African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 129-140)
www.abjournals.org

GLOBAL SOLUTION EXISTENCE AND EVENTUAL STABILITY FOR NONLINEAR SYSTEMS WITH IMPULSES

Atsu J. U.[1], Ante J. E.[2*], Ambrose U. U.[3], Essang S. O.[4],

Aigberemhon E. M.[5], Oyong S. B.[6], and Francis R. E.[7]

[1]Department of Mathematics, University of Cross River State, Calabar.
Email: jeremiahatsu@unicross.edu.ng

[2]Department of Mathematics, Topfaith University, Mkpatak, Nigeria.
Email: jackson.ante@topfaith.edu.ng

[3]Department of Mathematics, University of Calabar, Calabar.
Email: ambroseuwem@gmail.com

[4]Department of Mathematics and Computer Science, Arthur Jarvis University, Akpabuyo.
Email: samuelessang@arthurjarvisuniversity.edu.ng

[5]Department of Electrical Engineering, University of Cross River State, Calabar.
Email: moscomag2k2@gmail.com

[6]Department of Computer Science, Topfaith University, Mkpatak, Nigeria.
Email: samuel.oyong@topfaith.edu.ng

[7]Department of Statistics, Federal Polytechnic, Ugep, Nigeria.
Email: runyifrancis@fedpolyugep.edu.ng

[*]Corresponding Author's Email: jackson.ante@topfaith.edu.ng

Cite this article:

Atsu, J. U., Ante, J. E., Ambrose, U. U., Essang, S. O., Aigberemhon, E. M., Oyong, S. B., Francis, R. E. (2025), Global Solution Existence and Eventual Stability for Nonlinear Systems with Impulses. African Journal of Mathematics and Statistics Studies 8(2), 129-140. DOI: 10.52589/AJMSS-USLHJ4TC

Manuscript History

Received: 13 May 2025

Accepted: 15 Jun 2025

Published: 11 Jul 2025

ABSTRACT: *In this paper, the existence of maximal solution of a comparison differential system for a class of piecewise continuous Lyapunov function is examined. Alongside, the eventual stability for nonlinear impulsive differential equations with fixed moments of impulse is established using the Lyapunov functions. Together with comparison results, sufficient conditions for eventual stability are presented. Results obtained are extension and improvements on existing results in the literature.*

KEYWORDS: Eventual stability; impulsive differential equation; Lyapunov functions.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

INTRODUCTION

Undoubtedly, one of the central concerns in the qualitative theory of differential equations is the stability of solutions. Stability allows for the comparison of the behavior of solutions originating from different initial conditions (see [1]).

Traditionally, the Lyapunov method has been extensively used to investigate the stability of differential equation solutions. In many practical scenarios, it becomes necessary to study the stability of sets that are invariant under the dynamics of a given system. However, this requirement often excludes classical Lyapunov stability [23]. To address this limitation, a new concept known as *eventual stability* was introduced (see [14] and the references therein). This concept posits that a set may not be invariant in the traditional sense but is invariant in an asymptotic sense (see also [25]).

The theory of impulsive differential equations (IDEs) is more comprehensive than that of conventional differential equations [11], as it models numerous real-world phenomena more accurately. Many natural and engineered processes experience sudden state changes at specific time moments. These abrupt shifts are due to short-term disturbances whose durations are negligible relative to the total duration of the process. Therefore, it is reasonable to model such disturbances as instantaneous impulses. For example, impulsive effects are observed in biological threshold phenomena, bursting rhythms in medical and biological systems, economic optimal control models, pharmacokinetics, and frequency-modulated systems [11].

Effective application of impulsive differential systems hinges on the ability to establish stability criteria for their solutions and one of the most robust tools for analyzing the stability of impulsive systems is the Lyapunov function method. Among various methods for assessing stability—such as the Razumikhin technique, matrix inequalities, variational methods, Banach's fixed point theorem, and monotone iteration—the second method of Lyapunov stands out. Its primary advantage lies in enabling the study of stability without needing to solve the system explicitly. This method involves finding a suitable continuously differentiable Lyapunov function that is positive definite and has a non-positive derivative along system trajectories. The stability of trivial solutions in impulsive systems has been thoroughly explored in works such as [3, 6, 20, 21].

