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A NEW TRIANGULAR PROBABILITY MODEL: A COMBINATORIAL MODIFICATION OF PROBABILITY DISTRIBUTIONS

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ABSTRACT: *The paper projects innovating means to probability modeling using integration method and the concept of normalization. It brings to light a probability distribution obtained by combining a probability density function of a distribution and a cumulative distribution function of another continuous probability model, to produce a triangular trend and a half bell-curve trend. Gumbel and exponential distributions are the root-distributions used for this development. The properties of the distribution include moments and related measures, and estimation, studied alongside simulation.*

KEYWORD: Gumbel distribution, Exponential distribution, Product arrangement, Triangular trend.

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INTRODUCTION

Probability models are developed to forecast various life events, providing handy possibilities simulated as data prototype of occurrences. This great advancement is due to some veteran researchers for their great stride in the development of probability models that exhibit various shapes including increasing, decreasing, bathtub, inverted bathtub, uniform, skewed, and bell-curve shapes. These make up different possible trends there is, and can be projected by some distributions including exponential, Gumbel, gamma, Weibull, Pareto, beta, Lomax, logistic, Burr XII, Gompertz, uniform Kumaraswamy, Lindley, Fréchet distribution among others, for the continuous dimension.

Amidst the various trends there is, Weibull (1939) championed the modern formulation and widespread use of the triangular probability model in statistics; and De Moivre (1718) and Laplace (1812) also made notable contributions in this regard. This probability model is categorized under the continuous distribution with various expressions of the probability density function (PDF) given as

$$\begin{cases} 0 & for\ x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & for\ a \leq x < c \\ \dfrac{2}{b-a} & for\ x = c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & for\ c \leq x < b \\ 0 & for\ b < x \end{cases}$$

where a is the lower limit, b is the upper limit, and c is the mode, such that $a \in (-\infty, \infty)$, $a < b$, $a \leq c \leq b$, and the variable support $a \leq x \leq b$. The trends of this kind of PDF can vary; however, symmetricity can be realized when $c = (a+b)/2$. Triangular distribution (TD) is typically used as a subjective description of a population for which there is only limited sample data, and especially in cases where the relationship between variables is known but data is scarce probably due to high cost of collection. Another application of TD alongside PERT distribution is found in project management to model events which occur within an interval defined by a minimum and maximum value. More so, in corporate finance, it is often used for business decision making, particularly in simulations; whereas the symmetric version of the distribution is commonly applied in audio dithering.

This research is not intended to leverage the limitations of the triangular distribution reviewed above, for a possible sufficiency; however, it seeks to show an alternative development for the realization of the same patterns of trend. This will be achieved combining two continuous distributions over a method of integration and normalizing constant. Howbeit, we consider exponential distribution and Gumbel distribution for this combinatorial development, recalling that the former is a decreasing trend memoryless distribution, and the latter is right skewed, and has strong relationship with extreme value theory.

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MATERIALS AND METHOD

Here, we investigate a probability model development technique, which composes the integration approach and normalizing constant. The mathematical combination of functions and operators will be studied, amidst the various methods captured in Lai (2013). This widely used method of probability distribution development allows us the privilege to exhaustively maximize possible means to achieve convergence of the combined probability functions. It is interesting to note that in this combinatorial development, we have plethora of choices with respect to the employment of mathematical operations.

The selection of the models could be homogeneous, in the sense that a model can be used to form these combinations; and at the other hand non-homogeneous, such that different probability models can be adopted. More so, the choice of the models for these constructions cuts across either of the pdf(s) or the cumulative distribution function CDF(s). The enabling factor to this developmental possibility is in the integrability of the continuous probability models.

Homogeneous Combinations include:

$$g(x) = f(x)\, f(x), \qquad \frac{f(x)}{f(x)}, \qquad f(x) \pm f(x)$$

$$g(x) = f(x)\, F(x), \qquad \frac{f(x)}{F(x)}, \qquad f(x) \pm F(x)$$

$$g(x) = F(x)\, F(x), \qquad \frac{F(x)}{F(x)}, \qquad F(x) \pm F(x)$$

where Non-homogeneous Combinations include:

$$g(x) = f(x)\, t(x), \qquad \frac{f(x)}{t(x)}, \qquad f(x) \pm t(x)$$

$$g(x) = f(x)\, T(x), \qquad \frac{f(x)}{T(x)}, \qquad f(x) \pm T(x) \tag{1}$$

$$g(x) = F(x)\, T(x), \qquad \frac{F(x)}{T(x)}, \qquad F(x) \pm T(x)$$

where $f(x), t(x),\ F(x)\ and\ T(x)$ are the PDFs and CDFs of arbitrary probability models.

