African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 233-246)
www.abjournals.org

COMPARABILITY OF TWO STOCHASTIC MODELS IN THE ANALYSIS OF THE NIGERIAN FEMALE MORTALITY

Lawal Fatai Kolade[1*], Olumi Toba Timothy[2], Akor Enejo Felix[3],

Mathias Ahiaba[4] and Kabir Jamiu[5]

[1,3,4,5]Department of Statistics Kogi State Polytechnic, Lokoja Kogi State.

[2]Department of Statistics Federal Polytechnic, Orogun, Delta State.

[*]Corresponding Author's Email: lawalfkray@yahoo.com

Cite this article:

Lawal Fatai Kolade, Olumi Toba Timothy, Akor Enejo Felix, Mathias Ahiaba, Kabir Jamiu (2025), Comparability of Two Stochastic Models in the Analysis of the Nigerian Female Mortality. African Journal of Mathematics and Statistics Studies 8(3), 233-246. DOI: 10.52589/AJMSS-4I8J4GJK

Manuscript History

Received: 11 Jul 2025

Accepted: 14 Aug 2025

Published: 3 Sep 2025

ABSTRACT: *The study examines the strength of two stochastic models: the Lee Carter Model and Functional Data Time Series Analysis in modelling Nigeria Female Mortality. The data used for the analysis were obtained from Nigeria Bureau of Statistics from 1998-2024. Based on various errors of measurement, it was discovered that the Functional Time series Data Analysis (FDTA) performs better in the modelling of the Nigerian Female Mortality.*

KEYWORDS: Mortality, Stochastic, Lee-Carter, FDTA, Modelling.

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

INTRODUCTION

The practice of forecasting mortality dates back to 1725, when the French mathematician, De Moivre, introduced the concept of a survival function. Later, in 1825, British actuary, Gompertz, observed that for individuals aged between 20 and 60, the force of mortality tends to rise exponentially. This observation, based on demographic data from England, Sweden, and France, proved applicable to many countries. However, Gompertz's model was found to have limitations, particularly for certain age groups (Olshansky, 1997).

To address these limitations, several alternative models have been proposed, including those by Makeham (1867), Thiele (1872), Wittstein (1883), Pearson (1895), and Perk (1932). Stoto (1983) reviewed historical mortality forecasting models and identified significant prediction errors. Keilman (1988) further noted that many earlier forecasts failed to account for key future trends, particularly underestimating declines in old-age mortality and increases in life expectancy.

Mortality forecasting inherently involves a number of subjective decisions, with expert judgment required at nearly every stage (Alders & De Beer, 2005). According to Booth (2006), when the goal is to predict the number of future deaths, this is best achieved through forecasting mortality rates in combination with population projections.

Recently, more sophisticated methods for mortality forecasting have emerged. This study introduces a new approach using Functional Data Time Series Analysis (FDTA), a stochastic modelling technique, focusing on female mortality in Nigeria. The study aims to support gender-specific resource planning and policy-making. The primary goal is to apply functional data analysis techniques to forecast trends in the Nigerian female mortality and compare these results with those obtained from the well-established Lee-Carter (1992) model.

LITERATURE REVIEW

Since Graunt (1662) first studied mortality in London to generate the first publication that was primarily focused on public health statistics, demographers, insurance mathematicians, and medical experts have been actively researching modelling human mortality. Graunt's research demonstrates that although life expectancy varied among individuals, group mortality and the reasons of death followed a more predictable pattern. Halley (1693) successfully demonstrated how to create a non-deficient mortality table using empirical birth-death data. He also provided a way to calculate a life annuity using this table. These early tables required a lot of time to calculate and they were empirical.

DeMoivre (1725), who proposed a uniform distribution of deaths model and demonstrated streamlined annuity calculating techniques, was the first to introduce theoretical mortality modelling. Gompertz (1825) used a biological approach to mathematical modelling, assuming that the force of mortality μ_x in adulthood increases almost exponentially. The two parameters of his model are positive and change according to the level of mortality and the pace at which mortality increases with age.

Most commonly used as the standard models for adult mortality in humans (Kirkwood, 2015; Olshansky and Carnes, 1997), the Gompertz model and its modified version by Makeham

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

(1867), which adds a constant to account for background mortality from causes unrelated to age, were later expanded to include animal species in general (Sacher, 1977).

Danish scientists who were involved in insurance businesses at the time and exposed to actuarial science, such as Oppermann (1870) and Thiele (1871), produced early discoveries of significant concepts in the field of mortality modelling. However, a number of variables, including the publication of the novel concepts in inaccessible locations and in a language that was unusual, like Danish, as well as the absence of multidisciplinary collaboration, contributed to the acknowledgement of their work only many years later (Hoem, 1983).The need to predict future shifts in the makeup and structure of populations led to the creation of cohort-component forecasting techniques, which have gained popularity since Malthus (1798). Cannan (1895) was the first to employ these methods, creating a cohort component projection for Wales and England.

At the close of the 1920s, Tarasov (1922) had also made such predictions for the Soviet Union, Wiebols (1925) for the Netherlands, Wicksell (1926) for Sweden, and Whelpton (1928) for the United States.

