African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 67-84)
www.abjournals.org

ASSESSING THE VARIANCE OF MAXIMUM LIKELIHOOD ESTIMATES FOR TRUNCATED PSEUDO-LINDLEY POISSON DISTRIBUTION: A SIMULATION-BASED APPROACH

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Cite this article:

Obodo, O. E., Umeh, E. U. (2025), Assessing the Variance of Maximum Likelihood Estimates for Truncated Pseudo-Lindley Poisson Distribution: A Simulation-Based Approach. African Journal of Mathematics and Statistics Studies 8(2), 67-84. DOI: 10.52589/AJMSS-55E9RVCV

Manuscript History

Received: 4 Mar 2025

Accepted: 2 Apr 2025

Published: 13 May 2025

ABSTRACT: *This paper introduces a new lifetime probability distribution called the Truncated Pseudo-Lindley-Poisson Distribution (TPLPD), which generalises the Pseudo-Lindley Distribution and the Poisson distribution. The distribution is a flexible distribution used to model count data with varying degrees of dispersion. This study also investigates the performance of Maximum Likelihood Estimation (MLE) for estimating the parameters of the Truncated Pseudo Lindley Poisson Distribution (TPLPD) through a simulation-based approach. The variance of MLE estimates is assessed under various sample sizes and parameter combinations. The results of the simulation study reveal that the variance of MLE estimates decreases as the sample size increases and that the choice of parameter combinations significantly affects the variance.*

KEYWORDS: Truncated Pseudo-Lindley- Poisson distribution, Maximum Likelihood Estimation, Variance, Simulation Study.

INTRODUCTION

The Pseudo-Lindley Poisson Distribution (PLPD) has various applications in biology in modelling gene expression, species abundance, or count data in ecological studies. Quality Control is used to monitor and control product quality in manufacturing processes. Reliability engineering in modelling failure times, reliability, and maintainability of complex systems. Data has a non-negative integer value (count data). Data requires a flexible distribution with both Poisson and Lindley characteristics. The Truncated Pseudo-Lindley Poisson Distribution (TPLPD), as a truncated version of the Pseudo-Lindley Poisson Distribution, offers a more flexible and realistic approach to modelling count data with upper limits or censoring. A study on the general mathematical properties of the Lindley distribution was offered by Ghitany and Nadarajah (2008), who also applied it to the area of reliability analysis. Since the work of Ghitany and Nadaraja (2008), various generalisations, extensions and modifications of the Lindley distribution have been carried out to suit one or two purposes of real life situations. Indeed, the Lindley distribution has become one of the most widely applied distributions in lifetime analyses. However, due to the fact that the Lindley distribution possesses just a single parameter, the distribution is highly inadequate in modelling the random behaviour of many real life data. Nedjar and Zeghdoudi (2017) offered a two-parameter Lindley distribution and called it the Pseudo-Lindley distribution to increase the flexibility of the Lindley distribution. More parameters in a statistical distribution imply that the distribution can cover a wide range of shapes (Burr, 1942). This enables various data types to be accommodated within the fitting space of a well-parameterized model. In the literature, parameter(s) are added to a given distribution using several approaches. It can be through exponentiation, transformation or by combining two or more probability distributions to form a new one, in which case the collective contribution of each combined distribution makes the new model have more parameters and, in turn, exhibit the behaviour of each of the individual models as well as the combined properties of the generalised model. This is a good realisation. The new distribution formed by the addition of parameters is usually termed “generalised distribution”, “compounded distribution”, “extended distribution”, or “modified distribution” (Mudholkar & Srivastava,1998; Cordeiro, Ortega & Cunha, 2013; Adamidis & Loukas, 1998). For example, Azzalini A. 1985, introduced the skew-normal distribution by introducing an extra parameter to the normal distribution to add more flexibility to the normal distribution.

METHODS

Zeghdoudi and Nedjar (2016) introduced a new variant of the Lindley distribution called the Pseudo-Lindley (PL) distribution with cumulative distribution function (cdf) and probability density function (pdf) given respectively by

$$G_{PL}(x) = 1 - \frac{(\theta + \beta x)e^{-\beta x}}{\theta}\ ,\ \ g_{PL}(x) = \frac{\beta(\theta - 1 + \beta x)e^{-\beta x}}{\theta}, \qquad x > 0, \beta > 0, \theta \geq 1,$$

In this work, a new compound probability distribution is defined by compounding the PL distribution and the truncated Poisson distribution. The truncated Poisson distribution has the probability mass function (pmf) given by$P(N = n) = \frac{\lambda^n}{n!(e^{\lambda}-1)}$; $n = 1,2,3 \dots \dots, \lambda > 0$, where λ is the rate parameter (Noack, 1950). The new compound probability distribution is called the new Truncated Pseudo-Lindley Poisson (TPLP) distribution.

Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

Construction of the Pseudo-Lindley Poisson (TPLP) Distribution

Let $x_1, x_2 \dots \dots \dots x_n$ be independent and identically distributed (iid) random variables from the Pseudo- Lindley (PL) distribution whose cumulative distribution function (cdf) and probability density function (pdf) are given as $G_{PL}(x) = 1 - \frac{(\theta+\beta x)e^{-\beta x}}{\theta}$, $g_{PL}(x) = \frac{\beta(\theta-1+\beta x)e^{-\beta x}}{\theta}$, $x > 0, \beta > 0, \theta \geq 1.$

Suppose N is discrete and follows the Truncated Poisson distribution with pmf given as

$$P(N = n) = \frac{\lambda^n}{n!(e^\lambda - 1)} \quad ; n = 1,2,3 \dots \dots, \lambda > 0$$

Let the conditional cdf of $X_{(1)|N=n}$ be expressed as $G_{X_{(1)|N=n}}(x) = 1 - \prod_{i=1}^{n}[1 - G_{PL_i}(x)]$

Where, then, the $X_{(1)} = min\{x_1, x_2, \dots \dots x_N\}$, Then, the $G_{PL_i}(x) = \quad 1 - \frac{(\theta+\beta x)exp^{-\beta x}}{\theta}$

Since the $X_i{}'$s are identically distributed, it follows that

$$G_{X_{(1)|N=n}}(x) = 1 - [1 - G_{PL}(x)]^n = 1 - \left(1 - \left[1 - \frac{(\theta+\beta x)e^{-\beta x}}{\theta}\right]^n\right)$$

$$= 1 - \left(\frac{\theta+\beta x}{\theta}\right)^n e^{-n\beta x} \qquad x > 0, \beta > 0, \theta \geq 1 \tag{1}$$

