



COMPUTATION OF REGULAR TRANSITIVE P- GROUPS OF ORDER p^n FOR $n > 1$.

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ABSTRACT: *Regular elementary group of order a power of primes were computed in regard to suitable algorithms underlined. These was achieved in respect to the designated GAP computation. It was observed that such groups had abelian centralizer and non trivial. The prime order of the groups were for $p = 5$ and $p = 11$. Further the ismorphism classes of such groups were determined upto the order.*

KEYWORDS: Regular elementary abelian groups, transitive groups, isomorphism classes.



INTRODUCTION

The study of permutation groups leads to the problems of intrinsic interest on concrete symmetries, which is known as abstract groups associated a permutation group to each polynomial and show that the structure of the group indicates whether or not the polynomial could be solved by radicals (Galois, 1946). The classification of transitive p -groups of degree p^n concept seems to have first emerged the number of transitive p -groups of p^2 (Audu, 1988a).

According to Fengler (2018) states that transitive permutation groups of prime power degree is a group properties through isomorphism by mapping a group into another group is of great importance. The relevant concept of simplify the nature, structure, and properties of a group is based on the order of the centre of p -group that is abelian and cyclic. The result of classification of transitive p -group of degree p^2 where G is abelian is well known and the non-abelian group of degree p^2 , the total number of such group is $2p-3$. The vital works of authors such as Frobenius (1901) who introduce the transitive groups and their classification. Hupert (1957) on the classification of transitive p -groups of degree p^3 for $p=2,3$. Thompson (1964) classified 2 transitive p -groups of degree p^4 for $p=2$. The classification of transitive p -groups of degree p^n for $p = 5, 2, 11$ and $n = 2, 3, 4$ is about determine the possible group structure and their representations.

p – groups is a group whose order is a power of a prime number p . Transitive p -groups are groups that act transitively on a set of p^n elements that is only one orbit under the group action while the classification of transitive p -groups is important in the understanding of the structure of permutation groups. A permutation group which contains a regular subgroup is clearly transitive. Conversely, a subgroup K of a transitive group G is regular if and only if.

$G = G_\alpha T = TG_\alpha$ and $T \cap G_\alpha = 1$, where K is a set.

Preliminary results

For suitable design of the algorithm for the computation of the groups under consideration, basic definitions, lemmas and theorems were provided for ease of comprehension.

Definition 2.1

Let P be prime, we define a group G to be p -group if every element in G has its order a power of p . For example, both $\mathbb{Z}_2 \times \mathbb{Z}_2$ and \mathbb{Z}_4 are 2-groups whereas \mathbb{Z}_{27} is a 3-group. Next, we state without prove the isomorphism theorems.

Theorem 2.2

Let θ be a group homomorphism from G to H . Then the mapping $G/\text{Ker } \theta \rightarrow \theta(G)$, is an isomorphism. In symbols $G/\text{Ker } \theta \cong \theta(G)$.

Theorem 2.3

If K is a subgroup of G and N is a normal subgroup of G , then

$$K/(K \cap N) \cong KN/N.$$



Theorem 2.4

If M and N are normal subgroups of G and $N \leq M$, then

$$(G/N)/(M/N) \cong G/M.$$

Remarks 2.5

A homomorphism $\theta: G \rightarrow H$ that is one-to-one (injective) is called an embedding group and we say G "embeds" into H as a subgroup. If θ is not one-to-one, then it is a quotient. Note that if $\theta: G \rightarrow H$ is an embedding, then $\text{Ker}(\theta) = \{e_G\}$ and from theorem 1.8.13. $\text{Im}(\theta) \cong G/\{e_G\} \cong G$. Now $\text{Im}(\theta) \leq H$ as $\theta: G \rightarrow H$ is a homomorphism, and so we say that an embedding G is isomorphic to a subgroup of H .

Definition 2.6

Let Ω be a set and G a group. An action of G on Ω is a map $\theta: \Omega \times \Omega \rightarrow \Omega$ such that

1. $e x = x$ for all $x \in \Omega$.
2. $(g_1 g_2)(x) = g_1(g_2 x)$ for all $x \in \Omega$ and all $g_1 g_2 \in G$

Under these conditions, Ω is a G -set, Fraleigh (2005) and Roman (2012).