The novelty of this paper is two dimensional: establishing the existence of the maximal solution for the associated comparison system and investigating the eventual stability for a nonlinear impulsive differential system. Using vector Lyapunov functions—generalized through a class of piecewise continuous Lyapunov functions—along with comparison principles, the paper derives sufficient conditions for ensuring the eventual stability of the system's solutions.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 129-140)
www.abjournals.org

PRELIMINARY NOTES AND DEFINITIONS

Let R^n be the n-dimensional Euclidean space with norm $\|.\|$. Let Ω be a domain in R^n containing the origin; $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, $t_0 \in R_+$, $t > 0$.

Let $J \subset R_+$. Define the following class of functions is a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t_k^+)$ exists.

Consider the impulsive differential system

$$\begin{aligned} x' &= f(t,x), t \neq t_k,\ t \geq t_0 \\ \Delta x &= I_k(x),\ t = t_k, k \in N \\ x(t_0^+) &= x_0 \end{aligned} \qquad 2.1)$$

under the following assumptions:

A_0 (i) $0 < t_1 < t_2 < \ldots < t_k < \ldots,$ and $t_k \to \infty$ $k \to \infty$;

(ii) $f : R_+ \times R^n \to R^n$ is continuous in $(t_{k-1}, t_k] \times R^N$ and for each $x \in R^n$, $k = 1,2,\ldots,$

$$\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x) \text{ exists;}$$

(iii) $I_k : R^n \to R^n$

In this paper, we assume that the function f is Lipschitz continuous with respect to its second argument, and $f(t,0) \equiv 0,\ I_k(0) \equiv 0$ for all k, so that we have the trivial solution for (2.1), and the points $t_k, k = 1,2,\ldots$ are fixed such that $0 < t_1 < t_2 < \ldots and \lim_{k\to\infty} t_k = \infty.$ The system (2.1) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t;t_0,x_0) \in PC([t_0,\infty), R^N)$. Note that some sufficient conditions for the existence and uniqueness of the global solutions to (2.1) are considered in [9, 15, 16, 18].

Remark 2.1.

The second equation in (2.1) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 2.1.

Let $V : R_+ \times R^N \to R_+^N$ be a continuous mapping of $R_+ \times R^N$ into R_+^N. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1}, t_k] \times R^N$ and for each $x \in R^N$, $k = 1,2,\ldots,$ and $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ exists;

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

(ii) V is locally Lipschitz with respect to its second argument x. For $(t_{k-1,}\ t_k]\times R^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^{+}V(t,x)=\limsup_{h\to 0^{+}}\tfrac{1}{h}[V(t+h,x+hf(t,x))-V(t,x)] \qquad (2.2)$$

Note that in (2.1), $D^{+}V(t,x)$ is a functional whereas V is a function.

Definition 2.2.

A function $g(t,u)$ is said to be quasimonotone nondecreasing in u, if $u\le v$ and $u_i=v_i$ for $1\le i\le N$ implies $g_i(t,u)\le g_i(t,v)$ for all i.

Definition 2.3.

The trivial solution $x=0$ of (2.1) is said to be

(ES1) eventually stable if for every $\in>0$ *and* $t_0\in R_+$ there exist $\delta=\delta(\in,t_0)>0$ such that for any $x_0\in R^n$ the inequality $\|x_0\|<\delta$ implies $\|x(t,t_0,x_0)\|<\in$ *for* $t\ge t_0$;

(ES2) uniformly eventually stable if for every $\in>0$ *and* $t_0\in R_+$ there exist $\delta=\delta(\in)>0$ such that for any $x_0\in R^n$, the inequality $\|x_0\|<\delta$ implies $\|x(t,t_0,x_0)\|<\in$ *for* $t\ge t_0$;

(ES3) asymptotically eventually stable if it is stable and if for each $\in>0$ *and* $t_0\in R_+$ there exist positive numbers $\delta_0=\delta_0(t_0)>0$ and $T=T(t_0,\in)$ such that for $t\ge t_0+T$ and $\|x_0\|\le\delta$ we have $\|x(t,t_0,x_0)\|<\in$.