The cdf of the new TPLP distribution is the marginal cdf of $X_{(1)}$, is given by

$$\mathrm{F_{PLP}(x)} = \sum_{n=1}^{\infty} \mathrm{P(N = n)} G_{X_{(1)|N=n}}(x) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!(e^\lambda - 1)}\left[1 - \left(\frac{\theta+\beta x}{\theta}\right)^n e^{-n\beta x}\right]$$

where $\sum_{n=1}^{\infty} \frac{\lambda^n}{n!(e^\lambda-1)} = 1$ $\mathrm{F_{PLP}(x)} = 1 - \frac{1}{e^\lambda - 1}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\lambda\left(\frac{\theta+\beta x}{\theta}\right)\right]^n e^{-n\beta x}$

Since $\frac{\lambda^n}{n!(e^\lambda-1)}$, $n = 1, 2, 3, \dots \dots \dots, \lambda > 0$ *is a probability mass function it follows that* $\mathrm{F_{plp}(x)} = 1 - \frac{1}{e^\lambda - 1}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}\right]^n$ *but the Taylor series expansion, it follows that*

$$\mathrm{F_{plp}(x)} = 1 - \frac{1}{e^\lambda - 1}\left[e^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}} - 1\right]: \quad x > 0, \lambda > 0, \beta > 0, \theta \geq 0 \tag{2}$$

Equation (2) gives the cdf of the proposed Truncated Pseudo-Lindley-Poisson distribution which has three parameters in contrast to the Lindley and Pseudo-Lindley distributions which has one and two parameters respectively.

The probability density function of the *Truncated Pseudo-Lindley Poisson distribution* can be obtained by taking the derivative of the cdf shown in equation (2) with respect to the random variable x;

Let the cumulative density function of the newly derived distribution be y such that ;

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$y = F_{plp}(x) \quad = \quad 1 - \frac{e^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}} - 1}{e^{\lambda} - 1}$, The derivative of the function with respect to x is = $\frac{\left(\lambda\beta[\theta - 1 + \beta x]e^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}} - \beta x\right)}{\theta\left(e^{\lambda} - 1\right)}$.Therefore, the probability density function (f(x)) of the distribution is

$$f(x) = \begin{cases} \frac{\left(\lambda\beta[\theta - 1 + \beta x]e^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}} - \beta x\right)}{\theta\left(e^{\lambda} - 1\right)} & for\ x > 0 \\ 0\ elsewhere & \end{cases}$$

Graphical Presentation of the Newly Derived Distribution: Figures 1,2,3 give the various shapes of the cdf of the proposed Truncated Pseudo-Lindley-Poisson distribution.

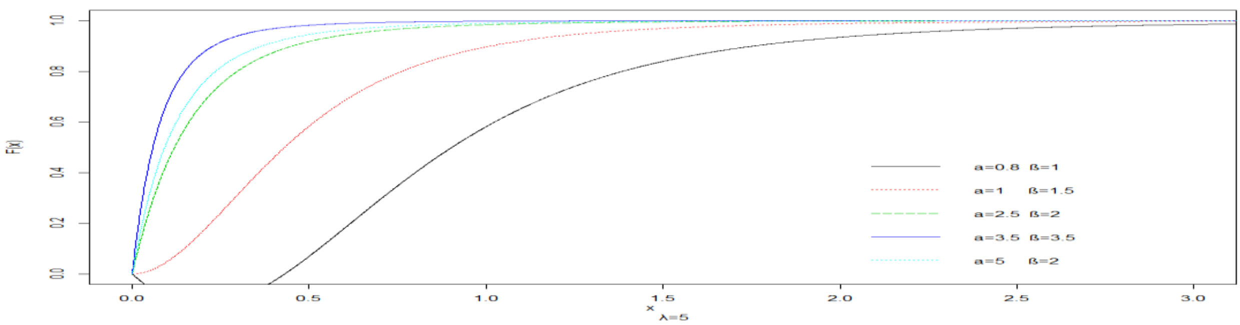

Figure 1: The TPLP cdf for various parameter values a, $\lambda, and\ \beta$

The plot shows that the TPLP distribution is **right-skewed**.

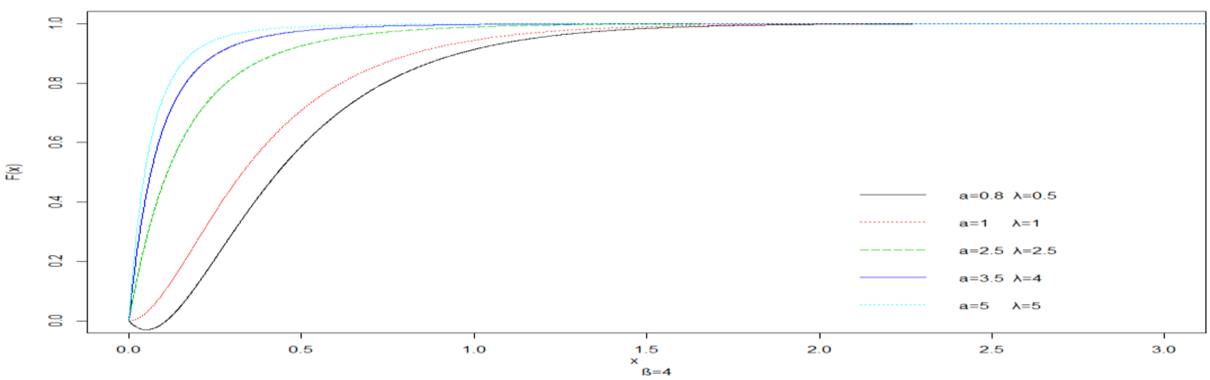

Figure2: The TPLP cdf for various parameter values a, $\lambda, and\ \beta$

The plot also shows that the TPLP distribution is **right-skewed**.

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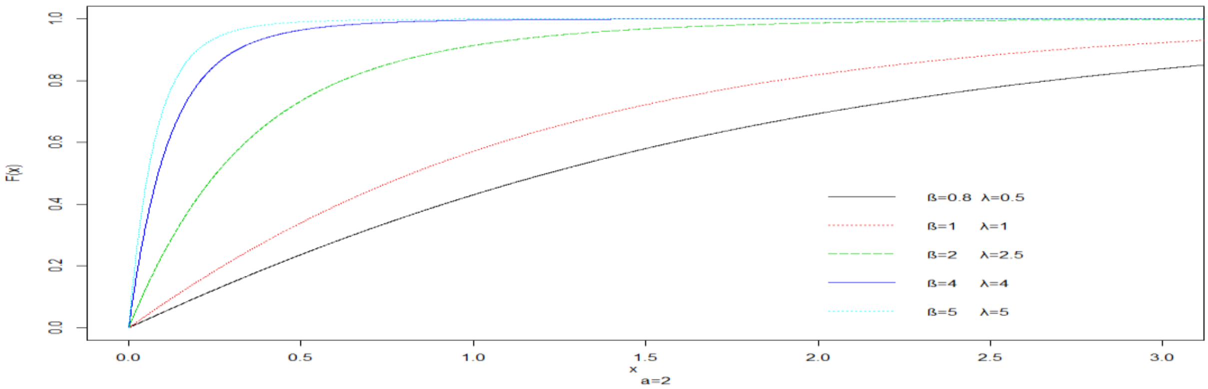

Figure 3: The TPLP cdf for various parameter values a, λ and β

The plot shows that the TPLP distribution is **right-skewed**. As evident from Figures 1-3, the density of the TPLP distribution is always right-skewed and unimodal. This shows that the distribution can be very efficient in fitting right-skewed unimodal data sets.

