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TOPOLOGICAL ISOMORPHISM OF THE SPACE OF FRÉCHET GENERALISED FUNCTIONS

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ABSTRACT: *The characterisation of the space of Fréchet generalised functions was established with respect to the weak-$\star$ topology, which confirms the completeness of the space, G_X, through the space of test functions , $D(X)$. The isomorphism bridges the gap between the Fréchet space of test functions and the space of generalised functions.*

KEYWORDS AND PHRASES: Generalised Functions, Fréchet Spaces, Test Functions, Weak $\star$ -topology, Isomorphism.

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INTRODUCTION

Fréchet spaces originated from Maurice Fréchet's 1906 thesis, where they represented a significant generalisation of Banach spaces, allowing more flexible structures while maintaining completeness properties. These spaces are defined using a countable family of seminorms, making them especially useful in studying function spaces where a single norm might be too restrictive. The development of Fréchet spaces has been crucial in expanding functional analysis, providing mathematicians with tools to handle infinite-dimensional spaces that naturally appear in mathematics and physics. Meanwhile, the theory of generalised functions was developed in the mid-twentieth century through the work of Laurent Schwartz and Paul Dirac, marking a profound extension of the classical concept of functions. The need for generalised functions stemmed from limitations in classical analysis, particularly in representing physical phenomena such as point charges, point masses, or instantaneous impulses. The Dirac delta "function" is the most famous example. Although not a function in the traditional sense, it acts as an object that is zero everywhere except at a single point, where it is "infinitely large" in a way that its integral equals one.

Mathematically, generalised functions are defined not by their pointwise values but by their action on test functions through integration. This approach allows a rigorous treatment of operations like differentiation on objects that may not be differentiable, or even continuous, in the classical sense. For instance, the derivative of the Heaviside step function can be properly defined as the Dirac delta distribution. The theory of generalised functions provides a framework for solving differential equations in a broader setting, accommodating solutions that would not exist classically, while offering more powerful analytical tools and more accurate modelling of physical phenomena.

In other words, the theories of generalised functions and Fréchet spaces are integral to modern functional analysis. Generalised functions extend classical function theory to handle objects like the Dirac delta, while Fréchet spaces provide a natural setting for these generalised functions. Specifically, the space of generalised functions can be understood as the dual space of smooth functions with compact support, a framework that aligns naturally with the topology of Fréchet spaces.

Many researchers have contributed significantly to the study of Fréchet spaces and generalised functions in recent decades. Among them is Dobson (2011) who investigates Fréchet geometry, extending earlier results on the structure of second tangent bundles to infinite-dimensional Banach and Fréchet manifolds. His work led to further insights into differential equations in Fréchet structures, along with results on the hypercyclicity of operators on Fréchet spaces. Ivanov *et al*. (2018) proves a surjectivity result in Nash-Moser-type Fréchet spaces, using uniform estimates over all seminorms. Their method applies to continuous and strong Gâteaux differentiable functions and extends to a multi-valued setting. Their key innovation was geometrising tameness estimates, reducing the problem to Banach space subproblems solvable via an abstract iteration scheme.

Furthermore, Curiel (2015) emphasises the importance of completeness and translation invariance in Fréchet metrics, which ensure compatibility between topological structure and vector space operations. Zaal (2015) notes that boundedness in Fréchet spaces differs from the usual metric sense. Instead, it defines via absorption by neighbourhoods: a subset (W) is

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bounded if, for every open neighbourhood (U) of 0, there exists (t > 0) such that W ⊂ tU. Allahyari *et al*. (2020) introduces a new contraction concept and extends the Tychonoff fixed-point theorem, applying it to infinite systems of integral equations using measures of noncompactness in Fréchet spaces.

Lupini (2022) on the one hand develops a theory of canonical approximations for Polishable subgroups of Polish groups, classifying their Borel complexity. His work also characterises the ranges of continuous linear maps between separable Fréchet and Banach spaces. Freyn (2013) on the other hand extends a result of Mashreghi and Ransford, proving that every separable infinite-dimensional Fréchet space with a continuous norm is isomorphic to a subspace of holomorphic functions on the unit disc or complex plane. This leads to examples of nuclear Fréchet spaces of holomorphic functions without the bounded approximation property. Shaviv (2018) defines Schwartz functions, tempered functions, and tempered distributions on manifolds definable in polynomially bounded o-minimal structures, showing that classical properties from the Nash category hold in this generalised setting, while Giordano *et al*. (2022) proves Picard-Lindelöf convergence for smooth normal Cauchy problems in PDEs under a Weissinger-like condition, encompassing non-analytic cases. They also derived an inverse function theorem for graded Fréchet spaces.