(ES4) uniformly asymptotically eventually stable if it is uniformly stable and $\delta_0=\delta_0(\in)$ and $T=T(\in)$ such that for $t\ge t_0+T$, the inequality $\|x_0\|\le\delta$ implies $\|x(t,t_0,x_0)\|<\in$.

Definition 2.4.

A function $a(r)$ is said to belong to the class K if $a\in C[[0,\rho),R_+],a(0)=0,$ *and* $a(r)$ is strictly monotone increasing in r.

In this paper, we define the following sets:

$$\bar{S}_{\psi}=\{x\in R^N:\|x\|\le\psi\}$$

$$S_{\psi}=\{x\in R^N:\|x\|<\psi\}$$

Remark 2.2.

The inequalities between vectors are understood to be component-wise inequalities.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

Definition 2.5.

A function $b(r)$ is said to belong to a class L if $b \in C[J, R_+], b(t)$ is monotone decreasing in t *and* $b(t) \to 0$ *as* $t \to \infty$.

Definition 2.6.

A function $a(t,r)$ is said to belong to the class KK if $a \in C[[0,\rho), R_+], a \in K$ for each $t \in J$, and a is monotone increasing in t for each $r > 0$ and $a(t,r) \to \infty$ *as* $t \to \infty$ for each $r > 0$.

Definition 2.7.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be positive definite if there exists a function $a \in K$ such that the relation $V(t,x) \geq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.8.

A function $V(t,x)$ with $V(t,0) = 0$ is said to be negative definite if there exists a function $a \in K$ such that the relation $V(t,x) \leq -a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.9.

A function $V(t,x) \geq 0$ is said to be decrescent if there exists a function $a \in K$ such that the relation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$.

Alongside (2.1), we shall consider a comparison system of the form

$$\begin{aligned} u' &= g(t,u), t \neq t_k,\ t \geq t_0, k = 1,2,... \\ \Delta u &= \psi_k(u(t_k)),\ t = t_k, \\ u(t_0^+) &= u_0 \end{aligned} \tag{2.3}$$

existing for $t \geq t_0$, where $u \in R^n$, elation $V(t,x) \leq a(\|x\|)$ is satisfied for $(t,x) \in J \times S_\rho$. existing for $t \geq t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g: R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t, t_0, u_0) \in PC([t_0, \infty), R^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [9,18, 24, 26].

Lemma 2.1. Assume that the hypotheses $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0$. Then the interval J can be extended to the maximal interval of existence $[t_0, \infty)$.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

Proof.

Since the conditions $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0,$ then from the existence theorem for the equation $x' = f(t, x(t))$ [18] with impulses, it follows that the solution $x(t) = x(t, t_0, x_0)$ of the IVP (2.1) is defined on each of the intervals $(t_{k-1,}\ t_k], k = 1,2,...$ Again, since $0 < t_1 < t_2 < ... and \lim_{k\to\infty} t_k = \infty,$ then we conclude that the interval J can be continued to $[t_0, \infty)$ for t_0.

MAIN RESULTS

In this section we begin by proving the comparison results, then proceed to establish the necessary conditions for the eventual stability of the set $x = 0$ of the impulsive differential systems with fixed moments of impulse.

Using (2.3), Definition 2.2 can be analogously defined as follows:

Definition 3.1.