Density Function

For lifetime data applications, the extra parameter in the proposed Truncated Pseudo-Lindley-Poisson distribution is aimed at adding flexibility to the Lindley and Pseudo-Lindley distributions in fitting data sets with different shape behaviour. Differentiating the cdf with respect to x and gives pdf of the proposed pseudo-Lindley-Poisson distribution as

$$f_{plp}(x) = \frac{\beta\lambda(\theta - 1 + \beta x)e^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}} - \beta x}{\theta(e^{\lambda} - 1)}; \quad where \quad x > 0, \lambda > 0, \beta > 0, \theta \geq 0$$

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NOW, the parameters $\lambda,\ \beta\ and\ \theta$ control the shape of the distribution.

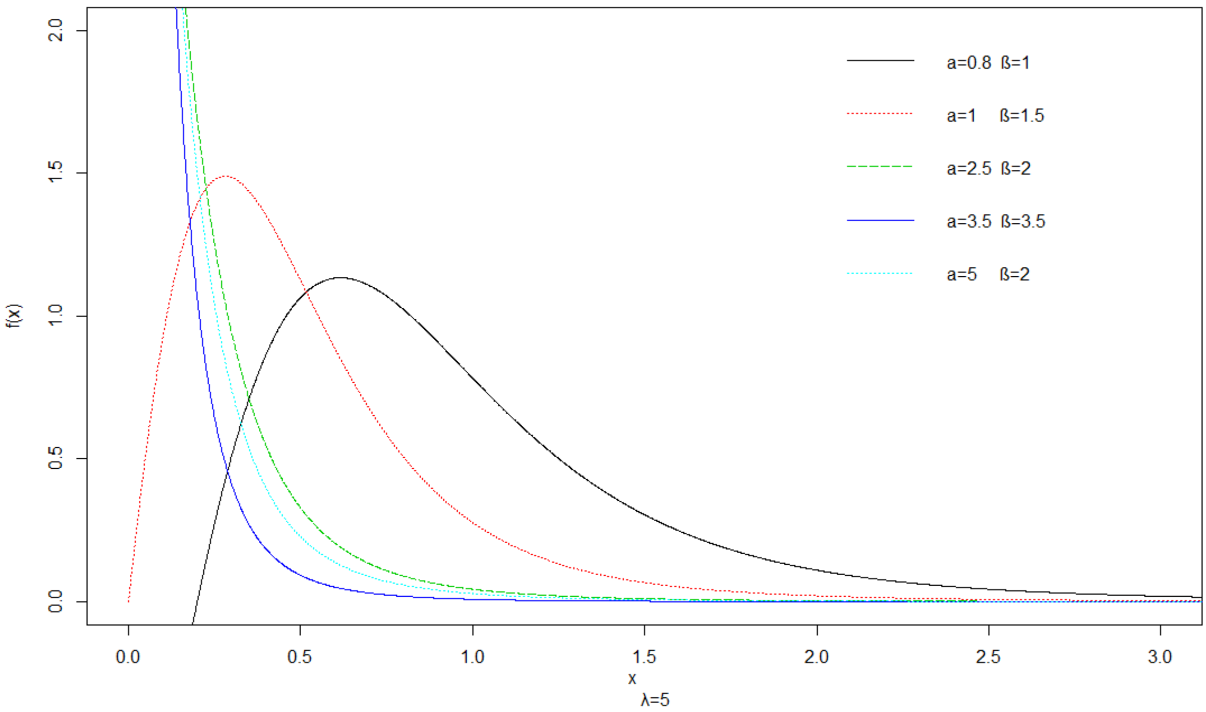

Figure 4: The TPLP density for various parameter values $a, \beta\ and\ \lambda$.

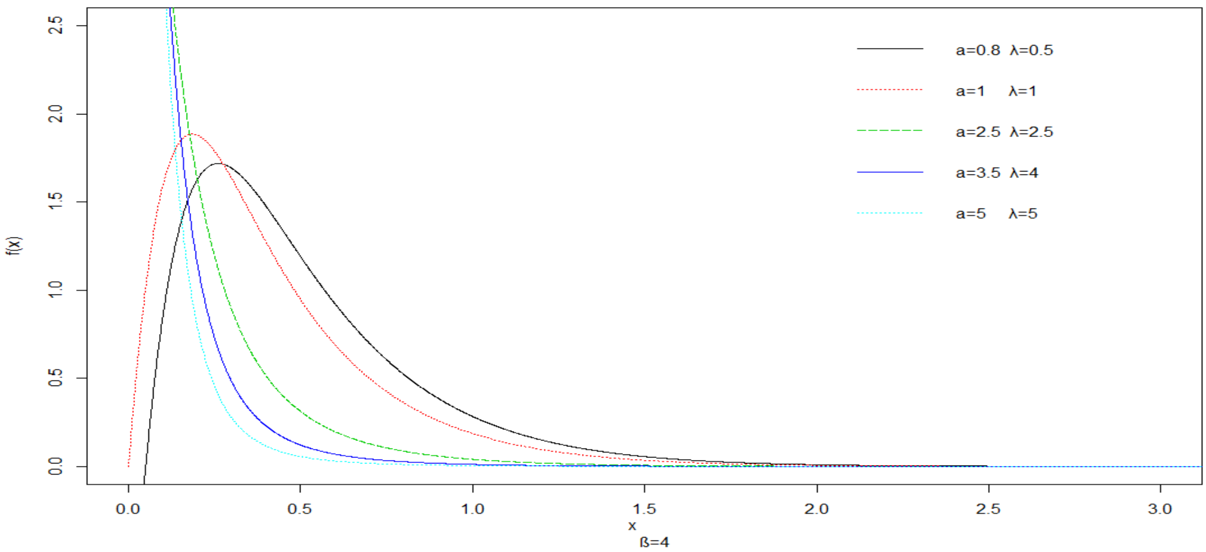

Figure 5: The TPLP density for various parameter values of a, $\lambda\ and\ \beta$

