{15} h^2 f_{n+2} - \frac{2}{225} h^2 f_{n+3}$$

$$g_{n+3} = g_n + \frac{3}{40} hf_n + \frac{24}{25} hf_{n+\frac{1}{2}} + \frac{9}{40} hf_{n+1} + \frac{57}{40} hf_{n+2} + \frac{63}{200} hf_{n+3}$$

$$y_{n+3} = 3h + y_n + \frac{17}{56} h^2 f_{n+\frac{1}{2}} + \frac{13}{45} h^2 f_{n+1} + \frac{9}{80} h^2 f_{n+2} + \frac{4}{75} h^2 f_{n+3}$$

Modifying the matrix inversion approach as used by Sirisena (1997) for $k=4$, we obtain the continuous form of the discrete methods as

$$y(x) = \alpha_{\frac{1}{2}}(x) y_{n+\frac{1}{2}} + \alpha_1(x) y_{n+1} + h^2\left(\beta_0(x) f_n + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2} + \beta_3(x) f_{n+3} + \beta_4(x) f_{n+4}\right) \quad (13)$$

where $g(x) = y'(x), \alpha_v(x), \ \alpha_{v-1}(x)$, $\beta_j(x)$ are continuous coefficients of the methods to be determined for $k=4$; using (6) and (13) we obtain D as

$$D = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & x^7_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} & x^7_{n+1} \\ 0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^5_n \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & 12x^2_{n+2} & 20x^3_{n+2} & 30x^4_{n+2} & 42x^5_{n+2} \\ 0 & 0 & 2 & 6x_{n+3} & 12x^2_{n+3} & 20x^3_{n+3} & 30x^4_{n+3} & 42x^5_{n+3} \\ 0 & 0 & 2 & 6x_{n+4} & 12x^2_{n+4} & 20x^3_{n+4} & 30x^4_{n+4} & 42x^5_{n+4} \end{bmatrix}$$

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Using Maple 18, the inverse of the D matrix $C = \mathrm{D}^{-1}$ is computed and the entries of the inverse matrix are obtained.

By using Maple 18, each column of the inverse matrix obtained above is being multiplied by the row matrix $\left[1, x, x^2, x^3, x^4, x^5, x^6, x^7\right]$ which gives the continuous coefficients of (13) and, with some manipulation, we obtain the continuous formulation of the form

$$
\begin{aligned}
y(x) = {} & \left(2 + \frac{2\zeta}{h}\right) y_{n+\frac{1}{2}} + \left(-1 + \frac{2\zeta}{h}\right) y_{n+1} + \left(\frac{19}{960} h^2 + \left(\frac{3397}{20160}\zeta\right) h + \frac{1}{2}\zeta^2 - \frac{49}{72}\frac{\zeta^3}{h} + \frac{15}{32}\frac{\zeta^4}{h^2} - \frac{1}{6}\frac{\zeta^5}{h^3} + \frac{7}{240}\frac{\zeta^6}{h^4} - \frac{1}{504}\frac{\zeta^7}{h^5}\right) f_n + \\
& \left(\frac{89}{420} h^2 + \left(\frac{5293}{8820}\zeta\right) h + \frac{128}{105}\frac{\zeta^3}{h} - \frac{80}{63}\frac{\zeta^4}{h^2} + \frac{18}{15}\frac{\zeta^5}{h^3} - \frac{32}{315}\frac{\zeta^6}{h^4} + \frac{16}{2205}\frac{\zeta^7}{h^5}\right) f_{n+\frac{1}{2}} + \left(\frac{11}{640} h^2 + \left(\frac{1307}{40320}\zeta\right) h - \frac{2}{3}\frac{\zeta^3}{h} + \frac{317}{36}\frac{\zeta^4}{h^2} - \frac{61}{120}\frac{\zeta^5}{h^3} + \frac{19}{180}\frac{\zeta^6}{h^4} - \frac{1}{126}\frac{\zeta^7}{h^5}\right) f_{n+1} + \\
& \left(\frac{1}{640} h^2 + \left(\frac{197}{40320}\zeta\right) h + \frac{1}{6}\frac{\zeta^3}{h} - \frac{43}{144}\frac{\zeta^4}{h^2} + \frac{23}{120}\frac{\zeta^5}{h^3} - \frac{17}{360}\frac{\zeta^6}{h^4} + \frac{1}{252}\frac{\zeta^7}{h^5}\right) f_{n+2} + \left(-\frac{1}{1920} h^2 + \left(\frac{197}{40320}\zeta\right) h - \frac{2}{45}\frac{\zeta^3}{h} + \frac{1}{12}\frac{\zeta^4}{h^2} - \frac{7}{120}\frac{\zeta^5}{h^3} + \frac{1}{60}\frac{\zeta^6}{h^4} - \frac{1}{630}\frac{\zeta^7}{h^5}\right) f_{n+3} \\
& + \left(\frac{1}{13440} h^2 + \left(\frac{37}{56448}\zeta\right) h + \frac{1}{168}\frac{\zeta^3}{h} - \frac{23}{2016}\frac{\zeta^4}{h^2} + \frac{1}{120}\frac{\zeta^5}{h^3} - \frac{31}{5040}\frac{\zeta^6}{h^4} + \frac{1}{3528}\frac{\zeta^7}{h^5}\right) f_{n+4}
\end{aligned}
$$