During the early 1900s, statisticians laid the groundwork for the so-called stationary processes as demographers developed the cohort-component forecasting system. White noise was transformed linearly as the basis for this hypothesis. The publication of Box and Jenkins (1970) marked the standardization of the theory's practical use in statistics, despite the fact that its key components had been substantially refined by the early 1950s (Doob & Doob, 1953). Saboia (1974, 1977) is one of the earliest instances of their application in demography. These techniques are popular because of their high degree of flexibility, their ability to create confidence intervals, and their ability to include the consequences of changes in behavioural and socioeconomic variables into forecasts (Tabeau et al., 2001).

Prior to the 1980s, life expectancy and mortality rates were directly predicted using mathematical models that were comparatively straightforward and that incorporated some subjective opinion.

Using stochastic models, several novel methods have been developed over the last three decades for predicting mortality, including Bell and Monsell (1991), McNown and Rogers (1989, 1992), Lee and Carter (1992, henceforth LC), Alho (1990), and Alho and Spencer (1991). Stochastic models' primary benefit is that their output is a distribution rather than a single number. LC suggests a log-bilinear model for death rates that takes into account the impact of both year and age.

According to Hollmann et al. (1999), the U.S. Bureau of the Census uses Lee and Carter's work as a benchmark model for long-term projections of age-specific mortality rates, and it has received numerous citations. Furthermore, since that study, the LC framework has served as the foundation for the majority of other models that aim to evaluate the age and time evolution of several levels of mortality.

Several studies have attempted to enhance Lee and Carter's (1992) model by including additional main components, a cohort effect, or any number of comparable statistical measures.

Many people have proposed various modifications to the LC approach. These include the LC method without correction, as well as Tuljapurkar et al. (2000), Lee and Miller (2001), and Booth et al. (2002) approaches. For the first time, Booth et al. (2005) evaluated the forecast

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

accuracy of the LC technique and its variants, and Booth et al. (2006) carried out further investigations on the cohort effect. Among these, statisticians and demographers have become more interested in the extension proposed by Hyndman and Ullah (2007). Their method combined the ideas of functional data analysis, nonparametric smoothing, and functional principal component regression to forecast mortality and fertility rates. This approach was utilised by Erbas et al. (2007) to forecast the death rates from breast cancer in Australia. This method has also been extended by Hyndman and Booth (2008) to improve variance estimation and include the forecasting of migration rates in Australia. Hyndman et al. (2013) recently extended it by comparing their coherent method with independent functional forecasts of mortality of Sweden by sex and for Australia by state. Booth (2020) used the method for low mortality standard and data of 21 countries by sex, showing reasonable gains in accuracy and bias. Richman and Wuthrich (2021) applied the neural networks approach and discovered an improvement over LC method on 15 years forecasts for several low mortality countries. Marie, Bergron and Kjaergaard (2022) employed different fitting and forecasting periods to evaluate the LC methods for ages 65 years and older by sex in four countries and found that LC under predicts life expectancy at older ages. Ugofilipo, Carlo and Heather (2023) worked thirty years on review of Lee-Carter methods and its various extensions with the aim of comparing their performance. Because of its relatively good performance and simplicity of use, the Lee-Carter (LC) model has been widely used for actuarial and demographic reasons in many countries.

Lee–Carter (LC) model structure is given as in Equation 2.1:

$$log\ log\ m_{x,t} = \alpha_x + b_x k_t + \varepsilon_{xt},\ \sum_x b_x = 1,\ \sum_t k_t = 0 \qquad (1)$$

where m_{xt} is the matrix of the central death rates at age X = (x_1, - - -,x_n) in year t, (t = t_1, - - - ,t_{n-1}). The term, ε_{xt}, represents the deviation of the model from the observed log-central death rates, and it is expected to be Gaussian Normal, i.e., $\varepsilon_{xt} \sim N(0, \delta^2)$.

The method assumes a pattern of change in the age distribution of mortality, such that mortality rates decline at different ages, maintaining the same time. However, in practice, the general speed of decline at different ages may vary. Wilmoth (1995) opined that the mortality rates at old ages were observed in Sweden to decline more gradually than at other ages 5 to 50, relative to the older and younger ages.

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

METHODOLOGY

The Lee-Carter Forecasting Model

Developed by Lee and Carter in 1992, this model was originally used to forecast mortality and life expectancy for the U.S. population. It relies on a numerical algorithm that operates on a matrix of age-specific mortality rates, where the rows represent different years and the columns represent age groups. The model is defined by the following equation:

$$ln\ (m_{x,t)}\ =\ \alpha_x + b_x k_t\ +\ \epsilon_{x,t} \quad (2)$$

or equivalently

$$m_{x,t} = exp\ exp\ \{\alpha_x + b_x k_t\ +\ \epsilon_{x,t}\} \quad (3)$$

where

k_t – time parameters indexed by $t\ =\ 1, 2, \ldots, T,$

α_x, b_x – age-specific parameters indexed by $x\ =\ 0, 2,7,12 \ldots, w,$

$\epsilon_{x,t}$– random errors assumed to be i id, $\pounds xt \sim N(0, \partial^2_{\pounds}\ )$.