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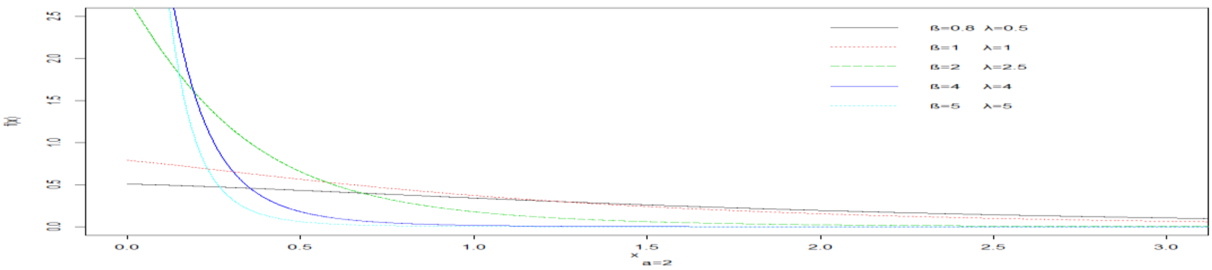

Figure 6: The TPLP density for various parameter values $\lambda,\ \beta\ and$ a

Proposition

The first proposition indicates that the new family has a PL distribution as a limiting case of the TPLP distribution. The PL distribution with parameters is a limiting case of the TPLP distribution when $\lambda \rightarrow\ 0^{+}$

Proof:

Using the Taylor series expansion of the exponential function, the cdf of the TPLP distribution in $F_{plp}(x) = 1-\frac{\exp^{\lambda\left(\frac{\theta+\beta x}{\theta}\right)\exp^{-\beta x}}-1}{\exp^{\lambda}-1}$

can be written as $F_{plp}(x) = 1-\frac{\sum_{n=1}^{\infty}\frac{1}{n!}\left[\lambda\left(\frac{\theta+\beta x}{\theta}\right) e^{-\beta x}\right]^{n}}{\sum_{n=1}^{\infty}\frac{\lambda^{n}}{n!}}$ Considering $\boldsymbol{\lambda}\rightarrow 0^{+}$. Taking the limits of both sides, we have

$$\lim_{\lambda\leftarrow 0^{+}} F_{plp}(x) = 1-\lim_{\lambda\leftarrow 0^{+}}\left[\frac{\sum_{n=1}^{\infty}\frac{1}{n!}\left[\lambda\left(\frac{\theta+\beta x}{\theta}\right) e^{-\beta x}\right]^{n}}{\sum_{n=1}^{\infty}\frac{\lambda^{n}}{n!}}\right]$$

$$1-\frac{\lim_{\lambda\leftarrow 0^{+}}\left\{\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}+\sum_{n=2}^{\infty}\frac{n}{n!}\lambda^{n-1}\left[\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}\right]^{n}\right\}}{\lim_{\lambda\leftarrow 0^{+}}\left\{1+\sum_{n=2}^{\infty}\frac{n\lambda^{n-1}}{n!}\right\}}$$

As 0 is substituted for $\lambda = 1-\frac{\left\{\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}+\frac{2}{2!}(0)^{2-1}\left[\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}\right]^{2}+\cdots\right\}}{\left\{1+\frac{2\,(0)^{2-1}}{2!}\cdots\right\}}$

$= 1-\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}$. $\lim_{\lambda\leftarrow 0^{+}} F_{plp}(x) = 1-\left(\frac{\theta+\beta x}{\theta}\right)e^{-\beta x}$ which is the cdf of the Pseudo Lindley (PL) distribution given by . Hence the proof is established. This shows that the cdf of (PL) of two parameter, $G_{PL}(x) = 1-\frac{(\theta+\beta x)e^{-\beta x}}{\theta}$, is also the same with the cdf of TPLPD with 3 parameter. When parameter λ *tends to zero from the right the new TPLPD will become the PL.*

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Quantile Function

The quantile function Q(p), defined by $F_{Tplp}\big(Q(p)\big) = P$ is the root of the equation; we have

$$1-\frac{e^{\lambda\left(\frac{\theta+\beta(p)}{\theta}\right)e^{-\beta(p)}}-1}{e^{\lambda}-1} = p \quad where \quad 0<p<1$$

We obtain the quantile function as follows: $1-\frac{e^{\lambda\left(\frac{\theta+\beta Q(p)}{\theta}\right)e^{-\beta Q(p)}}-1}{e^{\lambda}-1} = P$

$$= e^{\lambda\left(\frac{\theta+\beta Q(p)}{\theta}\right)e^{-\beta Q(p)}} - 1 = (1-P)\big(e^{\lambda}-1\big)$$

$$= e^{\lambda\left(\frac{\theta+\beta Q(p)}{\theta}\right)e^{-\beta Q(p)}} = (1-P)\big(e^{\lambda}-1\big)+1$$

Divide both sides by λ, $= \left(\frac{\theta+\beta Q(p)}{\theta}\right)e^{-\beta Q(p)} = \frac{1}{\lambda}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]$

Multiply both sides by θ, $= \big(\theta+\beta Q(p)\big)e^{-\beta Q(p)} = \frac{\theta}{\lambda}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]$

If we define $Z\,(p) = -\theta-\beta Q(p) \quad \Longrightarrow \quad z(p)+\theta = -\beta Q(p)$

and $-Z(p) = \theta+\beta Q(p), \quad -z(p)e^{z(p)+\theta} = \frac{\theta}{\lambda}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]$

Hence $z(p)e^{z(p)} = -\frac{\theta e^{-\theta}}{\lambda}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]$

It follows that $z(p) = W\left\{-\frac{\theta}{\lambda}e^{-\theta}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]\right\}$,

where W (.) is the Lambert-W function. (Corless et al.,1996)

Hence

$$-\beta Q(p)-\theta = W\left\{-\frac{\theta}{\lambda}e^{-\theta}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]\right\}, \qquad 0<p<1$$

where W (.) is the negative branch of the Lambert-W function. It follows that the quantile function Q (p) of the TPLP distribution is expressed as

$$Q(p) = -\frac{\theta}{\beta}-\frac{1}{\beta}W\left\{-\frac{\theta}{\lambda}e^{-\theta}\, log\, log\,\left[(1-p)\big(e^{\lambda}-1\big)+1\right]\right\}, \qquad \text{for } 0<p<1.$$