(15)

where $\zeta = x - x_n$

Evaluating (15) at the following points $\zeta = 0, 2h, 3h, 4h$ and also differentiating (15) to obtain $g(x) = y'(x)$ as

$$
\begin{aligned}
y'(x) = {} & \left(\zeta - \frac{49}{24}\frac{\zeta^2}{h} + \frac{15}{8}\frac{\zeta^3}{h^2} - \frac{5}{6}\frac{\zeta^4}{h^3} + \frac{7}{40}\frac{\zeta^5}{h^4} - \frac{1}{72}\frac{\zeta^6}{h^5}\right) f_n + \left(\frac{128}{35}\frac{\zeta^2}{h} - \frac{320}{63}\frac{\zeta^3}{h^2} + \frac{8}{3}\frac{\zeta^4}{h^3} - \frac{64}{105}\frac{\zeta^5}{h^4} + \frac{16}{315}\frac{\zeta^6}{h^5}\right) f_{n+\frac{1}{2}} + \\
& \left(-2\frac{\zeta^2}{h} + \frac{37}{9}\frac{\zeta^3}{h^2} - \frac{61}{24}\frac{\zeta^4}{h^3} + \frac{19}{30}\frac{\zeta^5}{h^4} - \frac{1}{18}\frac{\zeta^6}{h^5}\right) f_{n+1} + \left(\frac{1}{2}\frac{\zeta^2}{h} - \frac{43}{36}\frac{\zeta^3}{h^2} + \frac{23}{24}\frac{\zeta^4}{h^3} - \frac{17}{60}\frac{\zeta^5}{h^4} + \frac{1}{36}\frac{\zeta^6}{h^5}\right) f_{n+2} \\
& + \left(-\frac{2}{15}\frac{\zeta^2}{h} + \frac{1}{3}\frac{\zeta^3}{h^2} - \frac{7}{24}\frac{\zeta^4}{h^3} + \frac{1}{10}\frac{\zeta^5}{h^4} - \frac{1}{90}\frac{\zeta^6}{h^5}\right) f_{n+3} + \left(\frac{1}{56}\frac{\zeta^2}{h} - \frac{23}{504}\frac{\zeta^3}{h^2} + \frac{1}{24}\frac{\zeta^4}{h^3} - \frac{13}{840}\frac{\zeta^5}{h^4} - \frac{1}{504}\frac{\zeta^6}{h^5}\right) f_{n+4} \qquad (16)
\end{aligned}
$$

And evaluating (16) also at the following points $\zeta = 0, \frac{1}{2}h, h, 2h, 3h, 4h$ and manipulating the schemes obtained after evaluating (15) and (16), we obtain the following discrete schemes:

$$
\begin{aligned}
g_{n+\frac{1}{2}} = {} & g_n + \frac{4081}{23040} h f_n + \frac{77}{180} h f_{n+\frac{1}{2}} - \frac{121}{960} h f_{n+1} + \frac{313}{11520} h f_{n+2} - \frac{1}{144} h f_{n+3} \\
& + \frac{7}{7680} h f_{n+4}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} = {} & y_n + \frac{1}{2} g_n h + \frac{2599}{40320} h^2 f_n + \frac{311}{3528} h^2 f_{n+\frac{1}{2}} - \frac{2693}{80640} h^2 f_{n+1} \\
& + \frac{601}{80640} h^2 f_{n+2} - \frac{31}{16128} h^2 f_{n+3} + \frac{143}{564480} h^2 f_{n+4}
\end{aligned}
$$

