



## THE FOUNDATION OF ELECTROMAGNETISM: A COMPREHENSIVE STUDY OF MAXWELL'S EQUATIONS

Hazel Kiap and Mohsen Aghaeiboorkheili\*

School of Mathematics & Computer Science,  
Papua New Guinea University of Technology, Lae, Papua New Guinea

\*Corresponding Author's Email: [mohsen.aghaeiboorkheili@pnguot.ac.pg](mailto:mohsen.aghaeiboorkheili@pnguot.ac.pg)

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**ABSTRACT:** *In the history of physics, one of the deepest integrations that classically combined the phenomena of magnetism, optics and electricity into one theoretical structure is represented by Maxwell's equations. This analysis gives a thorough mathematical formulation of the four fundamental equations elegantly. It provides a deeper understanding beginning from their historical basis in the works of Faraday, Gauss and Ampère and finishing in Maxwell's vital input – the displacement current. We illustrate how these four equations beautifully surface from experimental laws when united with advanced vector calculus via comprehensive mathematical analysis. A disclosure that changed the concept of light is a consequence of Maxwell's equations which has surpassed classical electromagnetism that led immediately to the forecast of electromagnetic waves. James C. Maxwell, when synthesizing magnetic and electricity, he has proven that light alone is an electromagnetic phenomenon. Today, in this modernized world, Maxwell's equation has become the basis for electronic and electrical engineering. Nevertheless, examined here are some of their restrictions, notably in relativistic contexts and quantum mechanical where more enhanced theories become crucial. This analysis goals to give both thorough mathematical treatment and a well-defined understanding of the foundation equations of theoretical physics.*

**KEYWORDS:** Electromagnetism, Maxwell's equations, Gauss's law of Electricity, Ampère – Maxwell law, Quantum realm, Magnetic monopoles.

## INTRODUCTION

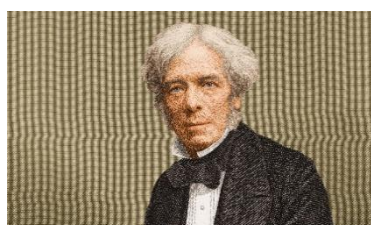
The combination of electricity and magnetism is one of the powerful accomplishments of the physics of the 19th century, and formulation of the Maxwell's Equations are the masterpiece of this unification. These equations didn't appear out of nowhere, but were forged in the fire of experimental and theoretical chemistry built up over decades by researchers all over Europe (Balanis, 2016).

### History of Maxwell's Equations

By the early 19th century, several central pillars of the science of electromagnetism had been discovered, notably, Coulomb's law (1780s), Ampère's force law (1825), Faraday's induction law (1831) and Lenz's and Neumann's contributions (1834), which pre-date Maxwell's formulation. This latter was later (1850s–1870s) developed in a collection of papers by James Clerk Maxwell (Zangwill, 2013).

Maxwell's masterpiece and his life's work reached fruition in *A Dynamical Theory of the Electromagnetic Field* (1865) where he quantitatively formulated the four equations which are now universally known as Maxwell's equations, establishing the first fundamental connection between magnetic, electrical and optical phenomena. These ideas were brought together in his *Treatise of 1873* (Purcell, 2013).

Later, Oliver Heaviside formulaically rewrote Maxwell's equations into a set of four elegant equations, and Heinrich Hertz experimentally demonstrated that the four equations accurately represented the nature of light. It was also on the basis of Maxwell's theories that Einstein based his special theory of relativity. Also, Maxwell, as well as Ludwig Boltzmann, formulated mechanical models (such as flywheel systems) to represent electromagnetic interaction (Einstein & others, 1905).



**Figure 1.** An English chemist and physicist became popular for his ground breaking work in electrochemistry and electromagnetism in 1791-1867, Michael Faraday



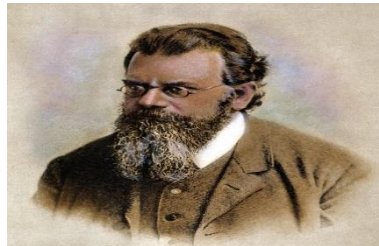
**Figure 2.** French engineer, physicist and officer named Charles-Augustin de Coulomb. He formulated the Coulombs Law which explains the electrostatic force between charged particles.