Using the quantile function of the TPLP distribution, the first three quantiles of the TPLP distribution are given respectively by

$$Q_1 = Q\left(\frac{1}{4}\right) = -\frac{\theta}{\beta}-\frac{1}{\beta}W\left\{-\frac{\theta}{\lambda}e^{-\theta}\, log\, log\,\left[\frac{3}{4}\big(e^{\lambda}-1\big)+1\right]\right\}$$

$$Q_2 = Q\left(\frac{1}{2}\right) = -\frac{\theta}{\beta}-\frac{1}{\beta}W\left\{-\frac{\theta}{\lambda}e^{-\theta}\, log\, log\,\left[\frac{1}{2}\big(e^{\lambda}-1\big)+1\right]\right\}$$

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$$Q_3 = Q\left(\frac{3}{4}\right) = -\frac{\theta}{\beta} - \frac{1}{\beta} W\left\{-\frac{\theta}{\lambda} e^{-\theta} \log\log \left[\frac{1}{4}\left(e^{\lambda} - 1\right) + 1\right]\right\}$$

The second quantile, Q,$\left(\frac{1}{2}\right)$ corresponds to the median of the TPLP distribution and is re-written as $M = -\frac{\theta}{\beta} - \frac{1}{\beta} W\left\{-\frac{\theta}{\lambda} e^{-\theta} \log\log \left[\frac{1}{2}\left(e^{\lambda} - 1\right) + 1\right]\right\}$

Samples can be simulated from the TPLP distribution by replacing P with U, where U is a uniform random variable defined on the interval (0, 1). Hence, if X is a random variable from the TPLP distribution, then

$$X = -\frac{\theta}{\beta} - \frac{1}{\beta} W\left\{-\frac{\theta}{\lambda} e^{-\theta} \log\log \left[(1 - U)\left(e^{\lambda} - 1\right) + 1\right]\right\}$$

Maximum Likelihood Estimation of the parameters of the Truncated Pseudo- Lindley-Poisson Distribution

For a random independent sample $x_1, x_2 \ldots\ldots x_n$ *of size n form the TPLP distribution, the maximum likelihood estimation of the parameters of the TPLP distribution involve the maximization of the log likelihood function defined by* $L = \sum_{n=1}^{n} \log f_{plp}(x_i)$

$$= \sum_{e=1}^{n} \log \left[\frac{\beta\lambda(\theta - 1 + \beta x_i) e^{\lambda\left(\frac{\theta + \beta x_i}{\theta}\right) e^{-\beta x_i}} - \beta x_i}{\theta(e^{\lambda} - 1)}\right]$$

$$= \sum_{i=1}^{n} \left\{\log\beta + \log\lambda - \log\theta - \log\left(e^{\lambda} - 1\right) + \log(\theta - 1 + \beta x_i) + \lambda\left(\frac{\theta + \beta x_i}{\theta}\right) e^{-\beta x_i} - \beta x_i\right\}$$

$$= n\log\beta + n\log\lambda - n\log\theta - n\log\left(e^{\lambda} - 1\right) + \sum_{i=1}^{n} \log(\theta - 1 + \beta x_i) + \lambda \sum_{i=1}^{n} \left(\frac{\theta + \beta x_i}{\theta}\right) e^{-\beta x_i} - \beta \sum_{i=1}^{n} x_i$$

Let $\theta = (\theta, \beta, \lambda)$ *be the unknown parameter vector .The associated score function is given by* $U(\theta) = \left(\frac{\partial L}{\partial\theta} \frac{\partial L}{\partial\beta} \frac{\partial L}{\partial\lambda}\right)$*, where* $\frac{\partial L}{\partial\theta}, \frac{\partial L}{\partial\beta}$ *and* $\frac{\partial L}{\partial\lambda}$ *are the partial derivatives of the log-likelihood function w.r.t to each parameter given by*

$$\frac{\partial L}{\partial\theta} = \frac{-n}{\theta} + \sum_{i=1}^{n} \frac{1}{(\theta - 1 + \beta x_i)} - \lambda \sum_{i=1}^{n} \frac{\beta x_i}{\theta^2} e^{-\beta x_i}$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{x_i}{(\theta - 1 + \beta x_i)} + \lambda \sum_{i=1}^{n} \frac{x_i}{\theta} e^{-\beta x_i}(1 - \theta - \beta x_i) - \sum_{i=1}^{n} x_i$$

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} - \frac{ne^{\lambda}}{e^{\lambda} - 1} + \sum_{i=1}^{n} \left(\frac{\theta + \beta x_i}{\theta}\right) e^{-\beta x_i}$$

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The maximum likelihood estimate of $\theta = (\theta, \beta, \lambda)$ can be obtained by solving the non-linear system of equations, $U(\theta) = 0$. Since the equations are not in closed form, the solutions can be found numerically using some specialised numerical optimisation.

Simulation of Data

The simulation study was conducted by generating random samples from the Truncated Pseudo Lindley-Poisson distribution using Monte Carlo simulation. We select eight different sample sizes (n = 20, 30, 50, 100, 200, 300, 500, 1000) to investigate the effect of sample size on the performance of MLE. Two different parameter combinations (θ = 0.2, 0.4; β = 0.3, 0.6, λ = 0.4, 0.8) were used to investigate the effect of parameter values on the performance of MLE. Generate 1000 random samples from the TPLPD for each sample size and parameter combination using the TPLPD generated using R software. The variance of the MLE estimates for each sample size and parameter combination were calculated.

For the determination of stability and homogeneity of the distribution, it is necessary to determine its effectiveness as sample size varies. Using the Monte Carlo simulation approach, varying parameter values and sample sizes yielded are presented in Table 1 below.

Table 1: Simulation Study for the TPLP distribution

θ = 0.2 , β = 0.3, λ = 0.4					
Sample size	**Parameters**	**Estimation**	**Biasness**	**Variance**	**MSE**
20	θ	0.211490	0.0114898	0.0002288	0.0003609
	β	0.246249	-0.0537507	0.0147249	0.0176140
	λ	0.430925	0.0309247	0.0034540	0.0044104
30	θ	0.212430	0.0124304	0.0001578	0.0003123
	β	0.229270	-0.0707295	0.0124339	0.0174365
	λ	0.464509	0.0645094	0.0071615	0.0113230
50	θ	0.216428	0.0164283	0.0001707	0.0004406
	β	0.227604	-0.0723962	0.0148859	0.0201271
	λ	0.431185	0.0311850	0.0062951	0.0072676
100	θ	0.213985	0.0139850	0.0000657	0.0002613
	β	0.227602	-0.0723978	0.0127325	0.0179739
	λ	0.464126	0.0641258	0.0127330	0.0168452
200	θ	0.210518	0.0105181	0.0001064	0.0002170
	β	0.282820	-0.0171799	0.0179628	0.0182580
	λ	0.418992	0.0189918	0.0127201	0.0130808
300	θ	0.202282	0.0022815	0.0000349	0.0000401
	β	0.277753	-0.0222475	0.0089753	0.0094703
	λ	0.478029	0.0780293	0.0063246	0.0124131
500	θ	0.198131	-0.0018686	0.0000873	0.0000908
	β	0.344009	0.0440091	0.0025124	0.0044492
	λ	0.368620	-0.0313799	0.0091554	0.0101401
1000	θ	0.201294	0.0012945	0.0000722	0.0000739