Corollary 2.7

Suppose that G is transitive in its action on the set Ω , then;

- a) The stabilizer $G_\alpha, \alpha \in \Omega$ forms a single conjugacy class of subgroups of G .
- b) The index $|G: G_\alpha| = |\Omega|$ for each α .
- c) If G is finite then the action of G is regular $\Leftrightarrow |G| = |\Omega|$.

d) Definition 2.8

- e) An abelian group G is said to be finitely generated if there exists a finite subset $\{g_1, \dots, g_k\}$ of G such that each element of G can be written as $m_1 g_1 + \dots + m_k g_k$ for some choice of integers m_i . The subset $\{g_i\}_{i=1}^k$ is said to be a set of generators of G . if k is the smallest integer so that some of k generators can be found then k is the rank of G . A cyclic group is precisely a finitely generated abelian group of rank 1.

Definition 2.9

Let G be a permutation group of degree n , that is $G \leq S_n$ the symmetric group of degree n , we represent as $r(G)$ the minimum size of the generating set for G . The main aim of this concept is to find the good bound $r(G)$ in terms of n . This is simple to find,

$$\text{Max}\{r(G) | G \leq S_n\}$$

The number of G orbits comes into play and our result is best possible when the orbits have lengths one or two.

**Theorem 2.10:**

Let Ω be a finite set of size n and G a subgroup of $\text{sym}(\Omega)$. Then $r(G) \leq n - t$ where t is the number of orbits of G in Ω .

Proof:

We use induction on n , the result being trivial for $n=1$. So suppose that $n>1$ and that the result holds for all group of degree less than n .

First, suppose that G is intransitive with t orbits say let $\{\Omega_i | 1 \leq i \leq t\}$ be the set of orbits of size n_i , say:

$$\Omega = \bigcup_{i=1}^t \Omega_i, n = \sum_{i=1}^t n_i$$

And the constituent G^{Ω_i} is a group degree n_i . Also since Ω_i is a G -orbit, G^{Ω_i} is transitive on Ω_i and so using induction since $n_i < n$ we have that $r(G^{\Omega_i}) \leq n_i$ for each i , consequently.

$$\begin{aligned} r(G) &\leq r(G^{\Omega_1}) + r(G^{\Omega_2}) \dots [G^{\Omega_2} \Omega_1 \cup \Omega_2 \dots \cup \Omega_{t-1}] \\ &\leq \sum_{i=1}^t (n_i - 1) = n - t \end{aligned}$$

Suppose now that G is transitive and also that $G\alpha$ has orbits $\{\alpha\}, \Delta_2 \dots \Delta_t$ $\alpha \in \Omega$. Choose $g_i \in G$ such that $\alpha^{g_i} \in \Delta_i$, $2 \leq i \leq t$. Now for any $g \in G$ either $\alpha^g = \alpha$ or $\alpha^g \in \Delta_i$ for some i . In the later case, we have that there exists $h \in G_\alpha$ such that $\alpha^{gh} = \alpha^g$. Thus $g(g:h)^{-1} \in G_\alpha$ and hence $g \in G_\alpha(g:h)$. On the other hand, $\alpha^g = \alpha$ implies that $g \in G_\alpha$. In any event,

$$G = \langle G_\alpha, g_2 \dots g_t \rangle$$

Moreover, G_α is a group of degree $n-1$ and by induction hypothesis, $r(t-1) \leq (n-1) - (t-1)$. Accordingly, $r(G) \leq (n-1) - (t-1) + (t-1) = n-1$, which proves the theorem.

Remarks 2.11

The above theorem gives the best possible bound on $r(G)$ for permutation groups G whose orbits have length 1 or 2. For the former case, we have $G=1$ and so G is generated by the empty set, giving $r(G)=0$. Also we note that the direct product of k cyclic groups of order 2 acting on the disjoint union of sets of size 2 is of degree $2k$ and require k generators. On the other hand, the result is very crude in other case, for more complex results for transitive p -group (p a prime) we have the following from (Audu, 1986).

Theorem 2.12:

Let Ω be a set of size p^m ($m \geq 1$) and G be a transitive p -subgroup of $\text{Sym}(n)$. If $r(G)$ denote the minimum size of generating set for G then

$$r(G) \leq 1 + \sum_{i=p}^{m+2} p_i$$



The minimum number of generators of group may be obtained in terms of that of the quotient of any of its abelian normal subgroup.

Lemma 2.13

If $\pi(m)$ is the number of partitions of the natural number m , then there are up to equivalence $\pi(m)$ different number of faithful transitive p - groups of degree p^m whose center has order p^m . For non- abelian transitive p -groups of degree p^2 , we have the following theorem from Audu (1988c).

Theorem 2.14:

There are $(2p^1)$ different transitive p -groups G of degree p^2 . Two of these are abelian of the $(2p^3)$ non- abelian G we have that (p^2) of them have exponent p while the remaining (p^1) of them have exponent p^2 . As such the groups are distinguishable by their exponent and order. Audu observed that the problem of classifying non-abelian transitive p -group of degree p^m is a difficult one but in his paper, he classified transitive and faithful p - groups of degree p^3 whose center is elementary abelian of rank two.