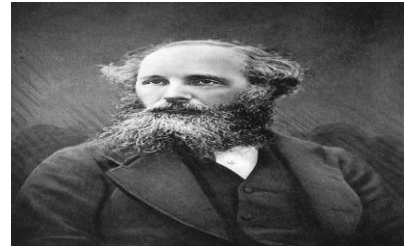
**Figure 3.** A brave French Physicist and Mathematician, André-Marie Ampère. He became one of the pioneers of the science of classical electromagnetism. He referred his foundational work to as electrodynamics.



**Figure 4.** An English mathematician and physicist named Oliver Heaviside is known for his work in which he reformulated the Maxwell's equations.



**Figure 5.** Austrian mathematical and theoretical physicist, Ludwig Edward Boltzmann. He has contributed his knowledge in developing the Maxwell-Boltzmann distribution.



**Figure 6.** Famously known for his exceptional work in formulating the electromagnetism theory and making the relationship between electromagnetic and light is a well-known Mathematician and Physicist named James C. Maxwell from Scotland.

### Role of Maxwell's Equations

The real strength and significance of Maxwell's equations is their capability of summarizing all classical electromagnetic effects in four compact formulas. In the language of vector calculus, these equations reflect profound relationships between electromagnetic fields and the corresponding sources which are electric charges and currents. The displacement current term, which is the essential new element that Maxwell's theory introduced, was necessary to prevent the appearance of serious inconsistencies in Ampère's law and to anticipate the existence of electromagnetic waves moving at light speed (Feynman, Leighton, Sands, & Hafner, 1965).

### Purpose and Outline

This report has three main objectives: first, to develop a careful-sounding derivation of each of the four equations of Maxwell from first principles; second, to discuss in depth the deep physical implications of these equations, in particular the prediction of electromagnetic Waves; third, to analyse both the great success that these equations have exhibited in the descriptions of the classical electromagnetic phenomena, as well as its limitation in the description of the modern physics. Our conversation will flow in a cascade, from the mathematical form of the equation of motion, to derivation, interpretation, application, and constraints (Griffiths, 2023).

In this analysis, the mathematical tools used are essentially vector calculus, especially the divergence and curl, and integral theorems of surface and volume integrals. With the aid also of these mathematical techniques as well as physical principles -- conservation laws of charge and energy -- we are able to write the experimental laws in the elegant differential form of Maxwell's equations (Heaviside, 1922).

### Understanding the Concept of Maxwell's Equations

Gauss's Law explains the relationship between electric fields and electric charges and how they are linked or connected to each other. It simply starts at the positive charge and terminates at the negative charge. The entire electric field is said to be linear if it flows from the closed surface to the enclosed charge. This includes the bound charge of the material polarization. Hence, electric fields can be formed in an empty space.



Gauss's law for Magnetism declares that the magnetic monopoles are non-existent. This simply implies that the existence of the north and south magnetic poles is not independent. Instead of the magnetic field being created due to a monopole, it emerges from a dipole. Thus, the magnetic flux through the surface is nil. Dissolution of the magnetic double into a loop or pair of characteristics "magnetic charges" opposing each other. In particular, the net magnetic flux via Gaussian surface is non-existent and the magnetic field is a solenoidal vector field (Jackson, 2021).

The Maxwell-Faraday form of Faraday's Law of Induction describes the concept that a non-uniform field (spatially varying) magnetic field is always linked with a flowing electric field, and especially, one which is not time varying. The quantity of work that is needed to move a charge around a closed part is proportional to the rate of change of the electromagnetic flux through any surface which is enclosed by that closed path.

The modification on Ampère's critical law by Maxwell is key, otherwise, the Ampère-Gauss laws would have to be modified to refer to static fields. Therefore, it expects a rotating magnetic field along with an alternating electric field. Another result is that of there being self propagating electromagnetic waves in the vacuum (Landau, et al., 2013).

The speed of electromagnetic waves proportional to the ratio  $c$  can be summarized from the experiments on charge and current, and that is the speed of light among others being X-rays, radio. Maxwell understood the relationship between tractions and light in 1861, when he married the electric and magnetic theories of light.

