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MODELLING DYNAMIC RESPONSES OF CLAMPED NON-UNIFORMLY PRESTRESSED BERNOULLI-EULER BEAMS ON VARIABLE ELASTIC FOUNDATIONS

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ABSTRACT: *This paper examines the dynamic response of a non-uniformly prestressed Bernoulli-Euler beam with clamped-clamped boundary conditions, resting on a variable bi-parametric foundation. The governing equation is a fourth-order partial differential equation with variable and singular coefficients. The primary objective is to derive an analytical solution for this class of dynamic problems. To achieve this, the Galerkin method is applied, utilizing a series representation of the Heaviside function to reduce the equation to a system of second-order ordinary differential equations with variable coefficients. These reduced equations are further simplified using two approaches: (i) the Laplace transform technique, combined with convolution theory, to address problems involving moving forces, and (ii) finite element analysis, integrated with the Newmark method, to solve analytically intractable moving mass problems with harmonic behaviour. We begin by solving the moving force problem using the finite element method and we validate its accuracy by comparing the results with analytical solutions. The numerical solution obtained from the finite element analysis demonstrates strong agreement with the analytical solution, confirming the method's reliability for tackling more complex moving mass problems that lack closed-form solutions. Finally, we generate displacement response curves for both moving distributed force and moving mass models at different time instances t, providing a comprehensive representation of the system's dynamic behaviour.*

KEYWORDS: Bernoulli-Euler beam, Prestressed, Clamped-Clamped, Newmark method, Bi-parametric foundation.

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INTRODUCTION

The dynamic behavior of elastic structures, particularly beams, has been a focus of engineering research due to its relevance in bridges, rails, mechanical systems, and aerospace applications. A significant body of literature has examined the vibration and response of these systems under various loading conditions. Among these, the Bernoulli-Euler beam theory, which assumes that plane sections remain plain and normal to the axis during deformation, has been extensively used for modeling slender beams subjected to dynamic loads (Inglis, 1934; Timoshenko, 1921; Wu *et al*., 2023). The theory provides a powerful yet simplified framework for studying beam deflection and stress distribution. Over time, numerous studies have refined the understanding of Bernoulli-Euler beams under different conditions, including non-uniform cross-sections, moving loads, and advanced boundary conditions (Stanisic *et al*., 1968; Sadiku & Leipholz, 1987; Oni, 1997; Li *et al*., 2024).

Dynamic analysis of beams under moving loads presents a unique challenge. Early studies, such as Krylov (1905) and Timoshenko (1921), explored simple harmonic loads on beams with ideal supports. However, real-world loads are often distributed rather than concentrated and may travel across the structure at varying speeds. Furthermore, the dynamic response of beams resting on elastic foundations is more complex, requiring models that incorporate both vertical stiffness and shear interaction, often referred to as bi-parametric foundations. Research has shown that non-uniformities in the beam, such as varying cross-sections or prestress, introduce additional complexities, requiring more sophisticated methods of analysis (Stanisic *et al*., 1968; Ahmadian *et al*., 2006; Adekunle *et al*., 2017; Huang *et al*., 2023).

The Bernoulli-Euler beam with non-uniform prestress is of particular interest because the internal stress varies along the beam's length, complicating the dynamic response. When such beams are subjected to clamped-clamped boundary conditions, they experience significant restraint at both ends, which further influences the vibrational behavior. This setup makes it challenging to derive exact solutions, particularly when resting on variable bi-parametric foundations, which account for both stiffness and shear effects along the beam's span (Esmailzadeh & Ghorashi, 1995; Zhao *et al*., 2023). Recent studies highlight the importance of capturing these complexities accurately.

The governing equation for the dynamic response of a non-uniformly prestressed Bernoulli-Euler beam resting on a bi-parametric foundation is typically a fourth-order partial differential equation with variable and singular coefficients. Analytical solutions to such problems are often difficult to obtain, especially when considering distributed moving loads or masses. Early studies by Muscolino and Palmeri (2007) focused on harmonic moving loads, while Dogush and Eisenberger (2002) extended the analysis to multi-span, non-uniform beams. However, these studies mostly employed simplified models of the moving load, often treating it as a point force, which overlooks the distributed nature of real loads. The present study addresses this gap by considering both distributed forces and moving masses, which more accurately reflect practical conditions.

The primary objective of this research is to derive analytical and numerical solutions for the dynamic response of a non-uniformly prestressed Bernoulli-Euler beam with clamped-clamped boundary conditions, resting on a variable bi-parametric foundation. To achieve this, the governing partial differential equation is reduced to a system of second-order ordinary differential equations using the Galerkin method, with a series representation of the Heaviside

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function to handle the complexity of distributed forces and masses. The following approaches are employed for solving the reduced equations:

1. Laplace transformation combined with convolution theory for problems involving moving forces.
2. Finite element analysis is integrated with the Newmark method for moving mass problems, which are analytically challenging due to their harmonic nature.

This study provides comprehensive analysis by generating displacement response curves for both moving force and mass models at various time intervals. The results offer new insights into the behavior of non-uniform Bernoulli-Euler beams under realistic loading and boundary conditions. Building on previous works such as Adekunle and Folakemi (2017) on *Dynamic Response of Non-Uniform Elastic Structure Resting on Exponentially Decaying Vlasov Foundation under Repeated Rolling Concentrated Loads*, this research highlights the importance of modeling non-uniform prestress and complex foundation effects in structural dynamics. The findings contribute to the broader understanding of dynamic systems, with potential applications in bridge design, rail systems, and structural health monitoring. Recent advancements in this field, including works by Zhang *et al*. (2023), Awodola *et al*., (2024) and Kumar and Patel (2024), further emphasize the evolving nature of research on Bernoulli-Euler beams under dynamic conditions, showcasing the necessity of refining existing models to incorporate contemporary challenges and developments.

The investigation considers critical aspects related to inertia terms, while also considering the beam's elastic properties such as its prestressed and bi-foundation both assumed not constant due to the non-uniform cross-section of the beam. It is important to note that damping effects are negligible in this scenario.

Problem Formulation

This study examines the problem of a variable-magnitude moving distributed load of non-uniformly prestressed clamped-clamped uniform Bernoulli-Euler beam resting on variable bi-parametric foundation. The properties of the beam, including its moment of inertia I, mass per unit length μ and axial force N, vary along its span length L.