To ensure the unique representation of (2) or (3), some additional constraints are imposed (See Lee-Carter, 1992.) in the following way:

$$\sum_{t=1}^{T} k_t\ = O\ \sum_{x=0}^{w} bx\ = 1\ so\ that\ ax = \frac{1}{T} \sum_{t=1}^{T} m_{x,t}\ )$$

Forecasting of mortality using LC model is based on time series prediction on the parameters k_t.

This is done using the Equation 3:

$$\hat{\mu}_x t_{n+s} = \exp(\hat{\alpha}_x + \hat{\beta}_x^{h(0)} it_{n+s-x} + \hat{\beta}^{(1)} kt_{n+s}), \text{ where s >0} \quad (4)$$

And it_{n+s-x} and kt_{n+s} represent the forecasted cohort and period effect, respectively. To produce mortality forecast, Lee and Carter assumes that β_x remains constant over time and they use forecast of $\hat{k}_t$ from a standard univariate time series model using ARIMA with drift, that is, the random walk with drift model for the parameter k_t, and the model is given as:

$$\hat{k}_t = \hat{k}_{t-1} + \hat{\theta}\ + \varepsilon_t \quad (5)$$

where $\varepsilon_t \sim$ N(0, σ_x^2) where $\hat{\theta}$ is known as the drift parameter and the maximum likelihood estimate is simply $\hat{\theta} = \hat{k}_{t-1} + \frac{\hat{k}_1}{t-1}$ which only depends on the first and last of the k estimates.

The performance of the Lee-Carter model has been covered in many studies. (See, e.g., Lee-Miller, 2001.) According to the concept underlying this methodology, parameters, a_x , b_x, k_t are estimated with the empirical data on the considered population. Lee and Carter have used in their paper the Singular Value Decomposition (SVD) method (Lee-Carter, 1992). Two other methods of estimation are also proposed, i.e., Weighted Least Squares (WLS) and the

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

Maximum Likelihood (ML) approach (See Wilmoth, 1993; Brouhns et al. 2002a, b.). The estimated values of k_t form a time series with one value for each year. Because of that, the statistical methods of time series modeling can be employed. Despite the usefulness of the model, it has its own limitations. First, the model was developed and used in advanced countries where there is enough information on mortality data. Another limitation is that it assumes errors to be constant with narrow predictive intervals (Wilmoth 1993). In order to overcome this, a new model is proposed, known as functional data analysis.

Functional Data Time Series Analysis

In functional data analysis, it is assumed that there is an underlying smooth function of age on the mortality rate in each year. Let the function $y_t(x_i)$ denote the log of the observed mortality rate of age x at year t. Assume $\mathfrak{f}_{t,F}(x)$ is the underlying smooth function obtained from the model when X represents the age continuum defined on a finite interval. We observed functional data on a set of grid points and the data are often contaminated by random noise.

$$y_t(x_i) = \mathfrak{f}_{t,i}(x) + u_{t,i}, \quad t = 1, \ldots n, \; i = 1, \ldots, p \tag{6}$$

where n denotes the numbers of years and p denotes the numbers of discrete data points of age observed for each function. The errors $(u_{t,i})$ are independently and identically distributed random variables with mean zero and variance. Smoothing techniques are thus needed to obtain each function $\mathfrak{f}_{t,i}(x)$ from a set of realizations. Hence, we used smoothen spline to observed the smoothing function, $\mathfrak{f}_{t,i}(x)$.

We used the underlying smooth function $\mathfrak{f}_{t,F}(x)$ obtained from smooth regression, as given by Equation 2.6, observing with error. Hence,

$$y_{t,F}(x_i) = \log\left[\mathfrak{f}_{t,F}(x_i)\right] + \sigma_{t,F}(x_i)e_{t,F,i} \tag{7}$$

where x_i is the middle of the age-class I, (I =1, . . . , p), and $e_{t,F,i}$ is the error, that is, standard normal distribution, independently and identically distributed random variable, and the value $\sigma_{t,F}$ allows the amount of the noise to vary with the age x. For the smoothness of the data, we used p-spline regression; we thereafter applied the product ratio method of functional data of time series analysis to the smoothed data.

We define the product ratio of functional time series analysis of the smoothed as:

$$p_t(x) = \left(F_{t,M}(x)F_{t,F}\right)(x)^{1/2} \tag{8}$$

We model these quantities via the smoothed log rate rather than the original sex-specific mortality rates. On the log-scale, these are sums and differences which are approximately uncorrelated (Turkey, 1977). We use functional time series models (Hyndman & Ullah, 2007) for p_t (x):

$$\text{Log}[p_t(x)] = \mu_p(x) + \sum_{k=1}^{K} \beta_{t,k}\, \phi_k(x) + e_t(x) \tag{9}$$

where the function ϕ_k(x) is the principal components obtained from the decomposition of p_t (x), and $\beta_{t,k}$ is the corresponding principal component score. The function μ_p(x) is the mean of set p_t (x). The error term given by e_t(x) has zero mean and is serially uncorrelated.