Giordano *et al*. (2014) introduces functionally compact sets in Colombeau's generalised functions, constructing test function spaces analogous to distribution theory and studying their topological properties. Khan and Lamb (2013) develop a summability theory for orthonormal sets in multi-normed spaces, applying it to test generalised function spaces, including almost-periodic generalised functions with uncountable exponential bases.

Fréchet spaces of generalised functions find natural applications in differential geometry and analysis in the work of Jiang *et al.* (2012) as they study differential operators on manifolds, showing that the space ($\mathrm{DO}(M)_k$) (operators of order (k)) forms a Fréchet space, providing a rigorous framework for analysis. Dave (2009) also introduces a Fréchet structure for analysing differential equations in mathematical physics, enabling novel approaches to regularity. For instance, the Sobolev regularity of a distribution (u) can be interpreted via tameness of maps between the spaces, C^{∞}(M) and $\varphi^{-\infty}$(M). Al-Omari (2019) successively explores local regularity properties (e.g., singular support, wavefront sets) using tameness conditions. He also extends integral transforms (e.g., Bessel-type integrals) between Fréchet spaces of Boehmians (a generalisation of distributions). Sheehy (2019) establishes convolution products and fundamental theorems in Boehmian spaces, extending integral transform methods. Additionally, he applies Fréchet space concepts to computational geometry, generalising the Fréchet distance for comparing complex objects.

This paper is divided into three parts: Part I – Introduction and Literature Review; Part II – Definitions and Propositions; and Part III – Results and Conclusion.

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Preliminaries

Definition (Fréchet Space)

A Fréchet space, X, is a topological vector space that satisfies two key conditions:

1. **Metrisability:** There exists a metric d on the space X such that X is homeomorphic to a metric space. This means the space can be described with a distance function.

2. **Complete:** The space is complete with respect to the topology induced by the metric. In other words, every Cauchy sequence in the space converges to an element in the space.

In other words, a Fréchet space can be characterised as a locally convex space whose topology is generated by a countable family of seminorms or a locally convex space that is complete with respect to a translation-invariant metric. (See Allahyari *et al.*, 2020; Lupini, 2022; Curiel, 2015.)

The Space of Generalised Functions

Generalised functions are defined through a dual-space approach, requiring first the establishment of appropriate spaces of well-behaved "test functions." Two fundamental test function spaces are commonly used: (i) the space D of infinitely differentiable functions with compact support, and (ii) the space S (sometimes denoted T) of rapidly decreasing smooth functions, also known as the Schwartz space.

We will take the definition of the test functions.

Definition (Test Functions)

A test function is a smooth function with compact support (functions that are infinitely differentiable, that are non-zero within a bounded interval). These functions are the elements of the space $C_c^\infty(X)$ which are called "sufficiently good" functions on which the generalised functions act. In the same vein, for the function $\phi : \mathrm{R}^n \to \mathrm{R}$, is expressed as

$$\phi(t) = \{exp^{\frac{1}{t}}\, if\ t < 0;\ 0\, if\ t \geq 0 \qquad (2.1)$$

is a member of $C^\infty(\mathrm{R})$, where $\phi(t) = \phi_0(||t||^2 - 1)$, and $\phi \in C_c^\infty(\mathbb{R})$, and the support of ϕ is the closed unit ball $\{x \in \mathrm{R} : |x| \leq 1\}$. (See Georgiev, 2015.)

Definition (Generalised Functions)

Let $X \subset \mathrm{R}^n$ be an open set. Then, the generalised functions are continuous linear functionals, $D'(X)$, over a space of infinitely differentiable functions $D(X)$ such that all continuous functions have derivatives which are themselves generalised functions. (See Al-Gwaiz, 1992.)