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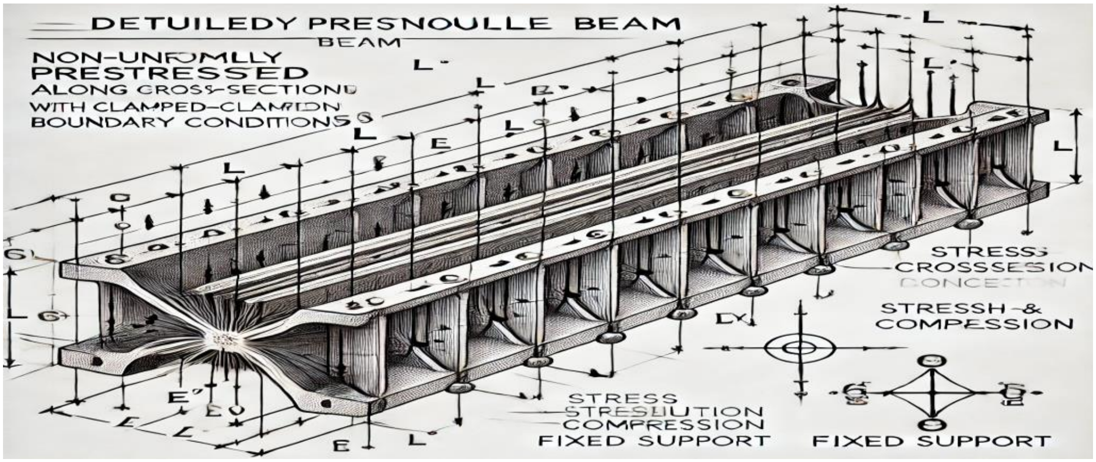

Figure 1: Geometry diagram of the non-uniformly prestressed clamped-clamped uniform Bernoulli-Euler beam

Figure 1 above depicts the transverse displacement *W*(*x*, *t*) of the beam as it moves at a constant speed. The equation of motion is given as:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} - N(x)\frac{\partial^2}{\partial x^2}W(x,t) + \mu\frac{\partial^2}{\partial t^2}W(x,t) + K(x)W(x,t) - G(x)\frac{\partial^2}{\partial x^2} = P(x,t), \quad (1)$$

In this problem, the time coordinate is represented by *t*, while *μ* denotes the mass per unit length of the beam. Additionally, *EI* refers to the flexural stiffness and *x* represents the spatial coordinate. *K(x)* represents variable foundation stiffness, *G(x)* signifies variable shear modulus, *N(x)* indicates variable axial force, and *P* (*x*, *t*) denotes variable magnitude moving distributed load acting on the beam. It is noteworthy that in this specific circumstance, the dispersed load traversing the beam bears a weight akin to that of the beam itself. Therefore, it must not be disregarded as its inertia exerts a considerable influence on determining the dynamical system's behavior. Therefore, *P* (*x*, *t*) will take on a specific form based on these factors as follows:

$$P(x,t) = \sum_{i=1}^{n} M_i g\, cos\omega t\, H(x - c_i t)\left[1 - \frac{1}{g}\frac{d^2W(x,t)}{dt^2}\right],$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\frac{df(t)}{dt}\frac{\partial^2}{\partial x\partial t} + \left(\frac{df(t)}{dt}\right)^2\frac{\partial^2}{\partial x^2} + \frac{d^2f(t)}{dt^2}\frac{\partial}{\partial x} \quad (2)$$

where g denotes the acceleration due to gravity, $\frac{d^2}{dt^2}$ is a convective acceleration operator, $\frac{\partial^2}{\partial t^2}$ is the support beam's acceleration at the point of contact with the moving mass, $\frac{df(t)}{dt}\frac{\partial^2}{\partial xdt}$ is the well-known Coriolis acceleration, $\left(\frac{df(t)}{dt}\right)^2\frac{\partial^2}{\partial x^2}$ is the centripetal acceleration of the moving mass and $\frac{d^2f(t)}{dt^2}\frac{\partial}{\partial x}$ is the acceleration component in the vertical direction when the moving load is not constant.

Likewise, with a steady velocity of *c*, the orientation and magnitude traversed by the weight on the beam at any particular instance *t* can be articulated as such.

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$$f(t) = c_i t, \tag{3}$$

Additionally, it is theorized that the portable load carries a mass represented by *M* and that the time *t* is limited to the period in which mass *M* remains on the beam. In simpler terms,

$$0 \leq f(t) \leq L, \tag{4}$$

The function *H* [*x*-*f* (*t*)] is the Heaviside function, commonly used in engineering applications to measure functions that are binary in nature, i.e., either "on" or "off". Its definition reads as follows:

$$H(x) = \begin{cases} 1, & x > ct. \\ 0, & x \leq ct. \end{cases} \qquad H[x - f(t)] = \begin{cases} 1, & x \geq f(t). \\ 0, & x < f(t). \end{cases} \tag{5}$$

For instance, consider the variable axial force denoted by *N(x)*, as defined in Adekunle *et al.* (2017), variable foundation stiffness and variable shear modulus, as defined in Adekunle and Folakemi (2017).

$$N(x) = N_0(1 + sin\frac{\pi x}{L}),$$

$$K(x) = K_0(4x - 3x^2 + x^3),$$

$$G(x) = G_0(12 - 13x + 6x^2 - x^3) \tag{6}$$

where N_0, K_0, and G_0 are constant axial force, constant foundation stiffness and constant shear modulus respectively, for the corresponding uniform beam. By substituting Equations (2), (3) and (6) into Equation (1) and conducting necessary simplification and rearrangement, we obtain the desired result as follows:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} - N_0(1 + sin\frac{\pi x}{L})\frac{\partial^2 W(x,t)}{\partial x^2} + \mu\frac{\partial^2 W(x,t)}{\partial t^2} + K_0(4x - 3x^2 + x^3)W(x,t)$$

$$-G_0(12 - 13x + 6x^2 - x^3)\frac{\partial^2 W(x,t)}{\partial x^2} + cos\ cos\ \omega t$$

$$\left[\frac{\partial^2}{\partial t^2} + 2Ci\frac{\partial^2}{\partial x \partial t} + c_i^2\frac{\partial^2}{\partial x^2}\right]W(x,t) = cos\ cos\ \omega t \tag{7}$$

The boundary conditions of the problem are deemed arbitrary, thereby allowing for the adoption of any form of classical boundary conditions. In contrast, without sacrificing generality, the initial conditions are presented as follows

$$V(x,0) = \frac{\partial V(x,0)}{\partial t} = 0. \tag{8}$$

Equation (7) constitutes the fundamental equation in the dynamic problem.