Although these combinations represent probability models, they can be treated like any other arbitrary mathematical functions, provided they are integrable. The idea of upper and or lower censoring comes handy when we run into complex cases where some combinations prove difficult to converge (integrate). Now, by censoring, we mean the adjustment of either the lower or upper bound, or both of them. These are referred to as left, right and double censoring $(C_r,\ C_l\ and\ C_d)$ respectively; where for $\int_{-\infty}^{\infty} g(x)dx$, we possible have

$$\int_{-\infty}^{x_{max}} g(x)dx \rightarrow\ C_r \qquad \int_{x_{min}}^{\infty} g(x)dx \ \rightarrow\ C_l$$

$$\int_{x_{min}}^{x_{max}} g(x)dx \ \rightarrow\ C_d$$

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A New Triangular Distribution and Properties

In this section, we develop a new probability distribution using possible functions from exponential and Gumbel distributions. We employ the case of left censoring l_{cs} using the product combination in equation (1) $g(x) = f(x)\,T(x)$, which represents a combination of a PDF and CDF from different distributions. This choice of combination follows from many other failed attempts to realize convergence from $g(x,\vartheta)\,t(x,\beta)$ or $G(x,\vartheta)\,T(x,\beta)$. Now, let $f(x,\vartheta) \sim Exponential[\theta]$ and $T(x,\beta) \sim Gumbel[0,1]$; then we realize a new PDF as:

$$g(x,\theta) = f(x,\theta)\,T(x) = \left(\theta e^{-\theta x}\right)\left(1 - e^{-e^{x}}\right) \tag{2}$$

Integrating equation (2) and applying the concept of normalization, the PDF is obtained as

$$\rightarrow \qquad p(x,\theta) = \frac{\theta e^{-\theta x}\left(1-e^{-e^{x}}\right)}{1-\theta\,\Gamma[-\theta,\ 1]}; \quad x > 0,\ \theta > 0 \tag{3}$$

where $\Gamma[-\theta,\ 1]$ is an incomplete gamma function.

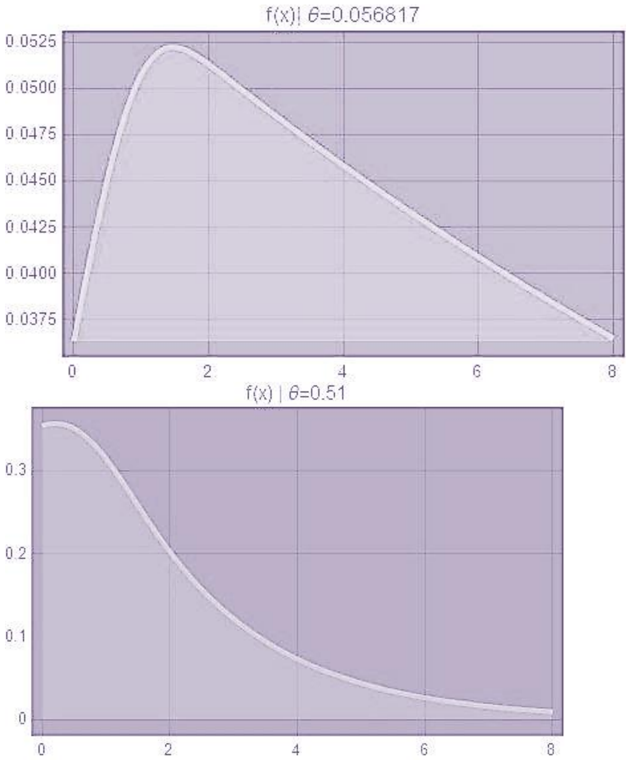

Figure 1: Graph plots of the PDF

Remark

The shapes obtained in Figure 1 are products of the combination of the CDF of Gumbel and the PDF of exponential distributions. The derived distribution is left censored, with consideration given to the Gumbel distribution with the support range $x \in R$. These distributions are primarily characterized by right skewed and monotone decreasing trend respectively; however, their combination yields triangular and half-bell-curve trend, which is majorly not a kind in their individual trends. The half-bell-curve trend can co-model

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scenarios where truncated normal distribution fits. We would refer to this development as Gumbel-Exponential Distribution (GED).