Many classical spaces of test functions, such as $C_c^\infty(X)$ (smooth functions with compact support), can be viewed as Fréchet spaces. These spaces have the topology of smooth convergence, making them ideal spaces to test the generalised functions.

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Specifically, the topology of this space $C_c^\infty(X)$ can be defined by a family of seminorms that measure the size of a function and its derivatives up to a certain order, thus forming a Fréchet space. Generalised functions act as continuous linear functionals on such spaces.

Throughout this work, the space of generalised functions will be denoted by G_X.

Since Fréchet spaces are complete with respect to a topology defined by seminorms, they allow for smooth functions and their derivatives to converge in a manner that is consistent with the behaviour of generalised functions. The Fréchet topology enables the space of smooth functions to be refined enough to handle generalised functions like the test and Delta functions. This implies that the generalised functions can enjoy the properties of the Fréchet spaces – bornology and barallel.

Proposition: *G_X is a bornological space. (See Nwachukwu et al., 2025.)*

Nwachukwu *et al*. remarked that G_X, being a bornological space, allows the flow of an inductive limit topology. Also, it is a Fréchet space, though not metrisable but can converge in a weak*-topology.

The Space of Generalised Functions, G_X, versus the Fréchet Space, X

With the help of convergence of the space test functions, G_X can be seen as a Fréchet space when we equip G_X with a topology that makes it complete, metrisable and locally convex. We define a topology on G_X using the notion of convergence of test functions.

A sequence of generalised functions G_{Xn} is said to converge to another generalised function G_{X1} if for every test function $\phi \in D(X)$ of smooth functions with compact support, the sequence of numbers $\{G_{Xn}(\phi)\}$ converges to $G_{X1}(\phi)$ as n tends to infinity. With this, we define a weak $\star$-topology on G_X. In this topology, a sequence of generalised functions converges when applied to every test functions, $\phi \in D(X)$ and it makes G_X a locally convex topological vector space. Consequently, for G_X to be a Fréchet space, we show that G_X admits a complete metric due to the topology of convergence in the test functions. Sequel to this, we say that G_X is a sequentially complete metrisable locally convex space.

Theorem: *Let D(X) be a Fréchet space of smooth functions with compact support equipped with the topology induced by a countable family of seminorms $\{P_K\}$. Let G_X be the space of generalised functions equipped with weak $\star$- topology. Then, there exists a continuous linear map $\tau : D(X) \longrightarrow G_X$ that preserves convergence such that for any sequence $\{\phi_n\}$ in G_X which converges to 0 in D(X), with respect to the seminorms, the corresponding sequence $(\tau\{\phi_n\})$ in G_X converges to 0 in the weak $\star$ -topology.*

Proof: Let $\phi, \varphi \in D(X)$. Then, we define the linear function $\tau: D(X) \to G_X$ by $(\phi)(\varphi) = \phi(\varphi)$ $\forall \phi, \varphi \in D(X)$. Next, we show that τ is well-defined by choosing a function in $D(X)$ which has an equivalent class. By Proposition 2.2.3, we pick two functions φ and ψ in $D(X)$ such that $\varphi - \psi = 0$, which implies $(\varphi - \psi)(\phi) = 0\ \forall \phi \in D(X)$. Now, $\tau(\varphi)$ and $\tau(\psi)$ are equivalent elements in G_X, for any given generalised functions, $u \in G_X$, and by definition of τ, we have

$$u\big(\tau(\varphi)\big) = u(\varphi) \tag{3.1}$$

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Adding $u(\psi)$ to both sides, we have

$$u\big(\tau(\psi)\big) = u(\psi) \tag{3.2}$$

Now, in the sense of generalised functions, $\varphi - \psi = 0$, we have $u(\varphi - \psi) = 0$. By linearity of u, we rewrite this as

$$u(\varphi) - u(\psi) = 0 \tag{3.3}$$

$$u(\varphi) = u(\psi) \tag{3.4}$$

This shows that $\big(\tau(\varphi)\big) = u\big(\tau(\psi)\big)\ \forall u \in G_X$. This implies that $\tau(\varphi)$ and $\tau(\psi)$ are equivalent in G_X.