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Solution Procedure

Partial Differential Equation (7) exhibits non-homogeneous variable coefficients. Separation of variables method seems unfeasible, given the complexity in obtaining separate equations with functions dependent on a single variable. Consequently, we resort to a modified rendition of the approximate approach that is most fitting for addressing diverse concerns associated with structural dynamics, popularly referred to as Galerkin's Method. To reduce the fourth order partial differential equation into a sequence of second order ordinary differential equations, we employ Galerkin's method, as described by Oni and Awodola (2003, 2010). This approach leads us towards finding solutions in the form:

$$W(x,t) = \sum_{m=1}^{n} Y_m(t)V_m(x) \qquad (9)$$

The kernel function $V_m(x)$ is thoughtfully selected for Galerkin's method in Equation (9) to ensure that the specified boundary conditions are met. It should be noted that our analysis assumes general boundary conditions at $x = 0$ and $x = L$ for the beam in question. Therefore, we must carefully choose a suitable set of functions to represent the beam shapes in order to obtain the m^{th} normal mode of vibration

$$V_m(x) = sin\ sin\ \frac{\lambda_m x}{L} + A_m\ cos\ cos\ \frac{\lambda_m x}{L} + B_m\ sinh\ sinh\ \frac{\lambda_m x}{L} + C_m\ cosh\ cosh\ \frac{\lambda_m x}{L} \qquad (10)$$

is chosen such that the boundary conditions are satisfied. The kernel is chosen as:

$$V_k(x) = sin\ sin\ \frac{\lambda_k x}{L} + A_k\ cos\ cos\ \frac{\lambda_k x}{L} + B_k\ sinh\ sinh\ \frac{\lambda_k x}{L} + C_k\ cosh\ cosh\ \frac{\lambda_k x}{L} \qquad (11)$$

In Equations (10) and (11), λ_m and λ_k respectively denote the mode frequency. The constants A_m, B_m, C_m, A_k, B_k and C_k are determined by substituting Equations (6) and (7) into the relevant boundary condition. Consequently, upon substitution of Equation (9) into Equation (7), we obtain:

$$\sum_{m=1}^{n} \{[V_m(x)V_k(x)]Y_m(t) + \frac{EI}{\mu}[V_m^{IV}(x)V_k(x)]Y_m - \frac{N_0\pi}{\mu L}[\frac{cos cos\ \pi x}{L} V_m^{I}(x)V_k(x)]Y_m(t) - \frac{N_0}{\mu}[V_m^{II}(x)V_k(x)]Y_m(t) - \frac{N_0}{\mu}[\frac{sin sin\ \pi x}{L} V_m^{II}(x)V_k(x)]Y_m(t) + \frac{K_0}{\mu}[4xV_m(x)V_k(x)]Y_m(t) - \frac{K_0}{\mu}[3x^2V_m(x)V_k(x)]Y_m(t) + \frac{K_0}{\mu}[x^3V_m(x)V_k(x)]Y_m(t) + \frac{G_0}{\mu}[12V_m^{II}(x)V_k(x)]Y_m(t) + \frac{G_0}{\mu}[13xV_m^{II}(x)V_k(x)]Y_m(t) - \frac{G_0}{\mu}[6x^2V_m^{II}(x)V_k(x)]Y_m(t) + \frac{G_0}{\mu}[x^3V_m^{II}(x)V_k(x)]Y_m(t) + \frac{M}{\mu}[cos\ cos\ \omega t H(x-ct)V_m(x)V_k(x)]Y_m(t) + \frac{M}{\mu}(cos\ cos\ \omega t H(x-ct)V_m(x)V_k(x))Y_m(t) + 2C(cos\ cos\ \omega t H(x-ct)V_m(x)V_k(x))Y_m(t) + c^2(cos\ cos\ \omega t H(x-ct)V_m(x)V_k(x))Y_m(t)\} - \frac{M}{\mu} g\ cos\ cos\ \omega t H(x-ct)V_k(x) = 0$$

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To derive an expression for $Y_m(t)$, let us examine a mass M that moves uniformly at velocity c along the x-coordinate. The solution for any number of moving masses may be obtained by superimposing the individual solutions, as the governing equation is linear. In order to determine the expression for a single mass M_1, it is necessary that the left-hand side of Equation (12) be orthogonal to function $U_k(x)$. Therefore, utilizing Equations (10) and (11) in (12) produces:

$$I_0^*\ddot{Y}_m(t)+I_1^*Y_m(t)+\frac{\cos\omega t}{\mu_0}M\left[I_2^*\ddot{Y}_m(t)+2cI_3^*\dot{Y}_m(t)+c^2I_4^*Y_m(t)\right]=\frac{g\cos\omega t}{\mu_0}MI_5^0, \quad (13)$$

where

$$I_0^*=\sum_{m=1}^{n}\int_0^L\left(1+\sin\frac{\pi x}{L}\right)U_m(x)U_k(x)\mathrm{d}x, \quad I_1^*=I_{1A}+I_{1B}+I_{1C}-I_{1D}+I_{1E}-I_F, \quad (14)$$

$$I_{1A}=\frac{EI_0}{4\mu_0}\sum_{m=1}^{n}\int_0^L\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)U_m^{iv}(x)U_k(x)\mathrm{d}x, \quad (15)$$

$$I_{1B}=\frac{6\pi EI_0}{4\mu_0 L}\sum_{m=1}^{n}\int_0^L\left(5\cos\frac{\pi x}{L}+4\sin\frac{2\pi x}{L}-\cos\frac{3\pi x}{L}\right)U_m'''(x)U_k(x)\mathrm{d}x, \quad (16)$$

$$I_{1C}=\frac{3\pi^2 EI_0}{4\mu_0 L^2}\sum_{m=1}^{n}\int_0^L\left(3\sin\frac{3\pi x}{L}+8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}\right)U_m''U_k(x)\mathrm{d}x, \quad (17)$$

$$I_{1D}=\frac{N}{\mu_0}\sum_{m=1}^{n}\int_0^L U_m''U_k(x)\mathrm{d}x,\ I_{1E}=\frac{K_0}{\mu_0}\sum_{m=1}^{n}\int_0^L U_mU_k(x)\mathrm{d}x,\ I_{1F}=\frac{G_0}{\mu_0}\sum_{m=1}^{n}\int_0^L U_m''U_k(x)\mathrm{d}x, \quad (18)$$

$$I_2^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_mU_k(x)\mathrm{d}x, \quad I_3^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_m'U_k(x)\mathrm{d}x, \quad (19)$$

$$I_4^*=\sum_{m=1}^{n}\int_0^L H(x-ct)U_m''U_k(x)\mathrm{d}x, \quad I_5^0=\int_0^L H(x-ct)U_k(x)\mathrm{d}x. \quad (20)$$

Using the property of Heaviside function, it can be expressed in series form given by Adekunle *et al.* (2017), i.e.,

$$H(x-ct)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x\cos(2n+1)\pi ct}{2n+1}-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x\sin(2n+1)\pi ct}{2n+1}. \quad (21)$$

Thus, in view of (14)–(20) and (21), it can be shown that:

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$$\ddot{Y}_m(t) + \frac{I_1^*(m,k)}{I_0^*(m,k)} Y_m(t) + \frac{\varepsilon_0 \cos \omega t}{I_0^*(m,k)} \Bigg\{ \Bigg[L\psi_{1A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_5^*(m,k)$$
$$- \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} \times I_6^*(m,k) \Bigg] \ddot{Y}_m(t) + 2c \Bigg[L\psi_{2A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_7^*(m,k)$$
$$- \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_8^*(m,k) \Bigg] \dot{Y}_m(t) + c^2 \Bigg[L\psi_{3A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_9^*(m,k)$$
$$- \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{10}^*(m,k) \Bigg] Y_m(t) \Bigg] \Bigg\} = \frac{MgL \cos \omega t}{\mu \lambda_k I_0^*(m,k)} \times \Bigg[-\cos \lambda_k x + A_k \sin \lambda_k x$$
$$+ B_k \cosh \lambda_k x + C_k \sinh \lambda_k x + \cos \frac{\lambda_k ct}{L} - A_k \sin \frac{\lambda_k ct}{L} - B_k \cosh \frac{\lambda_k ct}{L} - C_k \sinh \frac{\lambda_k ct}{L} \Bigg], \tag{22}$$

where

$$\varepsilon_0 = \frac{M}{\mu L}. \tag{23}$$

Equation (22) stands as the fundamental governing equation for the dynamic problem. This coupled, non-homogeneous second-order ordinary differential equation applies to all variants of classical boundary conditions. Consequently, two distinct cases emerge from Equation (22): the moving force and moving mass problems.

Non-uniformly Prestressed Bernoulli-Euler Beam Traversed by Moving Distributed Force for Clamped-Clamped End Condition

In this segment, we derive an approximate model for the differential equation that characterizes the reaction of the elastic structure. This is achieved by disregarding inertia terms, specifically setting ε_0 to zero. Furthermore, we will focus solely on the clamped-clamped end condition as our example. Under these circumstances, both displacement and bending moment are negligible and vanish entirely.

$$V_m(0,t) = 0 = V_m(L,t), \quad \frac{\partial^2 V_m(0,t)}{\partial x^2} = 0 = \frac{\partial^3 V_m(L,t)}{\partial x^3}, \tag{24}$$

and hence for normal modes

$$U_m(0) = 0 = U_m(L), \quad \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^3 U_m(L)}{\partial x^3}, \tag{25}$$

which implies

$$U_k(0) = 0 = U_k(L), \quad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^3 U_k(L)}{\partial x^3}. \tag{26}$$

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It is easily shown that

$$A_m = \frac{\sinh \lambda_m - \sin \lambda_m}{\cos \lambda_m - \cosh \lambda_m} = \frac{\cos \lambda_m - \cosh \lambda_m}{\sin \lambda_m + \sinh \lambda_m} = -C_m \text{ and } B_m = -1 , \quad (27)$$

and

$$\cos \lambda_m \cosh \lambda_m = 1 , \quad (28)$$

substituting Equations (24)–(28) into Equation (22), yields

$$\ddot{Y}_m(t) + \delta_f^2 Y_m(t) = \rho_0 \, cos\, cos\, wt \, [- cos\, cos\, \lambda_k + A_k \, sin\, sin\, \lambda_k + B_k \, cos\, cos\, h\lambda_k + C_k \, sin\, sin\, h\lambda_k + cos\, cos\, \theta_c t - A_k \, sin\, sin\, \theta_c t - B_k \, cos\, cos\, h\theta_c t - C_k \, sin\, sin\, h\theta_c t] , \quad (29)$$

where

$$\delta_f^2 = \frac{P_0(m,k)}{B_0(m,k)}; \quad \rho_0 = \frac{MigL}{\mu\lambda_k B_0(m,k)}; \quad \theta_c = \frac{\lambda_k c}{L}, \quad (30)$$

Thus, by applying the Laplace transform technique and convolution theory with the given initial conditions (8), we can obtain a solution to Equation (29) as follows:

$$W(x,t) = \frac{\rho_0 coswt}{\delta_f^2\left(\theta_c^4 - \delta_f^4\right)} \{\beta_n(m,t)\left[\left(1 - cos\delta_f t\right)(\theta_c^4 - \delta_f^4)\right] - \delta_f^2\left(cos\theta_c t - cos\delta_f t\right) - A_k\delta_f\left(\theta_c sin\delta_f t - \delta_f sin\theta_c t\right) - \left(\theta_c^2 - \delta_f^2\right)\left[B_k\delta_f^2\left(cosh\theta_c t - cos\delta_f t\right) + C_k\delta_f(\delta_f sinh\theta_c t - \theta_c sin\delta_f t)\right]\} \times \left[Sin\frac{\lambda_k x}{L} + A_k cos\frac{\lambda_k x}{L} + B_k Sinh\frac{\lambda_k x}{L} + C_k cosh\frac{\lambda_k x}{L}\right]. \quad (31)$$

Equation (31) depicts the transverse-displacement reaction of a non-uniformly prestressed clamped-clamped Bernoulli-Euler beam, which is subjected to a variable-magnitude moving distributed force and rests on a variable bi-parametric foundation.

Non-uniform Bernoulli-Euler Beam Traversed by Moving Distributed Mass for Clamped-Clamped End Condition

In this section, we endeavor to determine a solution for Equation (22) without disregarding any of the terms in the coupled differential equation. It is evident that conventional methods cannot yield an exact solution for this equation. Even Struble's frequently employed technique (Struble, 1962) falters due to the varying magnitude of the moving load. Henceforth, we resort to utilizing finite element method (FEM) for modeling the structure and subsequently implement Newmark numerical integration method to solve the resultant semi-discrete time-dependent equation and obtain our desired responses.

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Finite Element Method (FEM)

The finite element techniques assume that the unknown transverse deflection of the non-uniformly prestressed beam $W(x, t)$ can be represented approximately by a set of piecewise continuous functions which are defined over a finite number of sub-regions called elements and composed of the numerical values of the unknown deflection within the region. Thus, the first step involved in the technique consists of dividing the special solution domain of the non-uniformly prestressed beam, which happens to be the length of the beam in this case, into several sub-domain known as finite elements. These elements are joined to each other at selected points called nodes. Subsequently, the weak or variational form corresponding to governing Equation (1) is constructed thus:

Let us consider a customary segment of dimension L, with its domain $\lambda_e = (0, L)$. By inserting Equations (2) and (3) into Equations (1), we have:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 W(x,t)}{\partial x^2}\right) + \mu\frac{\partial^2 W(x,t)}{\partial t^2} - N(x)\frac{\partial^2 W(x,t)}{\partial x^2} + K(x)W(x,t) - G(x)\frac{\partial^2 W(x,t)}{\partial x^2} + (cos\, cos\, wt\,)MH(x-ct)\left[\frac{\partial^2 W(x,t)}{\partial t^2} + \frac{2C\partial^2 W(x,t)}{\partial_x \partial_t} + \frac{C^2\partial^2 W(x,t)}{\partial x^2}\right] = M_g\, cos\, cos\, (wt)\, H(x-ct) \tag{32}$$