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$$g_{n+1} = g_n + \frac{29}{180} hf_n + \frac{24}{35} hf_{n+\frac{1}{2}} + \frac{53}{360} hf_{n+1} + \frac{1}{120} hf_{n+2} - \frac{1}{360} hf_{n+3} + \frac{1}{2520} hf_{n+4}$$

$$y_{n+1} = g_n h + y_n + \frac{1499}{10080} h^2 f_n + \frac{856}{2205} h^2 f_{n+\frac{1}{2}} - \frac{25}{504} h^2 f_{n+1} + \frac{83}{5040} h^2 f_{n+2} - \frac{11}{2520} h^2 f_{n+3} + \frac{41}{70560} h^2 f_{n+4}$$

(17)

$$g_{n+2} = g_n + \frac{19}{90} hf_n + \frac{128}{315} hf_{n+\frac{1}{2}} + \frac{14}{15} hf_{n+1} + \frac{22}{45} hf_{n+2} - \frac{2}{45} hf_{n+3} + \frac{1}{210} hf_{n+4}$$

$$y_{n+2} = 2g_n h + y_n + \frac{211}{630} h^2 f_n + \frac{2048}{2205} h^2 f_{n+\frac{1}{2}} + \frac{184}{315} h^2 f_{n+1} + \frac{11}{63} h^2 f_{n+2} - \frac{8}{315} h^2 f_{n+3} + \frac{13}{4410} h^2 f_{n+4}$$

$$g_{n+3} = g_n + \frac{3}{20} hf_n + \frac{24}{35} hf_{n+\frac{1}{2}} + \frac{21}{40} hf_{n+1} + \frac{51}{40} hf_{n+2} + \frac{3}{8} hf_{n+3} - \frac{3}{280} hf_{n+4}$$

$$y_{n+3} = 3g_n h + y_n + \frac{579}{1120} h^2 f_n + \frac{72}{49} h^2 f_{n+\frac{1}{2}} + \frac{369}{280} h^2 f_{n+1} + \frac{639}{560} h^2 f_{n+2} + \frac{3}{56} h^2 f_{n+3} + \frac{9}{7840} h^2 f_{n+4}$$

$$g_{n+4} = g_n + \frac{14}{45} hf_n + \frac{64}{45} hf_{n+1} + \frac{8}{15} hf_{n+2} + \frac{64}{45} hf_{n+3} + \frac{14}{45} hf_{n+4}$$

$$y_{n+4} = 4g_n h + y_n + \frac{232}{315} h^2 f_n + \frac{4096}{2205} h^2 f_{n+\frac{1}{2}} + \frac{704}{315} h^2 f_{n+1} + \frac{656}{315} h^2 f_{n+2} + \frac{64}{63} h^2 f_{n+3} + \frac{32}{441} h^2 f_{n+4}$$

Equations (12) and (17) are our eight (8) and ten (10) block schemes which are of order

$[6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6]^T$ and $[7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7 \quad 7]^T$

with error constants of

$$\left[\frac{49}{76800}, \frac{143}{806400}, \frac{1}{3600}, \frac{41}{100800}, \frac{1}{300}, \frac{13}{6300}, -\frac{3}{400}, \frac{9}{11200} \right]^T$$

and

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$$\left[\frac{-311}{774144},-\frac{3497}{30965760},-\frac{5}{24192},-\frac{1}{3780},-\frac{11}{7560},-\frac{1}{945},\frac{1}{896},-\frac{3}{2240},-\frac{8}{945},-\frac{4}{945}\right]^T$$

respectively.

RESULT AND DISCUSSION

A block method is said to be zero stable if as $h \to 0$, the roots $r_j = 1, \text{ j} = 1,\dots,k$ which means that

$\rho(r) = \det[\sum A^{(i)} R^{k-1}] = 0$ satisfying $|R| \le 1$ must have multiplicity equal to unity (Fatunla S.O., 1991).

For our method for k=3

$$\rho(\lambda) = \det(\lambda A - B) = 0 \tag{18}$$

$$\rho(\lambda) = Det\begin{bmatrix} \lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & \lambda & -1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & \lambda-1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-1 \end{bmatrix} = 0$$

The characteristics polynomial yield

$$(\lambda - 1)^2 \lambda^6 = 0$$

which yields the roots $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$ which are less than or equal to 1, i.e., $\lambda \le 1$

Therefore, the method is zero stable.