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

The Nigeria Bureau of Statistics' data on male mortality from 1998 to 2024 will be modelled using the LC and FTDA models. After applying the models to our data, we evaluate our model's forecast accuracy against that of the Lee-Carter forecasting model. Each model is fitted to the first h observations, and the death rates for the observations in years t+1, t+2, t+3,..., n are then predicted. The range of values for t is n1=50 to n-1. Following that, the predicted horizon fluctuates between n = 1 and 50. The two metrics used to quantify point forecast errors are mean squared prediction errors (MSPE) and mean absolute prediction errors (MAPE). We compute MAPE and MSPE for every population. By analysing these errors, we can gain insights into the model's performance across different population segments. This evaluation allows us to refine our predictive techniques and enhance the accuracy of future forecasts:

$$\text{MAPE (h)} = \frac{1}{(n-h-\ n_1+1)p} \sum_{n=h}^{50} \sum_{j=1}^{p} \left| \left[m_{i+h,j}(x_i) \right] - log\ log\ \left[\widehat{m}_{t+h|t,j}(x_i) \right] \right| \quad (10)$$

$$\text{MSFE (h)} = \text{MSFE (h)} = \frac{1}{(n-h-\ n_1+1)p} \sum_{h}^{50} \sum_{i=1}^{p} \left(\left[m_{i+h,j}(x_i) \right] - log\ log\ \left[\widehat{m}_{t+h|t,j}(x_i) \right] \}^2 \right) \quad (11)$$

where $\widehat{m}_{t+h|t,j}$ represents the h-steps ahead predictions using the first n+t-h years fitted in the model, and $m_{i+h,j}(x_i)$ denotes the true value. For our interval forecast, if $\hat{f}^{u}_{n+h/n}$ and $\hat{f}^{i}_{n+h/n}$ are the upper and the lower (1-α) 100% prediction bounds and $m_{i+h,j}$ is the realized value, the interval point x_i is given as:

$$s_x(x)_j = \left[\hat{f}^{u}_{n+\frac{h}{n}}(x_i) - \hat{f}^{i}_{n+\frac{h}{n}}(x_i) \right]$$

$$+ \frac{2}{\alpha)} \left[\hat{f}^{i}_{n+h/n}(x_i) - m_{i+h,j}(x_i) \ \right] || \left\{ m_{i+h,j}(x_i) < \hat{f}^{i}_{n+\frac{h}{n}}(x_i) \right\}$$

$$+ \frac{2}{\alpha)} \left[m_{i+h,j}(x_i) - \hat{f}^{u}_{n+\frac{h}{n}}(x_i) \ \right] || \left\{ m_{i+h,j}(x_i) > \hat{f}^{u}_{n+\frac{h}{n}}(x_i) \right\} \quad (12)$$

where α is the level of significance; the forecast mean interval across the horizon is given as:

$$\underline{s_\alpha}(\text{h}) = \frac{1}{(n-h1)p} \sum_{n=h}^{50} \sum_{i=1}^{p} \underline{s_\alpha} \left| \left[\hat{f}^{u}_{n+\frac{h}{n}}(x_i) \right], \hat{f}^{i}_{n+\frac{h}{n}}(x_i); m_{i+h,j}(x_i) \right| \quad (13)$$

where p denotes the number of age groups and h denotes the forecast horizons.

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 233-246)

ANALYSIS AND RESULTS

In this section, the results of the analysis from the models shall be presented. The results obtained from the various models will be discussed in detail, highlighting key findings and their implications. Additionally, there will be comparability of the performance of models to ascertain which of the models provides the most accurate predictions and insights. The results of the analysis are as presented below. The graphs of Figures 1 and 2 represent the three parameters of the LC model and that of the parameters of FDTA.