To address the resolution of Equation (32), we will examine a mass M that moves uniformly at a velocity c along the x-coordinate. As the governing equation is linear, finding solutions for any number of moving masses can be achieved through superposition of individual solutions. For the single mass M_1, let Galerkin's weight function $V(x)$ be utilized. By multiplying Equation (32) with this weight function and integrating over the domain λ_e, simplification and rearrangement lead to its solution.

$$\int_0^{L^l} EI\frac{\partial^2 W(x,t)}{\partial x^2}\frac{\partial^2 V(x)}{\partial x^2}dx + \int_0^{L^l}\mu\frac{\partial^2 W(x,t)}{\partial t^2}V(x)\,dx - \int_0^{L^l}\left(N(x)+G(x)\right)\frac{\partial^2 W(x,t)}{\partial x^2}V(x)dx + \int_0^{L^l}K(x)W(x,t)V(x)\,dx - M_g\, cos\, cos\,(wt)\int_0^{L^l}V(x)H(x-ct)\,dx + M\, cos\, cos\,(wt)\int_0^{L^l}\frac{\partial^2 W(x,t)}{\partial t^2}V(x)H(x-ct)dx + 2Mc\, cos\, cos\,(wt)\int_0^{L^l}\frac{\partial^2 W(x,t)}{\delta x\delta t}V(x)H(x-ct)dx + Mc^2\, cos\, cos\,(wt)\int_0^{L^l}\frac{\partial^2 W(x,t)}{\partial t^2}V(x)H(x-ct)dx - V(L^l)Q_3^l - V(0)Q_1^l + \frac{\partial V}{\partial x}/_{x=L^l}\,Q_4^l + \frac{\partial V}{\partial x}/_{x=0}\,Q_2^l = 0 \tag{33}$$

where

$$Q_1 = [\frac{\partial}{\partial x}\left(EI\frac{\partial^2 W(x,t)}{\partial x^2}\right)]\,/_{x=0};\; Q_2^l = EI\frac{\partial^2 W(x,t)}{\partial x^2}\,/_{x=0}\; Q_3 = -[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 W(x,t)}{\partial x^2}\right)]\,/_{x=L^l;}\quad Q_4^l = EI\frac{\partial^2 W(x,t)}{\partial x^2}\,/_{x=L^l} \tag{34}$$

Additionally, it can be easily demonstrated that

$$\int_0^{L^l} f(x)H(x-ct)\,dx = \int_0^{L^l} f(x)\,dx \tag{35}$$

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Thus, Equation (33) becomes

$$\int_0^{L^l} EI \frac{\partial^2 W(x,t)}{\partial x^2}\frac{\partial^2 W(x,t)}{\partial x^2} dx + \int_0^{L^l} \mu \frac{\partial^2 W(x,t)}{\partial t^2} V(x)\, dx - \int_0^{L^l} \left(N_{(x)} + G_{(x)}\right)\frac{\partial^2 W(x,t)}{\partial x^2} V(x) dx + \int_0^{L^l} K(x)\, W(x,t) V(x) dx - M_g \cos\cos(wt) \int_0^{L^l} V(x)\, dx + M \cos\cos(wt) \int_0^{L^l} \frac{\partial^2 W(x,t)}{\partial t^2} V(x)\, dx + 2Mc \cos\cos(wt) \int_0^{L^l} \frac{\partial^2 W(x,t)}{\delta x \delta t} V(x)\, dx + Mc^2 \cos\cos(wt) \int_0^{L^l} \frac{\partial^2 W(x,t)}{\partial x^2} V(x)\, dx - V(L^l) Q_3^l - V(0) Q_1^l + \frac{\partial V}{\partial x} /_{x=L^l} Q_4^l + \frac{\partial V}{\partial x} /_{x=0} Q_2^l = 0 \tag{36}$$

The weak form of the variable magnitude moving distributed masses of non-uniformly prestressed beams resting on variable bi-parametric foundation, can be found in Equation (36). In order to obtain an approximate solution for the element being analyzed and develop its corresponding shape function, we assume that the unknown deflection $W(x, t)$ can be expressed approximately.

$$W(x,t) \approx W_n(x,t) = H_1(x)W_1(t) + H_2(x)W_2(t) + H_3(x)W_3(t) + H_4(x)W_4(t) = \sum_{k=1}^{4} H_k(x)V_k(t) = \{H\}\{V(t)\}, \qquad j = 1,2,3,4 \tag{37}$$

where $H_j(x)$ represents the Hermite-cubic shape functions, $V_k(t)$ represents the modal deflection functions and H is a row vector defined as:

$$[H] = [H_1(x), H_2(x), H_3(x), H_4(x)]. \tag{38}$$

Utilizing the techniques outlined by Junkins and Kim (1993) for creating Hermite-cubic interpolation functions results in

$$H_1 = 1 - \frac{3x^2}{h^2} + \frac{2x^3}{h^3}, \quad H_2 = x - \frac{x^2}{h} + \frac{x^3}{h^2}, \quad H_3 = \frac{3x^2}{h^2} - \frac{2x^3}{h^3}, \quad H_4 = -\frac{x^2}{h} + \frac{x^3}{h^2}, \tag{39}$$

Substituting Equations (37)–(39) into the weak form (36), where x denotes the spatial coordinate, and conducting some simplification and rearrangement yields:

$$[K^e]\{V(t)\} + [C^e]\{\dot{V}(t)\} + [M^e]\{\ddot{V}(t)\} + \{f^e\} + \{Q^e\} = 0. \tag{40}$$

The Matrix Equation (40) functions as the primary governing equation that delineates the conduct of a conventional finite element within a non-uniformly prestressed beam under the influence of a harmonic moving load. The symbol $[K^e]$ denotes the stiffness matrix of said element, while $[M^e]$ represents its mass matrix. Furthermore, $[C^e]$ signifies its centripetal matrix, $\{f^e\}$ serves as an indicator for the force vector and $\{Q^e\}$ reflects upon the boundary term vector of said element.

The subsequent phase entails the assembly of the equations. Wu (2005) and Irvine (2010) have extensively discussed the process of amalgamating various matrices and vectors for multiple

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beam elements that form a mesh. This culminates in an assembled governing equation of motion, which accurately describes the dynamic behavior exhibited by problems involving moving loads with Pasternak foundation.

$$[K]\{V(t)\}+[C]\{\dot{V}(t)\}+[M]\{\ddot{V}(t)\}=\{F\}, \tag{41}$$

where $[K]$, $[M]$ and $[C]$ are the assembled (global or overall) stiffness, mass, centripetal and load vector.

To obtain a comprehensive and unique solution (41), it is crucial to apply the designated boundary conditions on both the deflection/slopes and shear force/bending moments. Ultimately, in a free vibration system without the centripetal matrix, (41) transforms into a harmonic form.

$$\left([K]-\omega_i^2[M]\right)\{V(t)\}=0, \tag{42}$$

The natural frequency is represented by ω^2 while the system's corresponding mode shape is denoted by $V(t)$. Several techniques can be employed to determine both the eigenvalue ω^2 and its corresponding $V(t)$. The dynamic response of a non-uniform beam subjected to a partially distributed moving load can be derived through direct solution of equation (41) using the Newmark method.