Similarly, the block method for k=4 is given by

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$$\rho(\lambda)= Det\begin{bmatrix} \lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & \lambda & -1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & \lambda-1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-1 \end{bmatrix} = 0$$

The characteristics polynomial yield

$$(\lambda-1)^2\lambda^8 = 0$$

which yields the roots $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 0$ which are less than or equal to 1, i.e., $\lambda \leq 1$

Therefore, the method is zero stable.

IMPLEMENTATION

We shall use the discrete schemes of the methods (12) and (17) in block forms to solve initial value problems. The results obtained (numerical or approximate solution) shall be carefully examined by computing the absolute error of the methods. We shall derive the solution directly without reducing to a system of first order equations and the solution is obtained at once.

Numerical Implementation

Problem 1

$$y_1'' = y_1 + \frac{1}{1000}\cos x$$
$$y_2'' = y_2 + \frac{1}{1000}\sin x$$

$$h = \frac{1}{10} = 0.1, y_1(0) = 1, y_1'(0) = 0$$
$$y_2(0) = 0, y_2'(0) = 0.9995$$

Exact Solution

$$y_1(x) = \cos x + 0.0005\,x\sin(x)$$
$$y_2(x) = \sin x - 0.0005x\cos(x)$$

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Problem 2

$$y''_1 = -y_2 + \sin \pi x, \qquad h = \frac{1}{10}, y_1(0) = 0, y'_1(0) = -1$$

$$y''_2 = -y_1 + 1 - \pi^2 \sin \pi x, \qquad y_2(0) = 1, y'_2(0) = 1 + \pi$$

Exact Solution

$$y_1(x) = 1 - e^x$$

$$y_2(x) = e^x + \sin \pi x$$

Problem 3

$$y'' = 3y' + 8e^{2x} \qquad h = \frac{1}{20} = 0.05, y(0) = 1, y'(0) = 1$$

Exact Solution

$$y(x) = 2 + 3e^{3x} - 4e^{2x}$$

Table 1: Table of Result for Problem 1

x	y_1 *Numerical* for k=3	y_2 *Numerical* for k=3	y_1 *Exact*	y_2 *Exact*	y_1 *Numerical* for k=4	y_2 *Numerical* for k=4
0.10	1.005009168	0.100116833	1.005009168	0.100116833	1.005009166	0.100116833
0.20	1.020086756	0.201236668	1.020086756	0.201236668	1.020086754	0.201236668
0.30	1.045383515	0.304372533	1.045383515	0.304372533	1.045383517	0.304372533
0.40	1.081152378	0.410557618	1.081152378	0.410557617	1.081152378	0.410557617
0.50	1.127750987	0.520855594	1.127750987	0.520855593	1.127750986	0.520855593
0.60	1.185645284	0.636371263	1.185645283	0.636371261	1.185645284	0.636371261
0.70	1.255414171	0.758261597	1.255414169	0.758261593	1.25541417	0.758261593
0.80	1.337755312	0.887747309	1.33775531	0.887747304	1.33775531	0.887747304
0.90	1.433492125	1.026125069	1.433492124	1.026125062	1.433492124	1.026125062
1.00	1.543582027	1.17478047	1.543582024	1.174780458	1.543582024	1.174780457

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Table 2: Absolute Error of Problem 1

x	y_1 $Error$ for k = 3	y_2 $Error$ for k = 3	y_1 $Error$ for k = 4	y_2 $Error$ for k = 4
0.10	5.71925E-19	1.15207E-19	2.0572E-19	1.11521E-20
0.20	2.92035E-19	7.56965E-20	1.708E-19	7.56965E-21
0.30	1.41362E-20	1.43812E-19	1.8586E-19	5.6188E-21
0.40	4.72627E-20	8.68339E-20	4.7263E-19	1.68339E-20
0.50	9.19613E-21	1.27555E-20	9.0804E-20	1.24445E-22
0.60	9.56066E-20	1.88846E-20	9.5607E-21	8.84561E-21
0.70	1.95988E-21	3.80409E-20	9.5988E-22	9.59148E-22
0.80	1.57668E-21	5.25783E-21	4.2332E-22	4.21734E-23
0.90	1.34264E-21	6.74664E-21	3.4264E-23	2.53362E-23
1.00	3.02028E-22	1.18486E-21	2.0283E-23	1.1514E-24