Article DOI: 10.52589/AJMSS-55E9RVCV
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	β λ	0.330686 0.360292	0.0306855 -0.0397083	0.0055191 0.0027592	0.0064607 0.0043359

In the Table 1, the parameters were fixed to be 0.2, 0.3 and 0.4 for θ, β and λ respectively. The distribution was used to estimate the parameter values and biasness of the model was estimated as shown in the columns of the Tables. Observation is that increase in sample size lead to reduction in the Mean Square Error which is an indication of stability of the distribution with respect to sample size.

Table 2: Simulation Study for the TPLP distribution

θ = 0.4 , β = 0.6 , λ = 0.8					
Sample size	**Parameters**	**Estimation**	**Biasness**	**Variance**	**MSE**
20	θ β λ	0.340314 0.692590 0.828832	-0.0596860 0.0925902 0.0288316	0.0380941 0.0109153 0.0709360	0.0416566 0.0194882 0.0717673
30	θ β λ	0.405985 0.686005 0.901725	0.005985 0.086005 0.101725	0.0556197 0.0139806 0.0931763	0.055656 0.021378 0.103524
50	θ β λ	0.323492 0.804089 0.849816	-0.076508 0.204089 0.049816	0.0349632 0.0315989 0.0446076	0.0408166 0.0732511 0.0470892
100	θ β λ	0.336141 0.702576 0.827339	-0.063858 0.102576 0.027339	0.0313021 0.0095763 0.0412628	0.0353801 0.0200982 0.0420102
200	θ β λ	0.345182 0.701375 0.815984	-0.054818 0.101375 0.015984	0.0300832 0.0230356 0.0278940	0.0330882 0.0333125 0.0281495
300	θ β λ	0.353536 0.658707 0.812425	-0.0464636 0.0587072 0.0124250	0.0239268 0.0236134 0.0220002	0.0260857 0.0270599 0.0221545
500	θ β λ	0.347279 0.628547 0.779948	-0.0527209 0.0285471 -0.0200517	0.0185051 0.0212394 0.0147505	0.0212846 0.0220543 0.0151526
1000	θ β λ	0.312692 0.701054 0.763291	-0.087308 0.101054 -0.036709	0.0224830 0.0247386 0.0183577	0.0301057 0.0349505 0.0197053

In Table 2, the parameters were fixed to be 0.4, 0.6 and 0.8 for θ, β and λ, respectively. It can be observed that an increase in sample size leads to a reduction in the Mean Squared Error, which is an indication of the stability of the distribution with respect to sample size. For a better understanding of the stability or homogeneity property of the parameters, values of variances of the parameters were plotted against the sample sizes. A parameter is stable if its variance reduces as the sample size increases. Otherwise, the parameter is not stable. See Figures 1 to 4 below;

Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

Plot of Variance of the Parameters for estimated values of θ = 0.2 , β = 0.3, λ = 0.4

Varying sample sizes were used in order to determine the homogenous property of the parameters in the distribution. Sample sizes 20, 30, 50, 100, 200, 500 and 1000 were used.

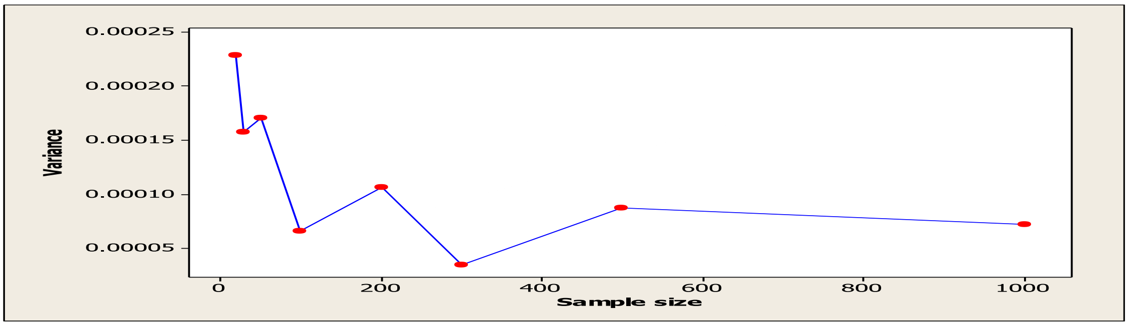

Figure 1: Plot of values of variance for the parameter θ at various sample sizes.

As shown in Figure1 , for all the sample sizes considered, the variance of the estimates for the parameter **are all less than 0.0** which is an indication that the parameter is stable in the distribution. **Also, increase in sample size lead to reduction in the value of the variance which implies the parameter performs better in the model and as sample size increases, the parameter becomes more stable.**

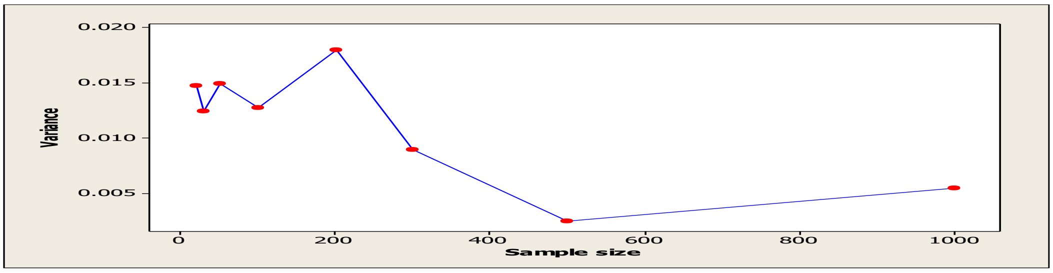

Figure 2: Plot of values of variance for the parameter β

For the parameter β, as shown in Figure 2, for all the sample sizes considered, the variance of the estimates for the parameter are **all less than 0.0** which is an indication that the parameter is stable in the distribution. Also, increase in sample size lead to reduction in the value of the variance which implies the parameter performs better in the model and as sample size increases, the parameter becomes more stable.