### Formulation of Maxwell's Equations

The Maxwell-Heaviside equations can be illustrated in either differential or integral forms and each of which has its advantages for various applications. In terms of vector calculus, each form represents fields at points in space (differential) and fields over regions of space (integral) (Maxwell, 1865).

| Name                                                           | Integral Equations                                                                                    | Differential Equations                                         |
|----------------------------------------------------------------|-------------------------------------------------------------------------------------------------------|----------------------------------------------------------------|
| <b>Gauss's Law</b>                                             | $\oint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho \, dV$     | $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$               |
| <b>Gauss's Law for Magnetism</b>                               | $\oint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0$                                                   | $\nabla \cdot \vec{B} = 0$                                     |
| <b>Maxwell – Faraday Equation (Faraday's Law of Induction)</b> | $\oint_{\partial\Sigma} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{\Sigma} \vec{B} \cdot d\vec{S}$ | $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ |



|                             |                                                                                                                                                                                    |                                                                                                            |
|-----------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|
| <b>Ampère – Maxwell Law</b> | $\oint_{\partial \Sigma} \vec{B} \cdot d\vec{l}$ $= \mu_0 \left( \iint_{\Sigma} \vec{J} \cdot d\vec{S} + \varepsilon_0 \frac{d}{dt} \iint_{\Sigma} \vec{E} \cdot d\vec{S} \right)$ | $\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$ |
|-----------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|

**Table 1: Differential and Integral Equations**

The differential equations are especially useful for:

- Examining field behavior at points in space
- Solving problems containing continuous charge and current distributions
- Deriving wave equations and boundary conditions

The integral equations are often more suitable for:

- Determining fields in symmetric configurations (spheres, cylinders, etc.)
- Handling macroscopic quantities (such as total charge, total current)
- Engineering applications where bulk properties are more important than discrete details

Both forms are mathematically equivalent through the divergence theorem and Stokes' theorem, but their practical utility depends on the specific problem being solved (Misner, Thorne, Wheeler, & Gravitation, 1973).

## RESULTS AND DISCUSSIONS

### Formulations of the Four Maxwell's Equations:

#### I. Gauss's Law

Gauss's Law is the first equation out of the four Maxwell's equations which declares that "an electric flux (the total electric across a given surface) via any closed surface is proportional to the total charge contained within that surface. The differential form is given by:

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$$

#### Integral Form Derivation

By definition, Gauss's law is given by:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enclosed}}{\epsilon_0} \quad (1)$$



Next, the theorem of Divergence is applied in order to transform the surface integral to a volume integral:

$$\oint_S E \cdot ds = \iiint_V (\nabla \cdot E) dv \quad (2)$$

Combining equations (1) and (2):

$$\iiint_V (\nabla \cdot E) dv = \frac{Q_{enclosed}}{\epsilon_0} \quad (3)$$

### Charge Density Relationship

The total charge can be expressed in terms of volume charge density  $\rho_v$ :

$$\rho_v = \frac{dQ}{dv} \quad \text{or} \quad dQ = \rho_v dv \quad (4)$$

Integrating over the volume gives the total enclosed charge:

$$Q_{enclosed} = \iiint_V \rho_v dv \quad (5)$$

### Final Derivation

Substituting equation (5) into equation (3):

$$\iiint_V (\nabla \cdot E) dv = \frac{1}{\epsilon_0} \iiint_V \rho_v dv$$

Since this must hold for any arbitrary volume  $V$ , the integrands must be equivalent:

$$\nabla \cdot E = \frac{\rho_v}{\epsilon_0}$$

This is the differential form of Maxwell's first equation, also known as Gauss's law for electrostatics (Peskin, 2018).

### Alternative Form

Using the electric displacement field  $D = \epsilon_0 E$ :

$$\nabla \cdot D = \rho_v$$

## II. Gauss's Law of Magnetism

Gauss's Law for Magnetism is the second equation out of the Maxwell's equations, which states that "the net magnetic flux through any closed surface is equal to zero". The mathematical expression is given by:

$$\nabla \cdot B = 0$$

or equivalently,

$$\nabla \cdot H = 0$$





## Derivation

By definition, Gauss's law for magnetism is expressed as:

$$\oint_S \vec{B} \cdot d\vec{s} = \phi_{enclosed} \quad (1)$$

For magnetic fields, magnetic monopoles do not exist. Therefore, enclosed magnetic flux is always equal to zero:

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \quad (2)$$

We can transform the surface integral to a volume integral simply by using the theorem of divergence:

$$\oint_S \vec{B} \cdot d\vec{s} = \iiint_V (\nabla \cdot \vec{B}) dv \quad (3)$$