COMMENTS ON THE CLOSED FORM SOLUTIONS

In principle, the deviations of a non-uniformly prestressed Bernoulli-Euler beam possess the capability to surpass acceptable thresholds. This occurrence signifies that the beam is undergoing resonance. The pace at which an external load triggers such resonance in the system is denoted as its critical velocity. As illustrated by (31), when subjected to a moving distributed force and bolstered by a Pasternak foundation with clamped-clamped supports, the said beam inevitably attains these resonant states.

$$\theta_c=\delta_f; \qquad \delta_f=\frac{\lambda_k c}{L} \tag{43}$$

Equation. (30) shows that the dynamic system will attain the state of resonance whenever velocity is

$$c=\frac{L\theta_c}{m\pi} \tag{44}$$

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ANALYSIS OF RESULT AND DISCUSSION

To illustrate the presented analysis, a non-uniform beam with a length of 5 meters is examined. The load velocity is set at 50 meters per second, while the young modulus amounts to 2.10924 x 10^9 Newtons per square meter and the moment of inertia measures at 0.00287698 cubic meters to the fourth power. The value of π is equal to approximately 22 divided by seven, and the mass per unit length of the beam equals 2758.291 kilograms per cubic meter; furthermore, the ratio between load mass and beam mass stands at 0.25.

The transverse deflection of this beam can be calculated for various values of axial force N, foundation stiffness K as well as shear modulus G, which are all subject to variation in this study: N varies between (0 - 4.5 x 10^7), K ranges from four times ten raised to three and nine times ten raised to eight units (4 x 10^3 - 4 x 10^9), whereas G varies from four times ten raised to three up until nine times ten raised to eight newtons per cubic meter cubed (N/m^3).

These calculations result in several graphs displayed across Figures Two through Seven that showcase our findings on these variables' impact on transverse deflection over time.

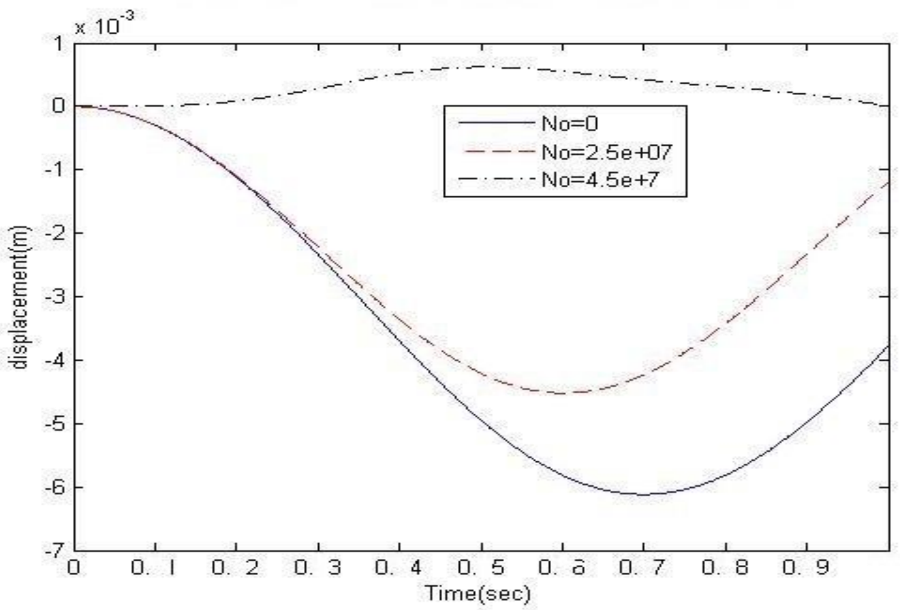

Figure 2: Transverse displacement of the non-uniform prestressed clamped-clamped Bernoulli-Euler beam for various values of axial force N_o and fixed values of K_o(4000) and G_o(4000) that traversed by moving distributed force.

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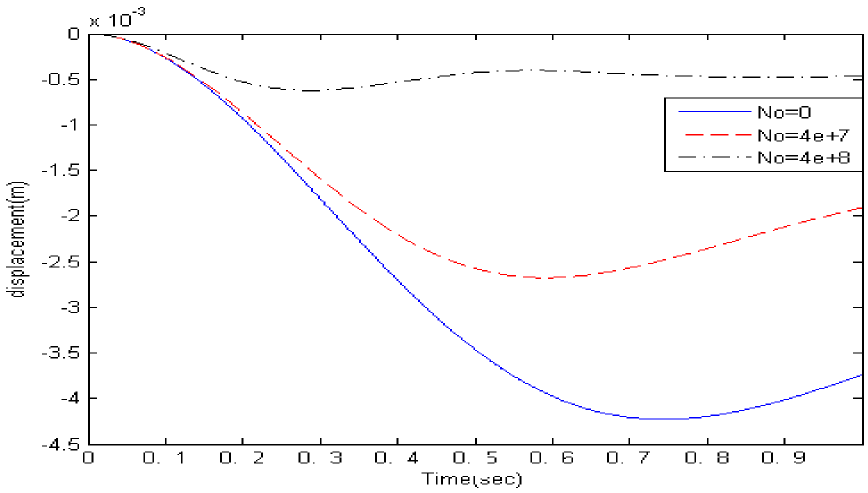

Figure 3: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of axial force N and fixed values of K(4000) and G(4000) that traversed by moving distributed mass.

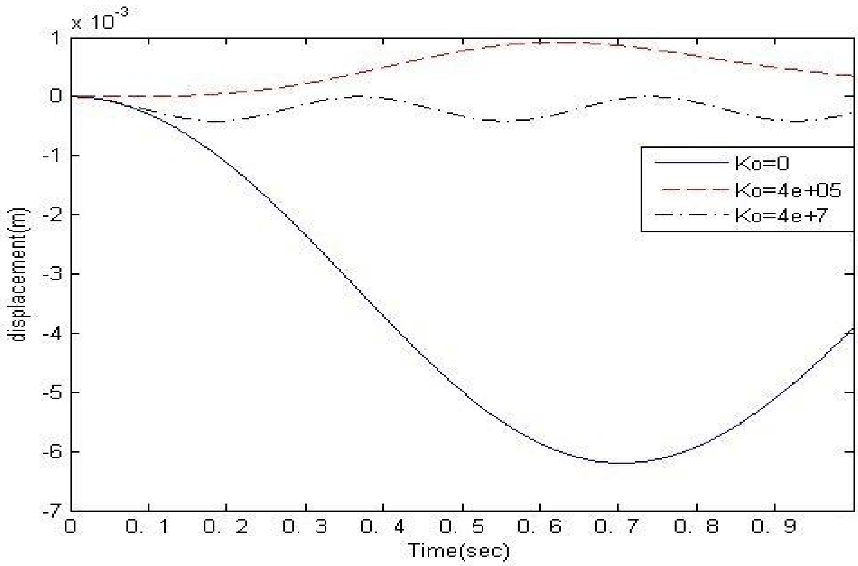

Figure 4: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of foundation stiffness K and fixed values of G(4000) and N(4000) that traversed by moving distributed force.