Moment

If X is a random variable from a continuous distribution and $g(x)$ is the density function, then the r^{th} moment about the origin of X is defined by

$$E(X^r) = \int_0^\infty x^r g(x)dx = \mu_r'.$$

Now, for the moments of GED, we recall equation (3) and obtain

$$E(X^r) = \frac{\theta}{1-\theta\,\Gamma[-\theta,\ 1]}\ \int_0^\infty x^r\left[e^{-\theta x}\left(1-e^{-e^x}\right)\right]dx, \tag{4}$$

From equation (4), no closed-form solution is available due to the complexity of the integrand. However, the integral can be approximated using numerical methods or analyzed using asyptotics.

Of course, the first four r^{th} moments of the GED can be obtained substituting $r = 1,2,3\ and\ 4$ in equation (4) before integrating, hence

$$\mu_1' = \frac{1-\theta^2\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,-\theta\},\{\}\},1]}{\theta-\theta^2\text{ExpIntegralE}[1+\theta,1]} = \mu,$$

$$\mu_2' = \frac{2(-1+\theta^3\text{MeijerG}[\{\{\},\{1,1,1\}\},\{\{0,0,0,-\theta\},\{\}\},1])}{\theta^2(-1+\theta\text{Gamma}[-\theta,1])},$$

$$\mu_3' = \frac{(6(-1+\theta^4\text{MeijerG}[\{\{\},\{1,1,1,1\}\},\{\{0,0,0,0,-\theta\},\{\}\},1]))}{(\theta^3(-1+\theta\text{Gamma}[-\theta,1]))}$$

$$\mu_4' = \frac{(24(-1+\theta^5\text{MeijerG}[\{\{\},\{1,1,1,1,1\}\},\{\{0,0,0,0,0,-\theta\},\{\}\},1]))}{(\theta^4(-1+\theta\text{Gamma}[-\theta,1]))}.$$

Therefore, the variance, skewness and kurtosis of the GED can further be obtained

$$Variance(\mu_2) = \mu_2' - \mu^2 = \sigma^2 \qquad Skewness\ (S_k) = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

$$Kurtosis\ (K_s) = \frac{\mu_4}{(\mu_2)^2} = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\mu_2' - \mu^2)^2}$$

Estimation

In this section, we obtain the parameter values of the probability function that maximizes the likelihood function. Kinaci et al. (2014) and Fang et al. (2015) studied maximum likelihood estimation (MLE) under different data conditions including uncensored data and censoring. Censoring is a concept that describes the timing, in which data are recorded during a procedural observation; in the sense that truncated events are not treated like exhaustive investigations. In general, the likelihood functions for the both data conditions are respectively given as:

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$$L(x,\theta) = \prod_{i=1}^{n}[g(x_i)]$$

$$L(x,\theta) = \frac{n!}{(N-n)!}\{\prod_{i=1}^{n} g(x_i)\}\{1 - G(x_T)\}^{N-n} \quad (5)$$

where $g(x_i)$ *and* $G(x_i)$ are the PDF and CDF of a distribution with an independent random observations $x_i, i = 1, 2, \dots, n$; N is the number of specimens being investigated or number of trials. Now, if the fixed time or cycle or count to an event (say failure time) is x_0, then for Type 1 censoring according to equation (5), the time of termination $x_T = x_0$; and $x_T = x_n$ for Type 2 case. However, for this study, we have our emphasis only on the complete or uncensored investigations. Let $x_i, i = 1,2, \dots, n$ is a vector of observations from GED, then the log-likelihood for the complete data is defined by:

$$l(x,\theta) = \log L(x,\theta) = \sum_{i=1}^{n} \log\{f(x,\theta)\}$$

$$l(x,\theta) = \sum_{i=1}^{n} \log\left\{\frac{\theta e^{-\theta x}\left(1-e^{-e^{x}}\right)}{1-\theta\,\Gamma[-\theta,\ 1]}\right\}$$

$$= \log\left[\frac{\theta}{1-\theta\,\Gamma[-\theta,1]}\right]^{n} + \sum_{i=1}^{n}\log\left[1 - e^{-e^{x_i}}\right] - \theta\sum_{i=1}^{n} x_i$$

$$= n\log\theta - n\log(1 - \theta\,\Gamma[-\theta,\ 1]) + \sum_{i=1}^{n}\log\left[1 - e^{-e^{x_i}}\right] - \theta\sum_{i=1}^{n} x_i \quad (6)$$