The trivial solution $u = 0$ of (2.3) is said to be

(ES_1^*) eventually stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|u_0\| < \delta$ implies $\|u(t, t_0, u_0)\| < \varepsilon\ for\ t \geq t_0;$

(ES_2^*) uniformly eventually stable if the δ in (S_1^*) above is independent of t_0 i.e. for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|u_0\| < \delta$ implies $\|u(t, t_0, u_0)\| < \varepsilon\ for\ t \geq t_0;$

(ES_3^*) equiasymptotically eventually stable if S_1^* is satisfied and given $\varepsilon > 0\ and\ t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that for $t \geq t_0 + T$ and $\|u_0\| \leq \delta$ we have $\|u(t, t_0, u_0)\| < \varepsilon,\ t \geq t_0 + T.$

(ES_4^*) uniformly asymptotically eventually stable if (ES_2^*) is satisfied (ES_3^*) is independent of t_0.

THEOREM 3.1. (Comparison results). Assume that:

i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,\ and\ g(t,u)$ is quasimonotone nondecreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+, u)$ exists;

ii) $r(t) = r(t, t_0, u_0) \in PC([t_0, T), R^n)$ is the maximal solution of (2.3) existing for $t \geq t_0$.

iii) $V \in C[J \times S_\psi, R_+^N], V \in L$ such that for $(t, x) \in J \times S_\psi$

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 129-140) www.abjournals.org

$D^{+}V(t,x) \le g(t,V(t,x)),\ t \ne t_k$

and

$V(t, x + I_k(x(t_k)) \le \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iv) $x(t) = x(t,t_0,x_0) \in PC([t_0,T],R^N)$ is a solution of (2.1) such that

$$V(t_0,x_0) \le u_0 \quad (3.1)$$

existing for $t \ge t_0$. Then

$$V(t,x(t)) \le r(t) \quad (3.2)$$

Proof.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1) existing for $t \ge t_0$, such that $V(t_0,x_0) \le u_0$.

Set $m(t) = V(t,x(t))$ for $t \ne t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t,x(t)) + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \le k\|x(t+h) - (x(t) + hf(t,x(t)))\| + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to0^+} \tfrac{1}{h}[m(t+h) - m(t)] \le \limsup_{h\to0^+} \tfrac{i}{h}\left[k\|x(t+h) - x(t) - hf(t,x(t)\|e\right] + \limsup_{h\to0^+} \tfrac{1}{h}[V(t+h, x(t) + hf(t,x(t)) - V(t,x)]$$

where k is the local Lipschitz constant and $e = (1,1,...,1)^T$.

It follows that,

$$D^{+}m(t) = D^{+}V(t,x(t)) \le g(t,m(t))$$

and using condition (ii) of Theorem 3.1 we arrive at

$$V(t,x(t)) \le r(t)$$

provided

$V(t_0,x_0) \le u_0$

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

Also,

$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k^+)) \le \psi_k(m(t_k^+))$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.1).

Corollary 3.2. Assume that:

i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

ii) $p(t) = p(t,t_0,u_0) \in PC([t_0,T), R^n)$ is the minimal solution of (2.3) existing for $t \ge t_0$.

iii) $V \in C[J \times S_\psi, R_+^N], V \in L$ such that for $(t,x) \in J \times S_\psi$

$$D^+V(t,x) \ge g(t,V(t,x)),\ t \ne t_k$$

and

$V(t, x + I_k(x(t_k)) \ge \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function $\rho_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iv) $x(t) = x(t,t_0,x_0) \in PC([t_0,T], R^N)$ is a solution of (2.1) such that

$$V(t_0,x_0) \ge u_0 \tag{3.3}$$

existing for $t \ge t_0$. Then

$$V(t,x(t)) \ge p(t) \tag{3.4}$$

Proof.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1) existing for $t \ge t_0$, such that $V(t_0,x_0) \ge u_0$.