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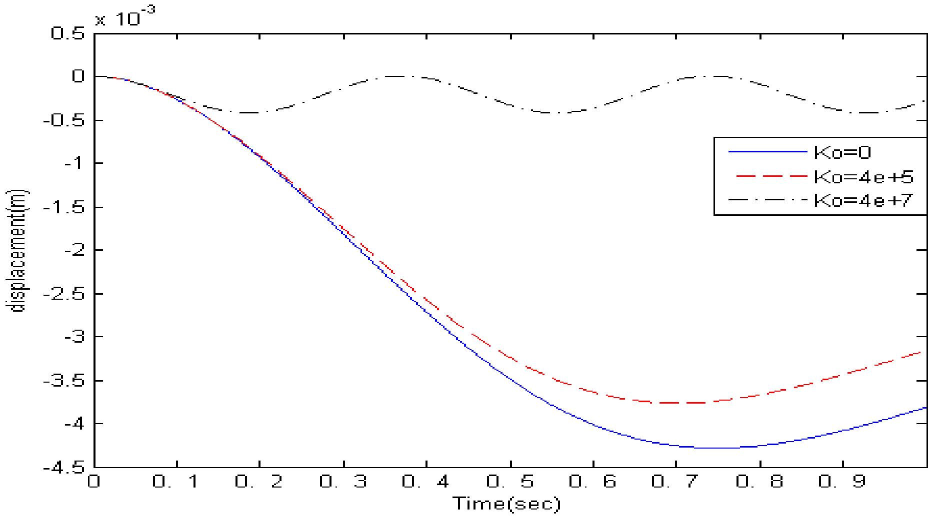

Figure 5: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of foundation stiffness K and fixed values of G(4000) and N(4000) that traversed by moving distributed mass.

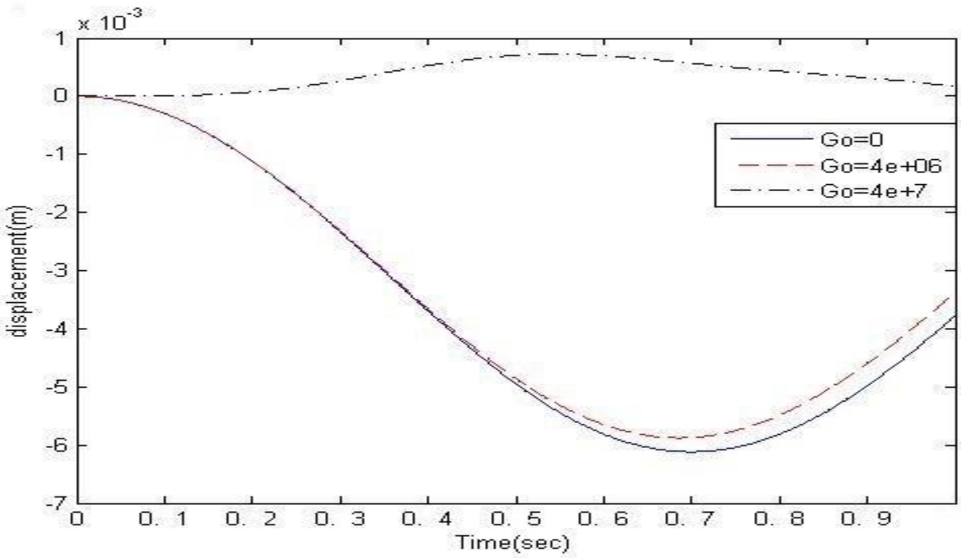

Figure 6: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of shear modulus G and fixed values of N(4000) and K(4000) that traversed by moving distributed force.

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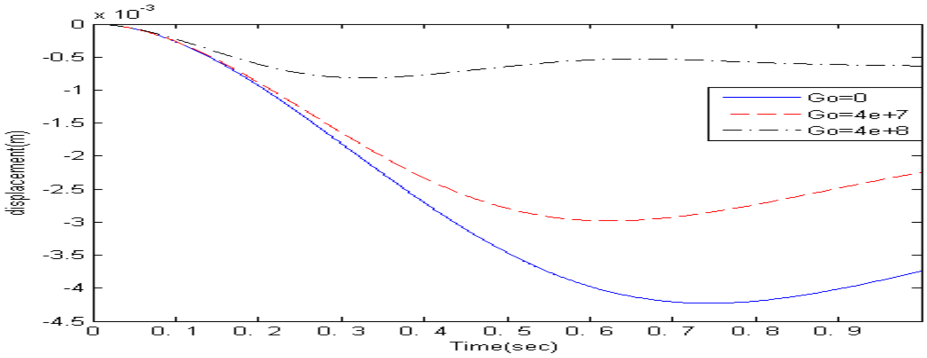

Figure 7: Transverse displacement of the non-uniform clamped-clamped Bernoulli-Euler beam for various values of shear modulus G and fixed values of N(4000) and K(4000) that traversed by moving distributed mass.

Figures 2–4 depict the transverse displacement responses of a non-uniform clamped-clamped Bernoulli-Euler beam subjected to distributed moving load traveling at constant velocity under the influence of moving distributed force. The figures display various values of (i) axial force N while other parameters remain fixed, (ii) foundation stiffness K while other parameters remain fixed, and (iii) shear modulus G while other parameters remain fixed. It is observed that as N, K, and G increase, there is a decrease in the deflection of the beam. Similar outcomes are achieved when the beam encounters moving mass, as shown in Figures 5–7.

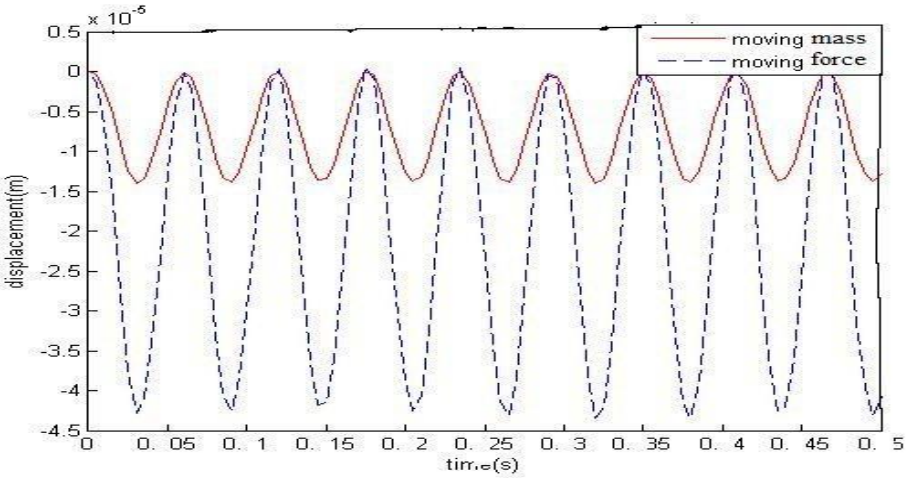

Figure 8: Comparison of the transverse displacement of the moving distributed mass and moving distributed force for the non-uniform clamped-clamped Bernoulli-Euler beam.