Table 3: Table of Result for Problem 2

x	y_1 $Numerical$ for k=3	y_2 $Numerical$ for k=3	y_1 $Exact$	y_2 $Exact$	y_1 $Numerical$ for k=4	y_2 $Numerical$ for k=4
0.10	-0.105170914	1.414226655	-0.105170918	1.414226653	-0.105170915	1.414226646
0.20	-0.22140275	1.809253917	-0.221402758	1.809253919	-0.22140277	1.809253920
0.30	-0.349858805	2.158947628	-0.349858808	2.158947626	-0.349858812	2.158947631
0.40	-0.491824697	2.442931550	-0.491824698	2.442931552	-0.491824690	2.442931548
0.50	-0.648721268	2.64872126	-0.648721271	2.64872125	-0.648721265	2.64872123
0.60	-0.822118791	2.773099765	-0.8221188	2.773099762	-0.822118788	2.773099767
0.70	-1.013752704	2.822602063	-1.013752707	2.822602066	-1.013752702	2.822602059
0.80	-1.225540926	2.813062502	-1.225540928	2.813062509	-1.225540923	2.813062510
0.90	-1.459603114	2.768271416	-1.459603111	2.768271416	-1.459603112	2.768271412
1.00	-1.718281826	2.717874487	-1.718281828	2.717874482	-1.718281821	2.717874481

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Table 4: Absolute Error of Problem 2

x	y_1 $Error$ for k = 3	y_2 $Error$ for k = 3	y_1 $Error$ for k = 4	y_2 $Error$ for k = 4
0.10	1.56E-18	4.00E-18	3.132E-20	4.00E-21
0.20	2.56443E-18	3.132E-18	5.563E-19	3.132E-20
0.30	1.0654E-17	3.00E-19	3.5762E-20	3.00E-21
0.40	3.09876E-18	1.20E-19	3.076E-20	2.208E-21
0.50	1.026426E-18	3.8762E-18	1.20E-19	3.8762E-21
0.60	1.05425E-19	1.2432E-19	1.055E-20	1.2432E-22
0.70	2.099873E-19	1.00E-19	2.099E-20	1.00E-22
0.80	1.98762E-18	3.00E-19	2.082E-21	3.00E-23
0.90	1.876252E-19	2.0082E-20	1.825E-21	2.0082E-23
1.00	1.233345E-19	3.098E-20	1.233E-21	3.098E-23

Table 5: Table of Result for Problem 3

x	y numerical for k = 3	y $Exact$	y numerical for k = 4
0.10	0.904837419	0.904837418	0.904837417
0.20	0.818730756	0.818730753	0.818730754
0.30	0.740818218	0.740818221	0.740818209
0.40	0.670320044	0.670320046	0.670320043
0.50	0.606530666	0.60653066	0.606530655
0.60	0.548811617	0.548811636	0.548811624
0.70	0.496585289	0.496585304	0.496585302
0.80	0.449328985	0.449328964	0.449328898
0.90	0.40656956	0.40656966	0.406569607
1.00	0.367879362	0.367879441	0.367879446

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Table 6: Absolute Error of Problem 3

x	y *error* *for* k = 3	y *error* *for* k = 4
0.10	5.6404E-19	1.13596E-20
0.20	2.62202E-19	1.22202E-21
0.30	2.58172E-19	1.12817E-21
0.40	1.63564E-19	3.03564E-21
0.50	6.18737E-19	4.31263E-21
0.60	1.8794E-20	1.1794E-21
0.70	1.43914E-20	1.89141E-21
0.80	2.05828E-20	6.59172E-22
0.90	1.00241E-20	5.23406E-22
1.00	7.87714E-20	4.72856E-22

CONCLUSION

We observe from the table that the method for the direct solution of Second Order Ordinary Differential Equations (ODEs) for step number k=4 performs better than for k=3.

The step number k=4 performs relatively better than block second order hybrid methods for step number k=3 for all the problems. Thus, we conclude that from the table of error, accuracy and efficiency of the modified block second order hybrid methods for k=3, 4 is guaranteed.

Finally, the block methods obtained from its continuous scheme with off-grid interpolation and collocation points at $x_{n+\frac{1}{2}}$ for k=4 performs better than k=3 from the table of absolute errors as they provide lesser errors when used to solve directly second order Ordinary Differential Equations (ODEs) without reducing to a system of first order ODEs.

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