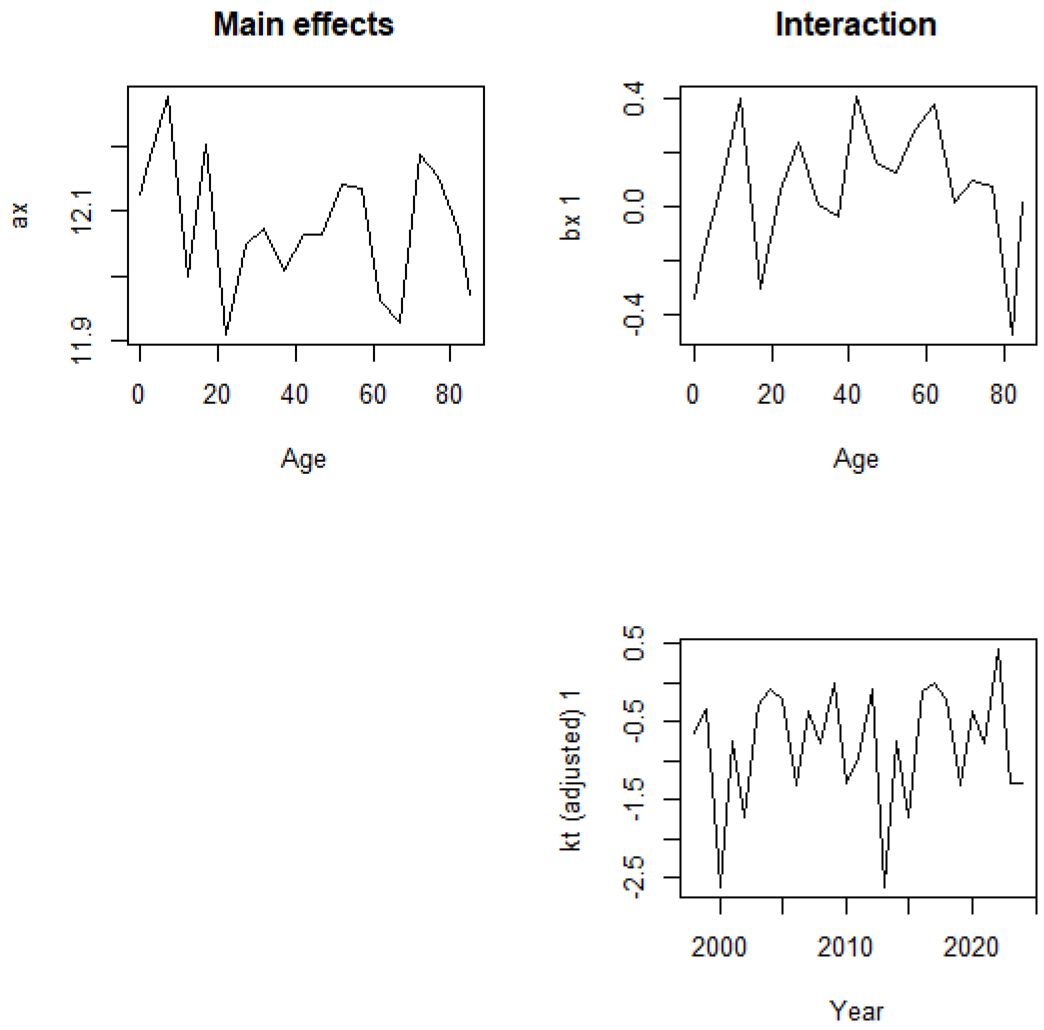

Figure 1: Basis and coefficients for female mortality of LC model

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 233-246)
www.abjournals.org