Set $m(t) = V(t,x(t))$ for $t \ne t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t,x(t)) + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \ge k\|x(t+h) - (x(t) + hf(t,x(t)))\| + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

$$\limsup_{h\to 0^+}\tfrac{1}{h}[m(t+h)-m(t)] \geq \limsup_{h\to 0^+}\tfrac{i}{h}\left[k\|x(t+h)-x(t)-hf(t,x(t)\|\right]e + \limsup_{h\to 0^+}\tfrac{1}{h}[V(t+h,x(t)+ hf(t,x(t))-V(t,x)]$$

where k is the local Lipschitz constant and $e=(1,1,...,1)^T$

It follows from condition (ii) of Cor 3.2 we arrive atat the estimate

$$D^+m(t)=D^+V(t,x(t))\geq g(t,m(t)),t\neq t_k,\ m(t_0^+)\geq u_0 \qquad (3.5)$$

Also,

$$m(t_k^+)=V(t_k^+,x(t_k)+I_k(x(t_k^+))\geq \psi_k(m(t_k^+)) \qquad (3.6)$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.5).

In what follows, we shall obtain sufficient conditions for the eventual stability of the system (2.3).

THEOREM 3.2. Assume that:

i) $g\in C[J\times R_+^n, R^n]$, $g(t,0)\equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for fixed $t\in J$.

ii) $V\in C[J\times S_\rho, R_+^n]$, $V(t,x)$ is locally Lipschitzian in x and $\sum_{i=1}^{N} V_i(t,x)\to 0$ *as* $\|x\|\to 0$ for each t and $(t,x)\in J\times S_\rho$,

$$D^+V(t,x)\leq g(t,V(t,x)) \qquad (3.7)$$

iii) for $(t,x)\in J\times S_\rho$,

$$b(\|x(t)\|)\leq \sum_{i=1}^{N} V_i(t,x)\leq a(t,\|x\|) \qquad (3.8)$$

where $b\in K$, $a(t,.)\in K$ whence $a\in C[J\times S_\rho, R_+]$

Then the eventual stability of the set of trivial solution $u=0$ of the system (2.3) implies the eventual stability of the set of trivial solution $x=0$ of the system (2.1).

Proof. Let $0<\varepsilon<\rho$ and $t_0\in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon)>0$ and $t_0\in R_+$, there exists a positive function $\delta=\delta(t_o,\varepsilon)>0$ which is continuous in t_0 for each $\in$ such that

$$\sum_{i=1}^{N} u_{i0}\leq \delta \text{ implies } \sum_{i=1}^{N} u_i(t,t_0,u_0)<b(\in), t\geq t_0 \qquad (3.9)$$

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 2, 2025 (pp. 129-140)

Since $V(t,x)$ is continuous and mildly unbounded i.e. $V(t,0) \to 0 \; as \; \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each $\in$ such that the inequalities

$$\|x_0\| < \delta_1 \;\; and \;\; V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t,t_0,x_0)\| < \in$ by the stability of $x(t)$.

Now suppose this claim is false, the there would exists a point $t_1 \in [t_0, t)$ and the solution $x(t,t_0,x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \;\; and \;\; \|x(t)\| < \varepsilon \;\; \text{for} \;\; t \in [t_0, t_1) \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)), \text{ implying}$$

$$b(\varepsilon) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)) \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t, x(t)) \leq r(t, t_0, u_0), \tag{3.13}$$

where $r(t,t_0,u_0)$ is the maximal solution of (3.2.3).

Then using equations (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \leq V_0(t_1, x(t_1) \leq \sum_{i=1}^{N} r_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the eventual stability of the set of trivial solution $u = 0$ of (2.3) implies that the set of trivial solution $x = 0$ of (2.1) is eventually stable in the sense of Lyapunov.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

CONCLUSION

This paper seeks to examine the global existence of solution for a given differential system. By using the vector Lyapunov functions—extended through a class of piecewise continuous Lyapunov functions—and applying comparison principles, the study establishes sufficient conditions for eventual stability of the given differential system. The results obtained extends the usual scalar case in the literature.