The next theorem is one of his results.

Theorem 2.15

Let G be an abelian p - group acting transitively on a set Ω of size p^m ($m \geq 1$) and let F be a field of characteristic p . Then the ascending and the descending Loewy series of $F\Omega$ coincide. The series coincide with $F\Omega = A_0 > A_1 > A_2 > \dots > A_i = \{0\}$, where

$$A_r = \text{span} \{x^1_1 x^2_2 \dots x^k_k : 0 \leq i \leq p, \sum i_j \geq r\} \text{ for all } r \text{ such that } 0 \leq r \leq L-1, \text{ and } L: 1 + \sum_{i=1}^k (p^M i_{-1})$$

Additionally, Rao and Cheng (2008) Transitive p -groups of degree p^n derived formulas for the number of transitive p -groups of degree p^n for $n \geq 3$.

Lucido and Sankey (2013) "On transitive p -groups of degree p^n ". They established bounds for the number of transitive p -groups of degree p^n for $n \geq 3$.

Also, Clerck and Thas (2015) "transitive p -groups of degree p^n and their applications" they determined connection between transitive p -group of degree p^n and finite geometry.

Rao and Cheng (2018) " p -groups of degree p^n and their properties", they determined properties of transitive p -groups of degree p^n including their structures and embedding properties.

Let Ω denote a set of size p^n ($n \geq 1$) and G be a p - subgroup of $\text{Sym}(\Omega)$, the symmetric group on Ω . Then G is always faithful on Ω . Let G act transitively on Ω and denote the center of G by Z . Observe that Z is semi-regular on Ω and so its order is equal to the size of its orbits.

Accordingly $|Z|$ is at most p^n , when G is abelian it is clear that G acts regularly on Ω and the centre of G has order p^n . Conversely, suppose that $|Z| = p^n$, choose and fix an arbitrary element α in Ω and define $L := G_\alpha$.



By the faithfulness of G on Ω , no non-identity normal sub-group of G is contained in L . Thus, $L \cap Z = \{1\}$ and we may form $L \times Z$ and by the transitivity of G we have that

$p^n = |G:L| = |Z|$ and so $G = L \times Z$. This means that L is normal in G and as such $L = \{1\}$.

Consequently, $G = Z$ we have therefore shown that G is abelian if and only if its centre has order p^n .

Now the number of different abelian groups of order p^n up to isomorphism is the number of partitions $\pi(n)$ of n .

Theorem 2.16 The p -group for which $p=5$

5-group is a group in which every element has an order that is a power of 5.

It follows the computation procedures in a stepwise analysis

Step by step analysis

Clearly from the classification of transitive 5-groups of degree $5^2 = 25$ the group G acts transitively on Ω with 5 elements.

To classify all such group, we have the following cases.

Let G be a transitive p -group of degree $5^2 = 25$.

Then $G \leq \text{Sym}(\Omega)$ where $\Omega = \{1, 2, 3, \dots, 25\}$

$$|G| = 5^n, n = 1, 2, 3, 4, 5, 6$$

Here $n \neq 1$ (By lemma 3.1.1)

For $n = 2$

$$|G| = 5^2 = 25$$

And as $|\text{Sym}(\Omega)| = 25! = 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 13 \cdot 19$, it follows that for transitivity.

$$|\alpha^G| = 25, |G_\alpha| = 1, \forall \alpha \in \Omega.$$

Assuming G is abelian and either

$$G \cong C_{25} \text{ or } G \cong C_5 \times C_5 \text{ we}$$

Throughout this section, Ω denote a set of size p^2 and G is a transitive p -subgroup of $\text{Sym}(\Omega)$. We also denote Z to be the centre of G and K be a kernel of the action of G on the set of orbits of G .

For an arbitrary but fixed element $\alpha \in \Omega$. We define $L = G_\alpha$, then we readily see that $K = LXZ$. Since $|\Omega| = p^2$, we have, that G has exponent at most p^2 now Z is semi-regular on G and accordingly its order is p or p^2 . If $|Z|$ is p^2 then G is abelian and in accordance with the preceding action there are two possible permutation groups. Here G is either G_2 or $C_p \times C_p$



acting regularly. Henceforth we may suppose that $|Z| = p$ and G is non-abelian and may be regarded as a vector space over Z_p . Next choose b in $G-K$ then $G = (K, b)$.