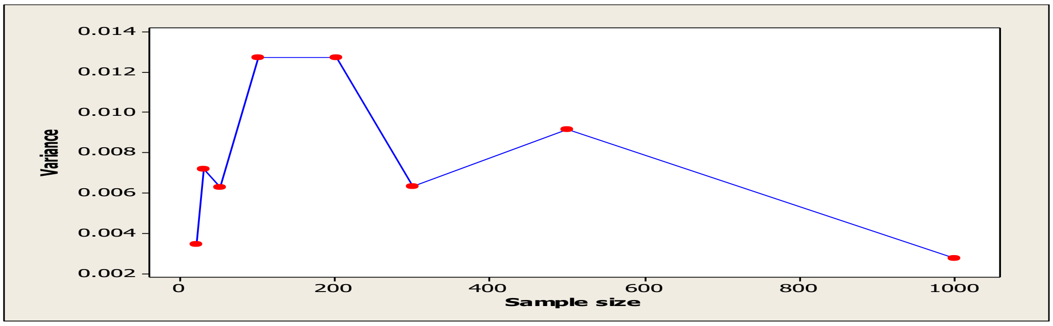

Figure 3: Plot of values of variance for the parameter λ

Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

For λ, the parameter is better for a large sample size as it has a very low variance at a sample size of 1000. Using the parameter values, λ has high fluctuation in terms of the value of variance, but all the values are **less than 0.0,** which is an indication of better stability of the parameter.

A. Plot of Variance of the Parameters for estimated values of θ = 0.4 , β = 0.6 , λ = 0.8

Changing the parameter values to 0.4, 0.6 and 0.8, the following graphs were obtained;

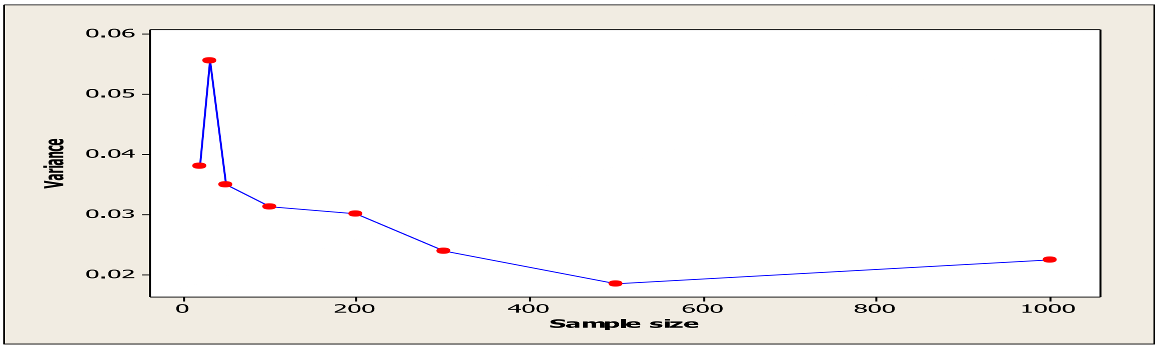

Figure 4: Plot of values of variance for the parameter θ

Figure 4 shows better stability of the parameter θ, and it can be observed that a higher sample size favours the parameter. Therefore, it can be concluded that the parameter is stable in the distribution.

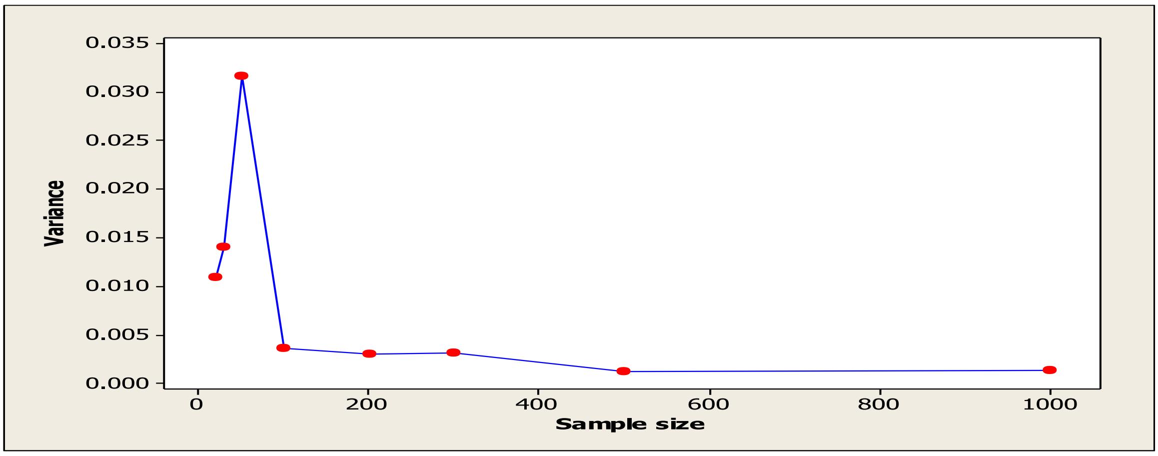

Figure 5: Plot of values of variance for the parameter β

Figure 5 shows better stability of the parameter β, and it can be observed that a higher sample size reduces the variability of the parameter. One can, therefore, conclude that the parameter is stable in the distribution.

Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

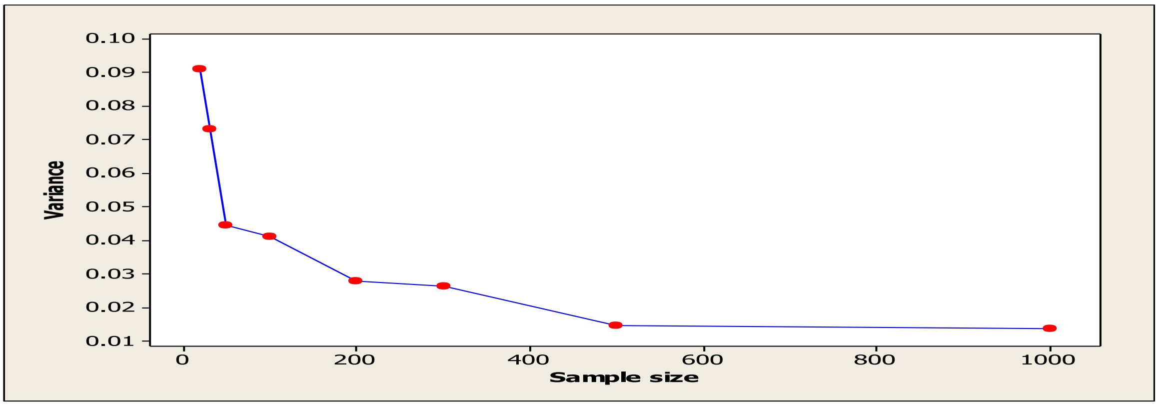

Figure 6: Plot of values of variance for the parameter λ

Figure 6 shows better stability of the parameter λ, and it can be observed that a higher sample size improves the stability of the parameter by lowering its variability. Therefore, it can be concluded that the parameter has a tendency for long-run stability in the distribution.