REFERENCES

[1] J. E. Ante, S. O. Essang, J. U. Atsu, E. O. Ekpenyong, R. D. Effiong (2024). On a class of piecewise continuous Lyapunov functions and eventual stability for nonlinear impulsive Caputo fractional differential equations via new generalized Dini derivative. International Journal of Mathematical Analysis and Modelling, 7(2), 423-439

[2] Michael Precious Ineh, Umar Ishtiaq, Jackson Efiong Ante, Mubariz Garayev and Ioan-Lucian Popa (2025). A robust uniform practical stability approach for Caputo Fractional hybrid systems. AIMS Mathematics, 10(3): 7001-7021. DOI: 10.3934/math.2025320.

[3] J. E. Ante, J. O. Achuobi, U. P. Akai, E. J. Oduobuk, A. B. Inyang (2024). On Vector Lyapunov Functions and Uniform Eventual Stability of Nonlinear Impulsive Differential Equations. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(4), 70 – 80.

[4] Michael Preious Ineh, Vivian Ndufu Nfor, Marshal Imeh Sampson, Jackson Effiong Ante, Jeremiah Ugeh Atsu and Okoi Okoi Itam (2025). A Novel Approach for Vector Lyapunov Functions and Practical Stability of Caputo Fractional Dynamic Equations on Time Scale in Terms of Two Measures. Khayyam J. Math. 11(1), 61-89. DOI: 10.22034/kjm.2025.487084.3362.

[5] Ante, J. E., Essang, S. O., Itam, O. O., John, E. I. (2024). On the existence of maximal solution and Lyapunov practical stability of nonlinear impulsive Caputo fractional derivative via comparison principle. Advanced Journal of Science, Technology and Engineering, 4(4), 92 – 110. DOI: 10.52589/AJSTE.9BWUJX90.

[6] Devi, J. V., Mc Rae, F. A., & Drici, Z. (2012). Variational Lyapunov method for fractional differential equations. Computers & Mathematics with Applications, 64(10), 2982-2989.

[7] Hale, J. (1977). Theory of Functional Differential Equations. Springer-Verlag, New York, Heidelberg, Berlin.

[8] Igobi, D. K. and Ineh, M. P. (2024). Results on Existence and Uniqueness of Solutions of Dynamic Equation on Time Scale via Generalized Ordinary Differential Equations. International Journal of Applied Mathematics, 37, No. 1, 1-20,

[9] Igobi, D. K. and Ante, J. E. (2018). Results on Existence and Uniqueness of Solutions of Impulsive Neutral Integro-Differential System. Journal of Mathematics Research, Vol 10, No. 4. http://doi:10.5539/jmr.v10n4p165.

[10] Lakshmikantham, V. (1974). On the Method of vector Lyapunov Functions. Technical report No. 16.

[11] Lakshmikantham, V., Bainov, D. D. and Simeonov, P. S. (1989). The Theory of Impulsive Differential Equations. Series in Modern Applied Mathematics, No. 6, 1989, World Scientific Publishing C. Pte, Ltd.

[12] Lakshmikantham, V. and Vatsala, A. S. (2008). Basic Theory of Impulsive Differential Equations. Nonlinear Analysis: Theory, Methods & Applications, 69(8), 2677-2682.

Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC

[13] Lakshmikantham V. and S. Leela, (1969). Differential and Integral Inequalities, Theory and Application, Vol. 1. Academic Press, Inc. New York and London.
[14] Atsu, Jeremiah Ugeh, Ante, Jackson Efiong, Inyang, Augustine Brendan, Akpan, Ubong Dominic (2024). A survey of the vector Lyapunov functions and practical stability of nonlinear impulsive Caputo fractional differential equations via new modeled generalized Dini derivative. IOSR Journal of Mathematics, vol. 20, issue 4, ser. 1. DOI: 10.9790/5728-2004012842.
[15] Lakshmikantham, V. and S. Leela (2009). Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers.
[16] Lakshmikantham, V. and A. S. Vatsala, Devi, J. V. (2007). General Uniqueness and Monotone Iterative Technique for Fractional Differential Equations, Applied Mathematics Letters 21, 828-834.
[17] Dodi Kanu Igobi, Jeremiah Ugeh Atsu, Jackson Efiong Ante, Joseph Oboyi, Udeme Emmanuel Ebere, Emmanuel Emmanuel Asuk, Paul Edet Okon, Samuel Okon Essang, Uwem Prospero Akai, Runyi Emmanuel Francis, Godwin Odido Igomah, Benedict Iserom Ita. (2025). On the vector Lyapunov Functions and Asymptotic Eventual Stability for Nonlinear Impulsive Differential Equations via Comparison Principle. Journal of NAMP 69(1), 57-70. https://doi.org/10.60787/jnamp.vol69no1.459
[18] Ante, J. E., Abraham, E. E., Ebere, U. E., Udogworen, W. K., Akpan, C. S. (2024). On the Global Existence of Solution and Lyapunov Asymptotic Practical Stability for Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle. Sch J Phys Maths Stat, 11(11), 160-172. https://doi.org/10.36347/sjpms.2024.v11i11.001.
[19] Ludwig, A., Pustal, B. Herlach, D. M. (2001). General Concept for a Stability Analysis of a Planar Interface under Rapid Solidification Conditions in Multi-Component Alloy Systems. Material Science and Engineering: A, 304, 277-28.
[20] Milman, V. D. and Myshkis, A. D. (1960). On Motion Stability with Shocks, Sibirsk. Mat Zh. 1 233-237 (in Russian).
[21] Jackson Efiong Ante, Jeremial Ugeh Atsu, Adhir Maharaj, Etimbuk Emmanuel Abraham, Ojen Kumar Narain (2024). On a class of piecewise continuous Lyapunov functions and uniform eventual stability of nonlinear impulsive Caputo fractional differential equations via new generalized Dini derivative. Asia Pac. J. Math. 2024, 11:99. DOI: 10.28924/APJM/11-99
[22] Ante, J. E., Itam, O. O., Atsu, J. U., Essang, S. O., Abraham, E. E., Ineh, M. P. (2024). On the Novel Auxilliary Lyapunov function and Uniform Asymptotic Practical Stability of Nonlinear Impulsive Caputo Fractional Differential Equations via New Modelled Generalized Dini Derivative. African Journal of Mathematics and Statistics Studies, 7(4), 11-33. DOI: 10.52589/AJMSS-VUNAIOBC.
[23] Ante, J. E.; Ineh, M. P.; Achuobi, J. O.; Akai, U. P.; Atsu, J. U.; Offiong, N.-A. O. (2024). A Novel Lyapunov Asymptotic Eventual Stability Approach for Nonlinear Impulsive Caputo Fractional Differential Equations. AppliedMath 2024, 4, 1600-1617. https://doi.org/10.3390/appliedmath4040085
[24] Ante, J. E., Akpan, U. D., Igomah, G. O., Akpan, C. S., Ebere, U. E., Okoi, P. O., Essang, S. O. (2024). On the Global Existence of Solution of the Comparison System and Vector Lyapunov Asymptotic Eventual Stability for Nonlinear Impulsive Differential Systems. British Journal of Computer, Networking and Information Technology 7(4), 103-117. DOI 10.52589/BJCNIT-ZDMTJB6G
[25] Ineh, M. P., Achuobi, J. O., Akpan, E. P., and Ante, J. E. (2024). On the Uniform stability of Caputo fractional differential equations using vector Lyapunov functions. Journal of NAMP 68(1), 51-64. https://doi.org/10.60787/10.60787/jnamp.v68no1.416.

 Article DOI: 10.52589/AJMSS-USLHJ4TC
DOI URL: https://doi.org/10.52589/AJMSS-USLHJ4TC