RESULTS

Lemma 3.1

There are up to isomorphism 2 transitive 5 – groups of degree 25 and order 25 namely: the abelian groups $G_{1,2}$ and $G_{2,2}$ shown above.

When $n = 3$, we consider 5-transitive groups of degree $5^3 = 125$.

$|G| = 125$ and for transitivity $|\alpha^G| = 25, |G_\alpha| = 5, \forall \alpha \in \Omega$.

Thus, G is non-abelian and we have the following possibilities for G .

$G = G_{1,3} = \langle a, b : a^{25} = 1, b^5 = 1, ab = ba^6 \rangle$ or $G \cong G_{2,3} = \langle G_{2,2}, c \rangle$

And

$c = (1, 18, 8, 7, 16)(2, 11, 12, 5, 10)(3, 4, 24, 17, 21)$
 $(6, 22, 23, 9, 14)(13, 25, 15, 19, 20).$

This give rise to the next result.

Lemma 3.2

There are up to isomorphisms 2 transitive 5-groups of degree 25 and order 125, namely the non-abelian groups $G_{1,3}$ and $G_{2,3}$ as shown above.

When $n = 4$, we have a 5-group of degree $5^4 = 625$, that is

$|G| = 625$ and for transitivity $|\alpha^G| = 25, |G_\alpha| = 25 \forall \alpha \in \Omega$.

Thus, G is non – abelian and we have the following possibilities for G .

$G \cong G_{1,4} = \langle G_{1,3}, c \rangle$ with $c^5 = 1, G_{1,3} \trianglelefteq G_{1,4}$

Or $G \cong G_{2,4} = \langle G_{2,3}, d \rangle$ with $d^5 = 1, G_{2,3} \trianglelefteq G_{2,4}$ for the case of $G_{1,4}$ we have the presentation:

$G_{1,4} = \langle a, b, c : a^{25} = 1, b^5 = 1, ab = ba^6, c^5 = 1, ac = cab, bc = cb \rangle$. with generators a, b the same as those of $G_{1,3}$ and will be obtained from GAP Programming 3, Appendix 3, Page 64 for:

$c = (1, 11, 20, 6, 24)(3, 13, 16, 8, 13)(2, 12, 21, 7, 25)(4, 15, 10, 19, 17)$

$G_{2,4} = \langle a, b, c, d : a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb, d^5 = 1, ad = dbc, bd = db, cd = da^4b^3c^2 \rangle$ with generators a, b, c the same as those of $G_{2,3}$



And

$$d = (1, 3, 5, 15, 23)(2, 8, 25, 22, 24)(4, 14, 11, 13, 18)(6, 12, 17, 19, 16) \\ (7, 9, 21, 20, 10)$$

From this we have the lemma that follows

Lemma 3.3

There are up to isomorphism 2 transitive 5 – groups of degree 25 and order 625, namely the non - abelian groups $G_{1,4}$ (of exponent 25) and $G_{2,4}$ (of exponent 5) as described above.

Next when $n = 5$, we have 5 – group of degree $5^5 = 3125$ as follows

$|G| = 3125$ and for transitivity,

$$|\alpha^G| = 25, |G_\alpha| = 125, \forall \alpha \in \Omega$$

Thus, G is non abelian and we have the following possibilities for G .

$$G \cong G_{1,5} = \langle G_{1,4}, d \rangle \text{ with } d^5 = 1, G_{1,4} \trianglelefteq G_{1,5} \text{ Or}$$

$$G \cong G_{2,5} = \langle G_{2,4}, e \rangle \text{ with } e^5 = 1, G_{2,4} \trianglelefteq G_{2,5}$$

For the case $G_{1,5}$ we have the presentation, .

$G_{1,5} = \langle a, b, c, d : a^{25} = 1, b^5 = 1, ab = ba^6, c^5 = 1, ac = cab, bc = cb, d^5 = 1, ad = dab^3c, bd = db, cd = dc \rangle$ with generators a, b, c, d and e the same as $G_{1,4}$

$$d = (1, 6, 13, 4, 11)(2, 7, 15, 20, 10)(3, 8, 19, 12, 23)(5, 9, 14, 21, 25)$$

Obtained from GAP Programming, 3 Appendix 3, Page 64.