CONCLUSION

This study provides an evaluation of the maximum likelihood estimate (MLE) for the Truncated Pseudo-Lindley Poisson Distribution (TPLPD), a widely used model for count data with truncation. Through an extensive simulation study, we assessed the performance of the MLE in terms of bias and variance, and explored the impact of sample size and parameter values on its accuracy. Simulation studies revealed that an increase in sample size leads to a reduction in the mean square error, which indicates stability of the distribution and improved accuracy and precision.

Our results demonstrate that the MLE is a reliable and efficient estimator for the TPLPD, particularly for large sample sizes. We found that the MLE exhibits low bias and variance, indicating its accuracy and precision. The variance of the MLE decreases as the sample size increases, highlighting the importance of large samples. Researchers should use large sample sizes when working with the TPLPD to minimise the variance of the MLE. Care should also be taken when interpreting results from small samples, as the variance of the MLE may be large and, hence, unreliable.

Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

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```
#Plotting the Pseudo-Lindley - Poisson (PLP)  Distribution
#=================================================
#Pseudo-Lindley - Poisson (PLP)  Distribution
dplp<-function(x,a,b,c)(b*c*(a-1+b*x)*exp(c*((a+b*x)/a)*exp(-b*x)-b*x)/
                    (a*(exp(c)-1)))
pplp<-function(x,a,b,c) 1-((exp(c*((a+b*x)/a)*exp(-b*x))-1)/(exp(c)-1))

library(LambertW)
library(lamW)
qplp<-function(p,a,b,c) (-a/b)-(1/b)*lambertWm1(-(a/c)*
                                log((1-p)*(exp(c)-1)+1)*exp(-a))
hplp<-function(x,a,b,c)  dplp(x,a,b,c)/(1-pplp(x,a,b,c))
Splp<-function(x,a,b,c) 1-pplp(x,a,b,c)
#a=θ,b=β,c= λ
#==================================================
#PLP Density Plot 1
x<-seq(0,5,0.001)
y1<-dplp(x,0.8,1,5)
y2<-dplp(x,1,1.5,5)
y3<-dplp(x,2.5,2,5)
y4<-dplp(x,3.5,3.5,5)
y5<-dplp(x,5,2,5)
plot(c(0,3),c(0,2),type="n",ylab="f(x)",xlab="x
  λ=5")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
```


Article DOI: 10.52589/AJMSS-55E9RVCV
DOI URL: https://doi.org/10.52589/AJMSS-55E9RVCV

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 67-84)


```
lines(x,y5,col=5,lty=9)
legend("topright",c(
  "α=0.8  β=1  ",
  "α=1    β=1.5 ",
  "α=2.5  β=2",
  "α=3.5  β=3.5",
  "α=5    β=2"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#==================================================
#PLP Density Plot 2
x<-seq(0,5,0.001)
y1<-dplp(x,0.8,4,0.5)
y2<-dplp(x,1,4,1)
y3<-dplp(x,2.5,4,2.5)
y4<-dplp(x,3.5,4,4)
y5<-dplp(x,5,4,5)
plot(c(0,3),c(0,2.5),type="n",ylab="f(x)",xlab="x
     β=4")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("topright",c(
  "α=0.8  λ=0.5  ",
  "α=1    λ=1 ",
  "α=2.5  λ=2.5",
  "α=3.5  λ=4",
  "α=5    λ=5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#============================================================
#PLP Density Plot 3
x<-seq(0,5,0.001)
y1<-dplp(x,2,0.8,0.5)
y2<-dplp(x,2,1,1)
y3<-dplp(x,2,2,2.5)
y4<-dplp(x,2,4,4)
y5<-dplp(x,2,5,5)
plot(c(0,3),c(0,2.5),type="n",ylab="f(x)",xlab="x
     α=2")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("topright",c(
  "β=0.8  λ=0.5  ",
  "β=1    λ=1 ",
```


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```
  "β=2     λ=2.5",
  "β=4     λ=4",
  "β=5     λ=5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#========================================================
#PLP CDF Plot 1
x<-seq(0,5,0.001)
y1<-pplp(x,0.8,1,5)
y2<-pplp(x,1,1.5,5)
y3<-pplp(x,2.5,2,5)
y4<-pplp(x,3.5,3.5,5)
y5<-pplp(x,5,2,5)
plot(c(0,3),c(0,1),type="n",ylab="F(x)",xlab="x
    λ=5")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("bottomright",c(
  "α=0.8  β=1  ",
  "α=1     β=1.5 ",
  "α=2.5  β=2",
  "α=3.5  β=3.5",
  "α=5     β=2"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#======================================================
#PLP CDF Plot 2
x<-seq(0,5,0.001)
y1<-pplp(x,0.8,4,0.5)
y2<-pplp(x,1,4,1)
y3<-pplp(x,2.5,4,2.5)
y4<-pplp(x,3.5,4,4)
y5<-pplp(x,5,4,5)
plot(c(0,3),c(0,1),type="n",ylab="F(x)",xlab="x
    β=4")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("bottomright",c(
  "α=0.8  λ=0.5  ",
  "α=1     λ=1 ",
  "α=2.5  λ=2.5",
  "α=3.5  λ=4",
  "α=5     λ=5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
```


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African Journal of Mathematics and Statistics Studies
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www.abjournals.org

```
#=========================================================
#PLP CDF Plot 3
x<-seq(0,5,0.001)
y1<-pplp(x,2,0.8,0.5)
y2<-pplp(x,2,1,1)
y3<-pplp(x,2,2,2.5)
y4<-pplp(x,2,4,4)
y5<-pplp(x,2,5,5)
plot(c(0,3),c(0,1),type="n",ylab="F(x)",xlab="x
    α=2")
lines(x,y1,col=1,lty=1)
lines(x,y2,col=2,lty=3)
lines(x,y3,col=3,lty=5)
lines(x,y4,col=4,lty=7)
lines(x,y5,col=5,lty=9)
legend("bottomright",c(
  "β=0.8  λ=0.5  ",
  "β=1    λ=1 ",
  "β=2    λ=2.5",
  "β=4    λ=4",
  "β=5    λ=5"),
  col=1:5, bty="n",lty=c(1,3,5,7,9))
#================================================================
```


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