Substituting equation (3) into (2):

$$\iiint_V (\nabla \cdot \vec{B}) dv = 0 \quad (4)$$

This equation must hold for any arbitrary volume  $V$ , which can only be satisfied if:

$$\nabla \cdot \vec{B} = 0$$

Since  $\vec{B} = \mu \vec{H}$ , where  $\mu$  is the magnetic permeability, we can also write:

$$\nabla \cdot \vec{H} = 0$$

## III. Faraday's Law of Induction

Faraday's Laws of Induction is the third equation out of the Maxwell's equations, which states that "the electromotive force (emf) around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path. The mathematical expression of the third equation is given by:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

## Derivation

By definition, Faraday's law of induction is given by,

$$emf_{alt} = -\frac{d\phi}{dt} \quad (1)$$

The sum of the magnetic field lines which pass through a given arbitrary or irregular surface  $S$  is:

$$\phi = \iint_S \vec{B} \cdot d\vec{s}$$

Substituting this equation into Faraday's law of induction, given in equation (1):



$$emf_{alt} = -\frac{d}{dt} \left( \iint_S B \cdot ds \right)$$

Assuming the surface  $S$  is stationary (not moving), the time derivative can be moved inside the integral:

$$emf_{alt} = \iint_S -\left(\frac{\partial B}{\partial t}\right) \cdot ds \quad (2)$$

The induced electromotive force (emf) around a closed loop is also defined as the line integral of the electric field:

$$emf_{alt} = \oint_C E \cdot dl \quad (3)$$

Equating equations (2) and (3) we have:

$$\oint_C E \cdot dl = \iint_S -\left(\frac{\partial B}{\partial t}\right) \cdot ds \quad (4)$$

Then, the line integral can be transformed to a surface integral by using the theorem of Stokes':

$$\oint_C E \cdot dl = \iint_S (\nabla \times E) \cdot ds$$

Substituting this into equation (4):

$$\iint_S (\nabla \times E) \cdot ds = \iint_S -\left(\frac{\partial B}{\partial t}\right) \cdot ds$$

The integrand should be equivalent as this holds for any arbitrary or irregular surface  $S$ :

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is Maxwell's third equation, also known as Faraday's law of electromagnetic induction.

#### IV. Ampère – Maxwell Law

The fourth and final equation out of the four Maxwell equations is derived from Ampere's circuital law with Maxwell's correction, which establishes the relationship between magnetic field to the electric current and changing electric field (Landau, et al., 2013). The mathematical expression is given by:

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$





### Derivation from Ampere's Law

By definition, Ampere's circuital law is given by:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 i \quad (1)$$

Then, the line integral is transformed to a surface integral by applying the Stokes' theorem:

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S (\nabla \times \vec{B}) \cdot d\vec{s} \quad (2)$$

From equations (1) and (2):

$$\iint_S (\nabla \times \vec{B}) \cdot d\vec{s} = \mu_0 i \quad (3)$$

The current  $i$  can be expressed as  $J$ , current density:

$$i = \iint_S \vec{J} \cdot d\vec{s} \quad (4)$$

Substituting (4) into (3):

$$\iint_S (\nabla \times \vec{B}) \cdot d\vec{s} = \mu_0 \iint_S \vec{J} \cdot d\vec{s}$$

$$\iint_S [(\nabla \times \vec{B}) - \mu_0 \vec{J}] \cdot d\vec{s} = 0$$

The integrand should be equivalent to zero as this holds for any arbitrary or irregular surface  $S$ :

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Using the relation  $\vec{B} = \mu_0 \vec{H}$ :

$$\nabla \times \vec{H} = \vec{J} \quad (\text{Original Ampere's Law})$$

### Maxwell's Modification

The modified Ampere's circuital law includes displacement current:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Where  $\Phi_E$  is the electric flux. The differential form becomes:

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d \quad (1)$$

The current density is  $J_d$ :



$$J_d = \epsilon_0 \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t} \quad (\text{since } D = \epsilon_0 E)$$

Substituting  $J_d$  into equation (1):

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

This is Maxwell's fourth equation in its complete form, incorporating both conduction current and displacement current.

### Physical Interpretation and Implications of the Maxwell Equations

The formulation of the four well known Maxwell's equations are one of the marvelous integrations in physics unveiling how magnetism, light and electricity are all characteristics of a single electromagnetic thing. The Ampère-Maxwell rule and the Faraday-Lenz rule, respectively, have bestowed forecast of the existence of electromagnetic waves by the supposed coupled time-evolution of the magnetic and electric fields which became the most extraordinary consequence. These wave solutions moved at light speed – a hint which made Maxwell come to a realization that light was in fact an electromagnetic wave. This vast uncovering has married electromagnetism to optics and paved a way for innovations such as X-rays, radio and microwaves. What makes it more pleasing is that the equations also imply a kind of symmetry between magnetic and electric waves – changing magnetic fields create electric fields (Faraday's Law) and changing electric fields create magnetic fields (Ampère's Law) – this makes electromagnetic waves keep going indefinitely (Feynman, Leighton, Sands, & Hafner, 1965).