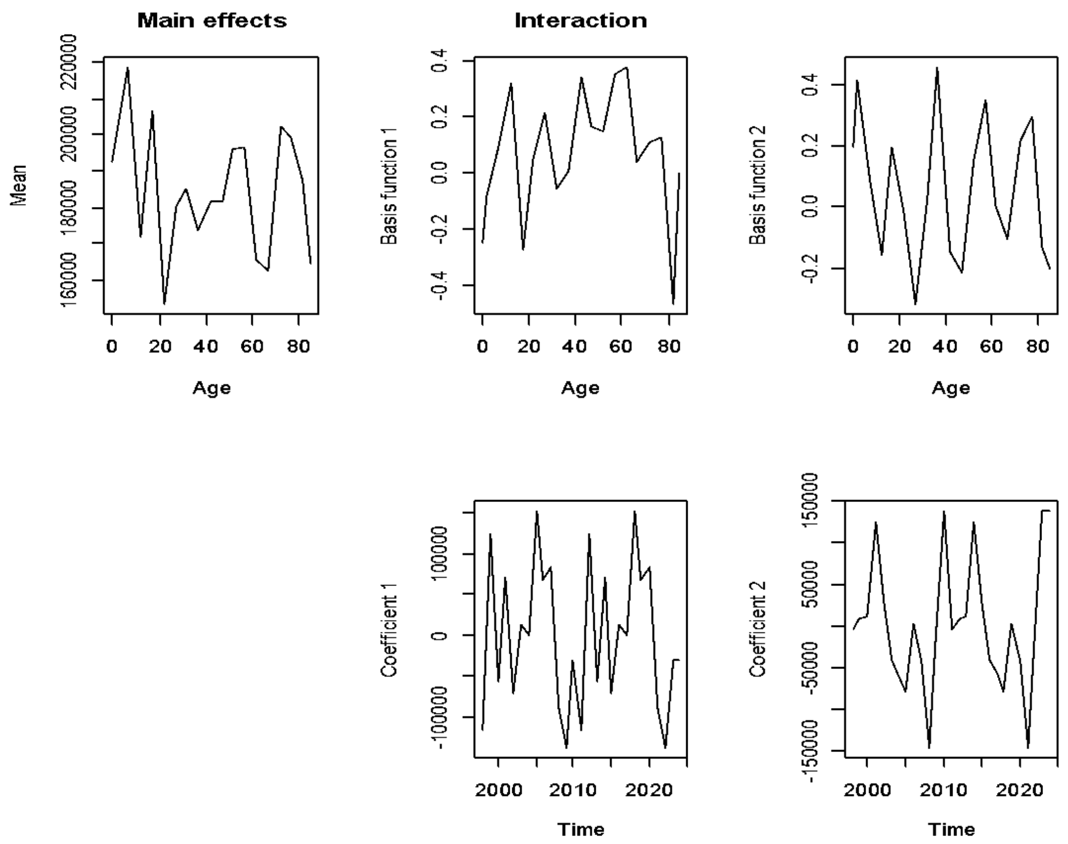

Figure 2: The parameters of FDTA model of Nigerian Female Mortality

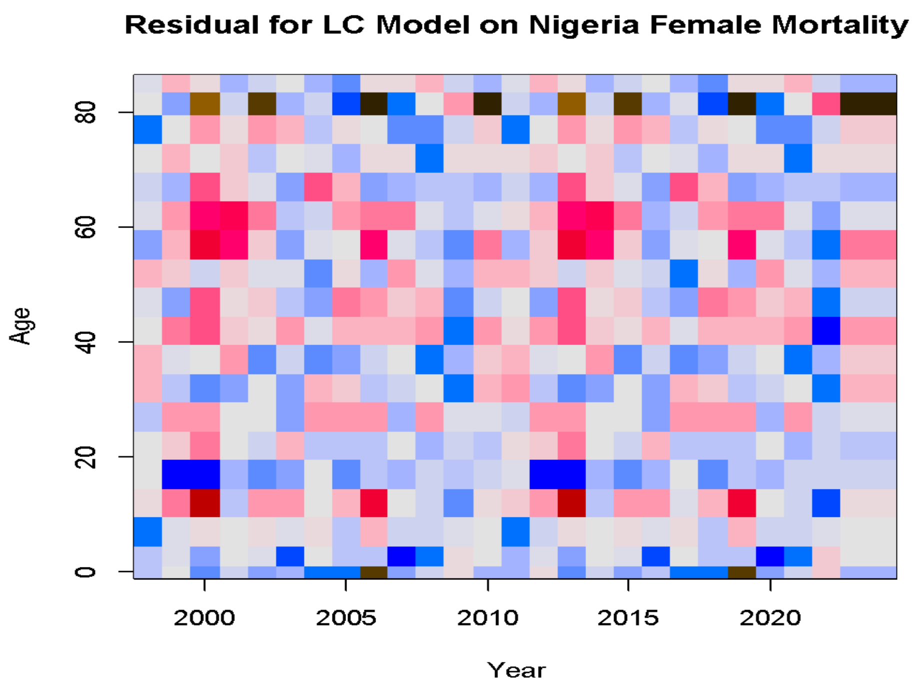

Figure 3: Residual for LC model on female mortality in Nigeria

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

Figure 3 is the LC errors heat map for female mortality. The deep red and blue scattered nature of the error map indicates that the model performance varies across different age groups or time periods.

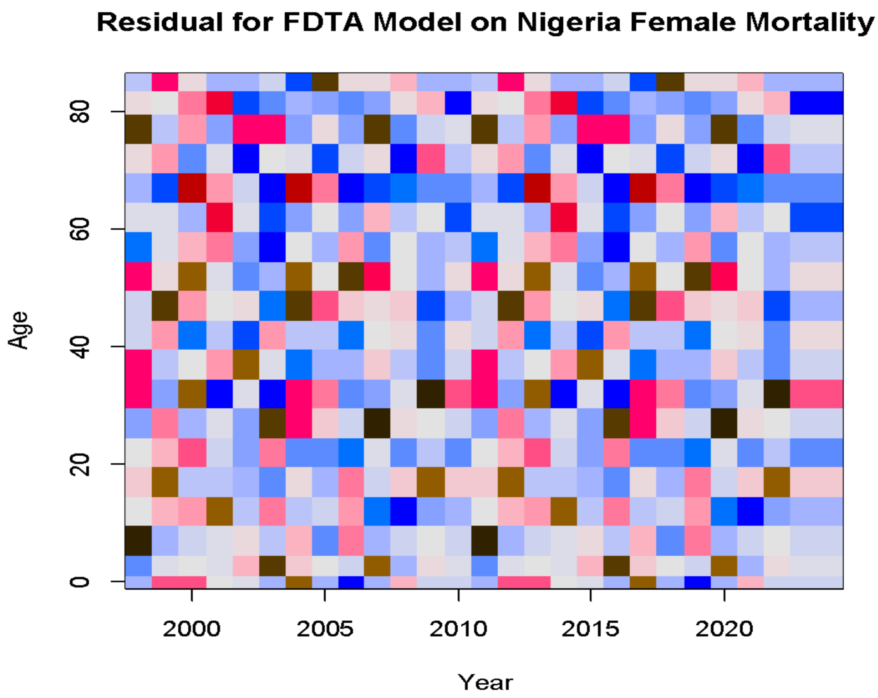

Figure 4: Residual for FDTA on female mortality in Nigeria

Figure 4 is the FDTA errors heat map for female mortality. The deep red and blue scattered nature of the error map indicates that the model performance varies across different age groups or time periods like the Lee-Carter model, but is more pronounced.