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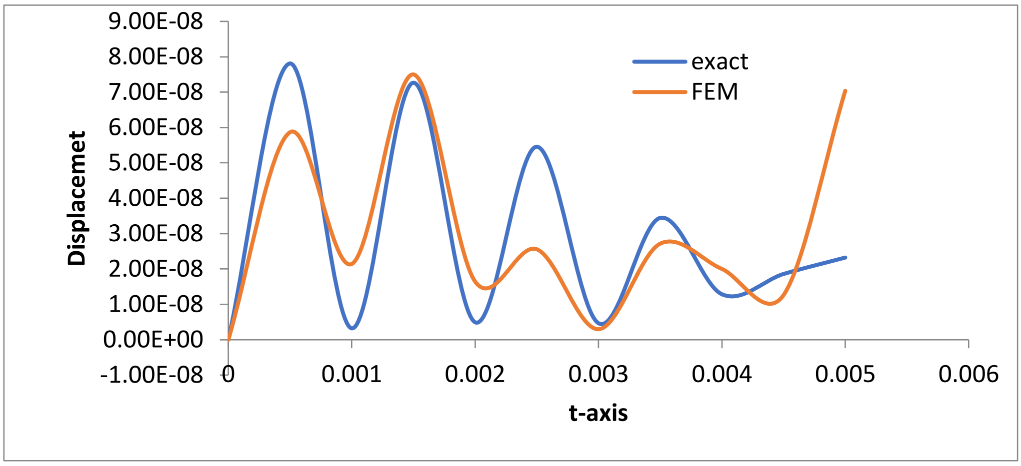

Figure 9: Comparison of the transverse displacement of the exact and numerical solutions for the non-uniformly prestressed clamped-clamped Bernoulli-Euler beam.

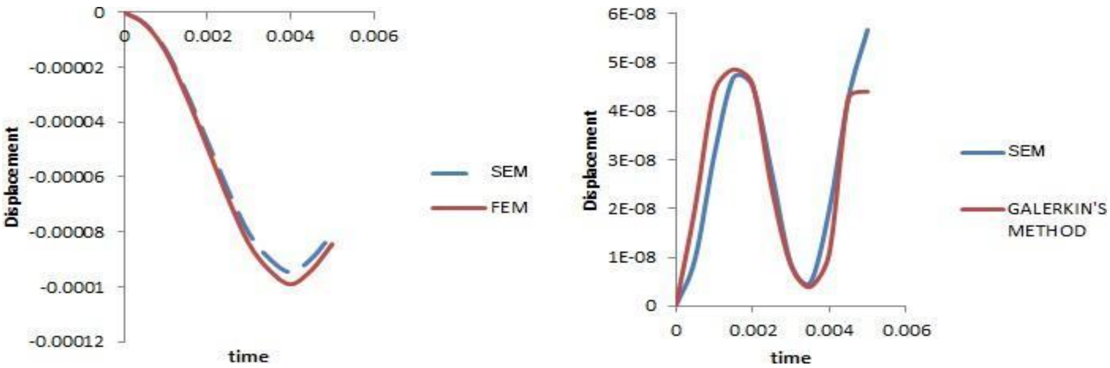

Figure 10: Comparison of the transverse displacement of the moving distributed mass and moving distributed with SEM force for the non-uniformly prestressed clamped-clamped Bernoulli-Euler beam.

Various comparisons of the lateral displacements are depicted in Figures 8–10. To authenticate the precision of the current approach, we compare the vibration caused by moving distributed masses with varying magnitudes on a non-uniform Bernoulli-Euler beam that is clamped-clamped and rests on a Pasternak elastic foundation, as obtained through our method and frequency-domain spectral element method (SEM) at two different velocities illustrated in Figure 10. The findings indicate that dynamic responses generated through our procedure are nearly identical to those acquired via SEM..

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CONCLUSION

The inquiry pertains to the oscillation of distributed masses that fluctuate in intensity and travel beneath a clamped-clamped non-uniformly prestressed Bernoulli-Euler beam supported by bi-parametric elastic foundation, which is governed by fourth-order partial differential equations with variable and singular coefficients. The principal aim is to derive a definitive solution for this dynamic predicament, particularly when addressing the non-uniformly prestressed Bernoulli-Euler beam that varies throughout its length. The complexity of the governing equation renders finite integral transform unsuitable for its resolution. As a result, Galerkin's method is commonly employed to transform the equation with singular and variable coefficients. The resulting equations from Galerkin are then solved using (i) Laplace transformation and convolution theory to obtain analytical solutions for one-dimensional dynamic problems caused by moving forces, and (ii) finite element analysis in conjunction with Newmark method for cases involving moving masses that cannot be analytically solved due to their harmonic nature. To ascertain the precision of the approach utilized in (i), Fig. 9 illustrates dynamic responses attained through finite element method (FEM) for a clamped-clamped non-uniform Bernoulli-Euler beam, while those obtained from frequency-domain spectral element method (SEM) are displayed in Fig. 10. A thorough analysis is performed on the acquired analytical solutions to identify resonance conditions pertinent to the problems at hand. This study features multiple intriguing aspects, as revealed by numerical analysis:

1. As the axial force values increase, the displacement amplitude of a non-uniformly prestressed Bernoulli-Euler beam that is clamped at both ends and subjected to a uniformly distributed force decreases. This finding remains valid for fixed shear modulus G and foundation stiffness K. The same conclusions and evaluations are derived in the presence of a moving mass.

2. In a dynamic setting, the displacement of a non-uniformly prestressed Bernoulli-Euler beam clamped at both ends and subjected to a moving distributed force decreases as the stiffness of its bi-parametric foundation increases. This correlation holds for constant axial force N and shear modulus G. Comparable results and analyses are demonstrated when dealing with moving mass scenarios.

3. The response amplitude of the clamped-clamped non-uniformly prestressed Bernoulli-Euler beam, under a constant axial force N and foundation stiffness K, decreases as the shear modulus G is increased when subjected to a moving force. Such results also apply in cases involving moving masses. This research presents valuable techniques for resolving dynamic problems related to clamped-clamped non-uniformly prestressed Bernoulli-Euler beams with variable magnitude distributed masses.

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