Suppose we have two distinct elements φ and ψ in $D(X)$ such that $\varphi \neq \psi$, and this implies that $\tau(\varphi) \neq \tau(\psi)$. By definition of τ,

$$\tau\big(\varphi(\phi)\big) = \varphi(\phi), \forall \phi \in D(X) \tag{3.5}$$

and

$$\tau\big(\psi(\phi)\big) = \psi(\phi), \forall \phi \in D(X). \tag{3.6}$$

Since φ and ψ are distinct, there exists a test function $\phi_0 \in D(X)$ such that $\varphi(\phi_0) \neq \psi(\phi_0)$. Therefore, $\tau\big(\varphi(\phi_0)\big) \neq \tau\big(\psi(\phi_0)\big)$. This implies $\tau(\varphi)$ and $\tau(\psi)$ are two distinct elements in G_X.

Thus, τ is injective.

Next, let $u \in G_X$. Then, there exists an element $\varphi \in D(X)$ such that $\tau(\varphi) = u$. Since $u \in G_X$, for any test function $\phi \in D(X)$, we have $u(\phi)$ is a real number. Now, given a function $\varphi \in D(X)$ such that $\tau(\varphi) = u, \Rightarrow \forall \phi \in D(X); \tau(\varphi)(\phi) = \varphi(\phi) = u(\phi)$

Also, if we consider $\phi_0 \in D(X)$, $\varphi(\phi_0) = u(\phi) \in D(X)$, since $u(\phi)$ is a real number for any test function ϕ. For φ_0, we have

$$\tau(\varphi_0)(\phi) = \varphi_0(\phi) = u(\phi). \tag{3.7}$$

Thus, we find an element $\varphi_0 \in D(X)$ such that $\tau(\varphi_0) = u$ for any $u \in G_X$. Then, τ is surjective. Now, let $\{\varphi_n\} \in D(X)$ be a sequence of test functions on $D(X)$ that converges to a test function $\varphi \in D(X)$. That is, for all seminorms P_K on $D(X)$, we have

$$\lim_{n\to\infty} P_K(\varphi_n - \varphi) = 0 \tag{3.8}$$

By Proposition 4.1.1, this implies that the sequence $(\varphi_n - \varphi)$ converges to 0 in $D(X)$ with respect to the seminorms. Then, $\{\tau(\varphi_n)\}$ converges to $\tau(\varphi)$ in the weak ⋆- topology. For any generalised function $u \in G_X$, the sequence $\{u\big(\tau(\varphi_n)\big)\}$ converges to $u\big(\tau(\varphi)\big)$. Now, let $u \in G_X$ be arbitrary such that

$$\lim_{n\to\infty} u\big(\tau(\varphi_n)\big) = u\big(\tau(\varphi)\big). \tag{3.9}$$

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By the definition of τ, we have

$$u\big(\tau(\varphi_n)\big) = \tau(\varphi_n)(u) \tag{3.10}$$

If we consider the sequence $(\varphi_n - \varphi)$, we have

$$\lim_{n\to\infty} (\varphi_n - \varphi)(u) = 0 \tag{3.11}$$

By linearity of τ, we have

$$\tau(\varphi_n - \varphi)(u) = \tau(\varphi_n)(u) - \tau(\varphi)(u) \tag{3.12}$$

$\Rightarrow$

$$\lim_{n\to\infty} [\tau(\varphi_n)(u) - \tau(\varphi)(u)] = 0. \tag{3.13}$$

We rearrange this and obtain

$$\lim_{n\to\infty} \tau(\varphi_n)(u) - \lim_{n\to\infty} \tau(\varphi)(u) = 0 \tag{3.14}$$

$\Rightarrow$

$$\lim_{n\to\infty} \tau(\varphi_n)(u) = \lim_{n\to\infty} \tau(\varphi)(u). \tag{3.15}$$

Thus, $\{\tau(\varphi_n)\}$ converges to $\tau(\varphi)$ in the weak$\star$ $-topology$. Therefore, τ is linear, injective, surjective and preserves convergence. There is a topological isomorphism between $D(X)$ and G_X.

CONCLUSION

This paper has established that G_X is the continuous dual of $D(X)$ endowed with the weak-$\star$ topology, confirming the completeness of the space of generalised functions, G_X. Furthermore, the isomorphism provides a key link between the Fréchet space of test functions and the space of generalised functions.

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