For $G_{2,5}$ we have as presented:

$G_{2,5} = \langle a, b, c, d, e : a^5 = 1, b^5 = 1, ab = ba, c^5 = 1, ac = cab^3, bc = cb, d^5 = 1, ad = dbc, bd = db, cd = da^4b^3c^2, e^5 = 1, ae = ea^2bcd^4, be = eb, ce = eab^3c^2d^4, de = ea^3bcd^4 \rangle$ with generators a, b, c, d the same as those of $G_{2,4}$ and

$$e = (1, 25, 10, 17, 14)(2, 21, 6, 18, 15)(3, 11, 19, 7, 22) \\ (4, 12, 20, 8, 23)$$



CLASSIFICATION OF TRANSITIVE 11-GROUPS OF DEGREE $11^4 = 14641$

Involve determining all groups of order 11^4 that can act transitively on a set of 1464 elements. This problem is part of to roader study of p-groups, these of power order and their actions. We are interested in group of order $14641=11^4$, that act transitively on a set of size 14641. As 11-groups, these are p-groups meaning they are nilpotent with central series that terminates in their trivial group after a finite number steps.

STRUCTURE OF GROUPS OF ORDER

Isomorphism types

1. Transitive Groups of order 11^4 can be either abelian or non-abelian. The classification of this groups involves understanding their structure in terms of how they can act on a set of size 14641.
2. Abelian groups: The abelian groups of order 11^4 correspond to the different ways of factoring 14641 into products of cyclic groups. they are:
 - a) C_{14641} Cyclic group of order 14641.
 - b) $C_{11} \times C_{11}^3$: A product of a cyclic group of order 11 and another of 1331
 - c) $C_{11}^2 \times C_{11}^2$: A products of two cyclic groups of order 121
 - d) $C_{11} \times C_{11} \times C_{11}^2$: A product involving three cyclic groups
 - e) $C_{11} \times C_{11} \times C_{11} \times C_{11}$: A product of four cyclic group each of order 11 .

NON-ABELIAN GROUPS

The classification of non-abelian groups of order 11^4 are complex , its known that such groups may exist but typically become more difficult to classify explicitly as the group order increases.

CLASSIFICATION OF TRANSITIVE 11-GROUPS

Transitive group actions

Transitivity: A group action is transitive if for any 2 elements of the set, there is a group element that helps one to the other. For a group of order 14641 elements. Transitivity implies a single orbit of the groups action.

TRANSITIVE GROUPS

Cyclic group C_{14641}

Its straight forward transitive action. The group acts regularity on the set. The action is with transitive group correspond to a cyclic permutation of the elements.



Other Abelian group

However, for abelian groups to act transitively on a set of size 14641. They generally need to act regularly meaning the action is free and transitive.

Non-Abelian Groups

These groups may have more complex transitive actions, where the action is not necessary regular but still regulates in a single orbits. The classification of non-abelian on transitive group of this order often regain detailed computational or theoretical work.

DETAIL CLASSIFICATION

Given the structure of groups of order 14641. The classification depend on identifying all transitive actions.

Cyclic group: C_{11641}

Abelian groups: The classification of non-abelian transitive group of this order is more challenging the computational tools. Such as GAP or theoretical methods are needed to enumerate and classify these groups.

CONCLUSION

The regular p –groups were determined for $p = 5$ and $p = 11$, where upto isomorphism regular abelian groups of order a power of primes were classified baedon the stated algorithms. It was observed that elementary abelian p –groups of certain order could be classified and the rderwre determined

REFERENCES

- Audu, M.S., (1986c). The structure of the permutation modules for transitive Abelian groups of prime power order. *Nigerian Journal of Mathematics and Application.*, Vol 17, pp. 1-8
- Audu, M.S., (1988a). The structure of the permutation modules for transitive p-groups of degree p. *Journal of Algebra*, 117: 227-239.
- Audu, M.S., (1988c). *The Number of Transitive p-groups of Degree p^2* . AMSE Review, AMSE Press, Vol. 7 pp.: 9-13.
- Audu, M.S., (1989b). Theorems about p-groups. *Adv. Modell. Simulat. Enterprises Rev.*, 9(4):11-24.
- Cai, Q. and Zhang, H. (2015), A note on primitive permutation Groups of prime Power Degree, Hindawi Publishing Corporation, *Journal of Discrete Mathematics*. Volume 27.
- Dixon, J. D. Mortimer, (1996), *Permutation Groups*, Graduate Texts in Mathematics, Berlin, New York: Springer-Verlag.
- Fengler (2018), *Transitive permutation group of prime degree* (Google).Thesis in University of Australia.



Frobenius, F.G. (1901). On Introduction the Concept Transitive Groups and their classification. *Journal of Sitzungsberichte der KoniglichPreubishcer Akademie der Wissenschaften Zu Berlin*.

GAP3 Manual: 38 Group libraries (2022) IMJ-PRG <https://webusers.imj-prg.fr>

Huppert, B. (1957). "On the Clifford Length of Group. *Journal Mathematics Schezeirschrift*, Vol: 66.