Table 1: Parameters of the Models

Errors Measure Based on Mortality Rates

PARAMETERS	LC	FDTA
ME	-6.322	-97.63083
MSE	36.07	0.8567
MPE	0.13388	0.03527
MAPE	0.35167	0.47585

Errors Measure Based on Log Mortality

PARAMETERS	LC	FDTA
ME	0.0302	-0.17778
MSE	0.17277	0.04345
MPE	0.01006	-0.02245
MAPE	0.05202	0.08073

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

DISCUSSION OF RESULTS

The graphs in Figure 1 show the value of the Lee-Carter parameters: $a_{x,t}$ which defined the slope of the model at age x in time t. The slope represents the logarithm of the central death rate for that specific age and time, indicating how mortality rates change over the years. Understanding these parameters is crucial for analysing trends in mortality and projecting future life expectancies. The value of k_t captured the mortality trend at time t, and a_x describes the amount of mortality change at a given age for a unit of yearly total mortality change. We can see that the movement of the graphs portrays zig-zag movements, showing that there have been variations in the volume of mortality experienced at different ages and years. The graphs in figure two are the parameters of the FDTA model on female mortality in Nigeria. From the graphs, it was observed that the FDTA model produced a smooth forecasted graph compared to that of the LC graph. The graph also shows that there is a high death rate at age zero, and also high death rates at the older ages.

Table 1 shows the parameters of the two models. It was observed that the error measures based on death rates among the females on FDTA model are lower than that of the LC model; so also are the obtained error measures based on log rate, except the Maximum Average Percentage Error (MAPE) of FDTA that is higher than that of LC for both errors of measurement.

SUMMARY, CONCLUSION AND RECOMMENDATION

Lee-Carter model and FDTA model have been used to analyze the Nigerian female mortality data. Though both models produce good results on female mortality, it was observed that there is an improvement on the FDTA model over the LC model with the identified smooth forecast curve and the least errors for both errors of measurements except one, which is the MAPE.

The model is recommended for modelling both male and female mortalities and comparing the result. Further study should be done to extend the model for projecting life table and life assurance risk for actuarial usage.

REFERENCES

Alders, M. & De Beer, J. (2005). An expert knowledge approach to stochastic mortality forecasting in the Netherlands. In (N. Keilman, ed.) Perspectives on mortality forecasting (Vol. II. Probabilistic models, pp39-64). *Swedish Social Insurance Agency,* Stockholm.

Alho, J. M. 1990. Stochastic methods in population forecasting. *International Journal of Forecasting* 6 (4), 521–530.

Alho, J. M. 2000. The Lee–Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal* 4 (1), 91–93.

Alho, J. M. and B. D. Spencer. 1991. A population forecast as a database: implementing the stochastic propagation of error. *Journal of Official Statistics* 7 (3), 295.

Bell W R, Monsell B. (1991), "Using principal components in time series modelling and forecasting of age-specific mortality rates." In: Proceedings of the American Statistical Association, Box, G.E.P., & Jenkins, G.M. (1970). Time Series Analysis: Forecasting and Control. Holden-Day , 154–159.

Booth, H., Hyndman, R. J., Tickle, L. & de Jong, P. (2006), 'Lee-Carter mortality forecasting: amulti-country comparison of variants and extensions', *Demographic Research* 15 (9), 289–310.

Booth, H. (2020). Coherent Mortality Forecasting with Standards: Low Mortality Serves as a Guide. In S. Mazzuco & N. Keilman (Eds), *Development in Demographic Forecasting* (Vol. 49, pp. 123–145)

Booth, H., Tickle, L. & Smith, L. (2005), 'Evaluation of the robustness of birth and death rates estimates. *Demographic Research*, 13(21), 357–566.

Booth, H., Maindonald, J. & Smith, L. (2002a), 'Applying Lee-Carter under conditions of variable mortality decline', *Population Studies* 56 (3), 325–336.

Box, G.E.P., & Jenkins, G.M. (1970). Time Series Analysis: Forecasting and Control. Holden-Day

Brouhns, N., Denuit, M. & Vermunt, J.K. (2002). A Poisson log-bilinear regression approach to the construction of projected life-tables. *Insurance: Mathematics and Economics*, 31, 373-393Carter L. R. Lee, R.D, (1992), Modelling and Forecasting US sex differentials in Modelling. *International Journal of Forecast*, 8, 393 – 411.

Cannan, E. (1895). The Probability of a cessation of the growth of Population in Ehland and Whales during the next century. *The Economic Journal* 5(20), 505–515.

De Moivre, A. (1725). Annuities on lives: Or, the valuation of annuities upon any number of lives as also of reversions. *William Person, London*

Doob, J. L. and J. L. Doob. 1953. Stochastic processes, Volume 7. Wiley New York

Erbas, B., Hyndman, R. J. & Gertig, D. M. (2007), 'Forecasting age-specific breast cancer mortality using functional data models', *Statistics in Medicine* 26(2), 458–470.

Gompertz B., (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. Philosophical t*ransactions of the Royal Society of London B*: Biological Sciences.; 182:513–85.

Graunt J. (1962) Natural and Political Observation Made upon the bills of Mortality with Reference to the government, religion, trade, growth, diseases, and the several changes of the said city London. The Roycraft for John Martin, James Allestry, Tho Dicas.

Halley, E. (1693). An Estimates of the Degrres of the Mortality of Mankind, Drawn from Curious Tables of the Births and Funerals at the City of Breslaw, with an Attempt to Ascertain the Price of Annuities upon lives. *Philosophical Transactions of Royal Society of London*, 17(196–209)

Hoem, J. M. 1983. The reticent trio: some little-known early discoveries in life insurance Mathematics by LHF Oppermann, TN Thiele and JP Gram. International Statistical Review/Revue Internationale de Statistique, 213–221.