The score function for equation (6) is defined by

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \frac{n\left(\theta\mathcal{M}_G - \Gamma(-\theta,1)\right)}{1-\theta\,\Gamma[-\theta,1]} - \sum_{i=1}^{n} x_i$$

$$(1 - \theta\,\Gamma[-\theta,\ 1]) - \theta\left(\theta\mathcal{M}_G - \Gamma(-\theta,1)\right) - \theta(1 - \theta\,\Gamma[-\theta,\ 1])\hat{x} = 0$$

Where $\mathcal{M}_G = \text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,-\theta\},\{\}\},1]$

A numerical analysis like Newton-Raphson iterative method, which is a root finding algorithm, can be used to obtain the MLEs of $\hat{\theta}$. This scheme is given by

$$\hat{\theta} = \theta - H^{-1}(\theta)\,S(\theta)$$

where $S(\theta)$ is the score function and $H^{-1}(\theta)$ is the second derivatives of the log-likelihood function termed the Hessian matrix, Bayal et al. 2022. Finally, it is expected that the different methods show efficiency with respect to the selected size of data samples under consideration.

Simulation Study

Furthermore, the asymptotic character of the maximum likelihood estimates of the parameters of Gumbel-exponential distributions is investigated, through Monte Carlo simulation study. For different sample sizes $n = 20,\ 50,\ 75,\ 100\ \&\ 250$, a 10000 times trials is carried out; and the steps are given by the algorithm:

i) Choose a value M (which represents the number of Monte Carlo trials).

ii) Select the values θ_0 within the domain of their parameter supports.

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iii) Simulate a sample of size n from the derived distributions.

iv) Compute the maximum likelihood estimates $\hat{\theta}_k$ of θ_k

v) Redo steps 3-4 for N number of times

vi) The computations of the following measures are obtained:

$$\text{Average Bias} = \left[\ \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i - \theta\right)\right]\ and$$

$$\text{MSE} = \left[\ \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \sigma)^2\ \right]$$

Table 1: Average Bias and MSE of the (GED) Estimator $\hat{\boldsymbol{\theta}}$

Parameter	n	Average Bias (θ)	MSE (θ)
	20	3.5e-15	4.7921
	50	3.1e-15	2.1045
$\theta = 0.1$	75	4.6e-15	5.2365
	100	1.9e-14	0.2004
	250	1.4e-14	0.0158
	20	-1.1e-14	2.0365
	50	-2.8e-14	1.5867
$\theta = 0.5$	75	-4.8e-14	0.5876
	100	-8.5e-14	0.2365
	250	-3.1e-14	0.4986
	20	3.8e-15	0.3437
	50	3.3e-15	0.3014
$\theta = 1.5$	75	2.5e-15	0.2358
	100	1.2e-15	0.1356
	250	-4.4e-16	0.0258
	20	5.1e-15	0.2145
	50	6.2e-15	0.3254
$\theta = 2.5$	75	-9.7e-15	0.0214
	100	-7.7e-14	0.0145
	250	-5.5e-13	0.0015

The estimates for the average bias and mean square error are presented in Table 1, and at different selected values of the parameter. Apparently, from the Table, we deduce that the estimates of the average bias and mean square error decrease as the sample size n increases. This simply indicates that the estimators of the derived distributions are consistent and asymptotically stable.

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CONCLUSION

The paper presents some methodical approaches to the development of probability models; and a rare trend in the theory of probability, which features triangular trend. The development is termed the New Gumbel-Exponential Distribution. This development stems from an integrative combination of two different continuous probability models, realized over the mathematical principle of normalization. The combination is composite of the Gumbel distribution with right skewed shape and the exponential distribution with decreasing shape; where half-bell curve trend is another shape obtainable from the proposed model. Properties like moments and estimation were studied alongside simulation, underscoring the behavior of the parameters. It is recommended that further studies be carried out on the properties and applications of this development.

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