Maxwell's equations go far beyond wave theory to the root of modern electrical engineering and communications technology. They prescribe the behaviour of circuits (as low frequency approximations and via Kirchhoff's laws), radiation from antennas, and optics involving lenses and fibre optics. The equations also preserve the conservation of charge connecting electromagnetism to the underlying physics. But their classical framework fails at quantum scales, where quantum electrodynamics (QED) is needed, and in strong gravitational fields, where relativity comes into play. In spite of these limitations, Maxwell's equations continue to be indispensable, underpinning technologies from wireless communication to medical imaging and serving as a wellspring to theoretical physics, including the work that led to Einstein's special relativity theory. Their beautiful unification of observed laws in one mathematical theory is a testament to the ability of theoretical physics to unveil nature's profound connections.

### Maxwell's Equations in Our Technological Society

Maxwell's equations are the basis of almost all modern electrical and communications technologies. Perhaps their most straightforward application is in electronics and circuit analysis, where Kirchhoff's voltage and current laws appear as simplified versions of Maxwell's equations at quasi-static or low-frequency. These principles allow engineers to create and optimize everything from devices in our home to power grids. Outside of the realm of circuits, Maxwell's equations fundamentally altered our understanding of electromagnetic waves, and in so doing, they predicted the existence of microwaves, radio waves, visible light, infrared and x-rays – all of which are solutions to the wave equations that emerged from Maxwell's



investigations. This insight is at the core of wireless communication, radar systems, medical imaging, and astronomy, connecting disparate phenomena ranging from Wi-Fi signals to starlight using a single framework (Misner, Thorne, Wheeler, & Gravitation, 1973).

The direct importance of the four equations also applies to the area of antenna design: the radiation patterns and the antenna's efficiency are directly derived from Maxwell's equations. That's vital for technologies such as cell networks, satellite-based communications and GPS. In optics, the equations describe the behavior of light through lenses, mirrors and optical fibers, leading to advances in microscopy, laser manufacture and the high-speed internet through fiber-optic cables. Also, electromagnetic compatibility (EMC) uses Maxwell's principles to reduce interference between electronic devices, so that systems, such as an aircraft's electronics or medical equipment, function

without crosstalk. Maxwell's equations are still shaping our technological world, from household appliances to sophisticated scientific instruments.

Their impact is only burgeoning in fields such as photonics, quantum computing and metamaterials, demonstrating this 19th-century breakthrough is still at the center of 21st-century innovation.

### Maxwell Equations Limitation

Despite their incredible success, Maxwell's equations suffer from some limitations:

- **Quantum Realm:** They do not describe atomic-scale phenomena, which are described by quantum electrodynamics (QED).
- **Relativistic effects:** Special relativistic effects are accounted for, but general relativistic effects need to be corrected-for.
- **Non-linear Media:** In strong fields, especially in certain materials, non-linear effects come into play.
- **Magnetic Monopoles:** Given the absence of magnetic monopoles ( $\nabla \cdot \mathbf{B} = 0$ ), and yet many theories suggest that such things should exist.

### CONCLUSION

While perusing in this way Maxwell's equations, I recovered an admiration for their elegance and their grasp to link phenomena in physics. These four equations showed me how electric and magnetic fields connect with each other through their origins (charges and currents) and how these fields propagate into electromagnetic waves—and that the light itself is nothing but an electromagnetic phenomenon. I learned to deduce their mathematical derivation from experimental laws (Coulomb, Ampère, Faraday) to differential expressions by means of vector calculus and to understand the important significance of Maxwell's displacement current in the sense of eliminating severe discrepancies. The derivations emphasized the role of symmetry, with the changing (E-) B-fields generating E-fields, and vice versa, for wave propagation.



On the practical side, I have come to appreciate the importance of these equations to modern technological advancements such as circuits and wireless communication as well as their limitations over quantum and relativistic domains. But more importantly, the study showed how Maxwell's unification of diverse phenomena into a unified framework showed the power of mathematical physics to unite and predict natural behavior, a lesson that has inspired advances across science and engineering.

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