Hollmann, F.W., Mulder, T.J. & Kallan, J.E. (1999). Methodology and assumptions for the population projections of the United States: 1999 to 2100. Working paper 38,Population Division, U.S. Bureau of the Census.

Hyndman, R. J. & Booth, H. (2008), 'Stochastic population forecasts using functional data models for mortality, fertility and migration', *International Journal of Forecasting* 24(3), 323–342.

Hyndman, R. J., Booth, H., Tickle, L. & Maindonald, J. (2009), *demography: Forecasting mortality and fertility data*. R package version 0.995. URL: *http:// robjhyndman.com/ software/ demography*

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

Hyndman, R. J., Booth, H., & Yasmen, F. (2013) Coherent mortality forecasting: The product-ratio method with functional time series models. Demography, 50(1), 261-283009), 'https://doi.org.10.1007/s13524-012.Forecasting
Hyndman, R. J. & Ullah, M. S. (2007), 'Robust forecasting of mortality and fertility rates: a functional data approach', *Computational Statistics & Data Analysis* 51(10), 4942–4956.
Kirkwood, T. B. 2015. Deciphering death: a commentary on Gompertz (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of Determining the value of life contingencies. Phil. Trans. R. Soc. B 370 (1666), 20140379.
Lee, R. D. & Carter, L. R. (1992), 'Modelling and forecasting US mortality', *Journal of the American Statistical Association* 87 (419), 659–671.
Lee, R. D. & Miller, T. (2001), 'Evaluating the performance of the Lee-Carter method for forecasting mortality', *Demography* 38(4), 537–549. *Institute of Actuaries (1886-1994) 63* (1), 12–57.
Makeham, W. M. 1867. On the law of mortality. *Journal of the Institute of Actuaries* (1866-1867) 13 (6), 325–358
Malthus, T.R. (1978). An Essay on the Principle of Population (reprint) . J.Johnson London
McNown, R., A. Rogers and J. Little (1995) Simplicity and complexity in extrapolative population forecasting models, *Mathematical Population Studies*, **5**(3), 235–257.
Olshansky, S. J. and B. A. Carnes. (1997). Ever since Gompertz. *Demography 34* (1), 1–15.
Oppermann, L.H.F. (1870). On the graduation of life tables, with special application to the rate of mortality in infancy and childhood. Insurance Record, 1870. pp. 42-47. Also reprinted in *Journal of Actuaries*, 16 (1872). pp. 335–353.
Oeppen, J., Vaupel, J. W., 2002. Broken limits to life expectancy. Science 296 (5570), 1029–1031.
Pearson, K. (1985). Contributions to the Mathematical Theory of Evolution. II. Skew Variation in Homogeneous Material Proceedings of the Royal Society of London, 58, 240-254
Perks, W. 1932. On some experiments in the graduation of mortality statistics. *Journal of the Institute of Actuaries (1886-1994) 63* (1), 12–57.
Richman, R., Wuthrich, M.V. (2021). A neural network extension of the Lee-Carter model to multiple populations. *Annals of Actuarial Science*, 15(2), 346–366. https://doi.org.10.1017/s1748499519000071
Saboia, J. L. 1974. Modeling and forecasting populations by time series: The Swedish case. Demography 11 (3), 483–492.
Saboia, J. L. M. 1977. Autoregressive integrated moving average (ARIMA) models for birth forecasting. Journal of the American Statistical Association 72 (358), 264–270.
Sacher, G. A. 1977. Life table modification and life prolongation. Handbook of the Biology of Aging 582.
Stoto, M.A. & Durch, J.S. (1993). Forecasting survival, health, and disability: report on a workshop. *Population Development and Review*, 19(3), 557-581.
Thiele, T. N. (1871). On a mathematical formula to express the rate of mortality throughout the whole of life. *Journal of the Institute of Actuaries and Assurance Magazine. Translated by T.B. Sprague in 1872. 16* (5), 313–329.
Taruvinga, T., & Gatawa, I. (2010). Understanding Population Dynamics in Africa. *UZ weekly*. II P.P 50-62
Tuljapukar S, Li N, Boe C. (2000). A universal pattern of mortality decline in theG7 countries. *Nature*. 405(6788)

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0

Whelpton, P. K. 1928. Population of the United States, 1925 to 1975. American Journal of Sociology 34 (2), 253–270.

Wiebols, G. A. H. 1925. De toekomstige bevolkingsgrootte in Nederland. PhD thesis. NV Dorsman Odé's bookshop and printing company.

Wilmoth, J. (1993). Computational Methods for Fitting and Extrapolating the Lee–Carter Model of Mortality Change. *Technical report. Department of Demography,* University of California, Berkeley, California.

Wilmoth, J. R. (1995), 'Are mortality projections always more pessimistic when disaggregated by cause of death? Mathematical Population Studies 5 (4), 293–319.

Article DOI: 10.52589/AJMSS-IICISLI0
DOI URL: https://doi.org/10.52589/AJMSS-IICISLI0