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ANALYSIS OF FRACTIONAL INPUT STABILITY AND GLOBAL ASYMPTOTIC STABILITY OF SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT: *Fractional input stability and global asymptotic stability of systems of fractional differential equations have been studied in the sense of Caputo. Systems of fractional differential equations have been studied using Lyapunov method and other known stability notions. Two stability theorems have been stated and proved. Examples are given to demonstrate the utilization of the theorems.*

KEYWORDS: Fractional Input Stability, Global Asymptotic Stability, Fractional Differential Equations, Caputo derivative.

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INTRODUCTION

In more than three centuries, differential calculus has attracted the interest of researchers from diverse fields of study. The literature of differential equation, be it integer-order differential equation or fractional-order differential equation has continued to grow geometrically due to increasing researches in the subject [1-10].

Fractional calculus is as old as the traditional integer-order calculus proposed independently by Newton and Leibniz at the end of the seventeenth century. However, for one reason or another, fractional calculus had its development accelerated only from the second half of the twentieth century, after the first international conference on the subject. Nowadays, fractional calculus is completely consolidated and is a subject of research in several fields of knowledge.

Stability is one of the most important issues in the analysis and design of dynamical systems. A system that is not stable may burn out, disintegrate or saturate when a signal is applied. Therefore, an unstable system is useless in practice and needs a stabilization process through an additional control parameter. The literature on the stability analysis of differential equations (integer order or fractional) is rich due to increasing interest and much work already done in the subject[11-20]. Stability analysis of fractional differential equation with input stability is recently occupying the attention of professionals in science and engineering fields.

Different types of input stability exist such as conditional stability, Mittag-Leffler input stability, input-to-state stability, etc. These stability conditions address suitable situations of stability analysis. The input-to-state stability with respect to fractional differential systems is known popularly in the literature as Fractional Input Stability. It is a novel notion for studying the behavior of fractional differential equations and was recently introduced in the literature by Sene [21,22]. This state of stability can be converging or bounding.

Another important stability type related to the fractional input stability is the Global Asymptotic Stability. Global asymptotic stability is of importance from a theoretical as well as an application point of view in several fields. Much work has been done on analyzing the fractional input stability and the global asymptotic stability of fractional differential equations.

Sene [21] studied fractional input stability and contributed to introducing a new stability notion in the stability analysis of fractional differential equations with exogenous inputs using Caputo fractional derivative. The fractional input stability of the electrical circuit equations described by the fractional derivative operators has been investigated [23].

Mostafa Adimy et al [24] investigated a system of two nonlinear age-structured partial differential equations describing the dynamics of proliferating and quiescent hematopoietic stem cell populations. By constructing a Lyapunov functional, a necessary and sufficient condition for global asymptotic stability of the trivial steady state was obtained. Stamova [25], studied nonlinear system of impulsive differential equations of fractional order and gave sufficient conditions for Mittag-Leffler stability and uniform asymptotic stability of the zero solution of the system under consideration.

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Sene[26] presented a global asymptotic stability criterion for the fractional differential equations in triangular form using Caputo generalized fractional derivative and introduced a new procedure for studying the global asymptotic stability of the fractional differential equations. A system of fractional differential equation in which one of the equations is fractional input stable and the other is globally asymptotically stable was considered.

In this work, a system of fractional differential equation in which the two equations are fractional input stable will be studied. Also, a system with both equations being globally asymptotically stable will be studied. Section one is the general introduction, section two is definitions and preliminaries and section three is the main result. Application and conclusion are considered in sections four and five respectively.

Definitions and Preliminary Analysis

In this section, some definitions and concepts that would be needed in the analysis are given.

Definition 2.1:

The Mittag-Leffler function with two parameters is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where the parameters $\alpha > 0,\ \beta \in \mathbb{R}$ and $z \in \mathbb{C}$.

Definition 2.2:

The class PD function denotes the set of all continuous functions $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class K function is an increasing PD function. The class K_∞ represents the set of all unbounded K functions.

Definition 2.3

A continuous function $\beta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class KL if $\beta(.,t) \in$ K for any $t \geq 0$, and $\beta(s,.)$ is non-increasing and tends to zero as its arguments tend to infinity.

Definition 2.4

Given a function $f : [0, +\infty[\to \mathbb{R}$. Then, the Riemann-Liouville fractional derivative of f of order α is defined by

$$(D_\alpha^{RL} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

for $t > 0, and \quad \alpha \in (0,1)$.

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Definition 2.5

The fractional differential equation defined by $D_{\alpha}^{c}x = f(x,u)$ is said to be globally asymptotically stable if there exists a class KL function β such that for any initial condition ξ, the following inequality holds

$$\|x(t,\xi)\| \leq \beta(\|\xi\|, t - t_0),$$

where $\|.\|$ represents the Euclidean norm.

Definition 2.6

The fractional differential equation defined by $D_{\alpha}^{c}x = f(x,u)$ is said to be fractional input stable if for any input $u \in \mathbb{R}^n$, there exists a class KL function μ , and a class K_{∞} function γ such that for any initial condition $x(t_0)$, its solution satisfies

$$\|x(t,x_0,u)\| \leq \mu(\|x_0\|, t - t_0) + \gamma(\|u\|_{\infty})$$

where γ is the asymptotic gain.

Definition 2.7

The trivial solution to system $D_{\alpha}^{c}x = f(t,x,0)$ is said to be stable if , for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for any initial condition $\|x_0\| < \delta$, the solution $x(t)$ of the system $D_{\alpha}^{c}x = f(t,x,0)$ satisfies the inequality $\|x_0\| < \epsilon$ for all $t > t_0$.

The trivial solution to system $D_{\alpha}^{c}x = f(t,x,0)$ is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to\infty} x(t) = 0$.

MAIN RESULTS

In this section, the main results of this paper are given.

In a system with two fractional differential equations in which one of the fractional differential equations is fractional input stable and the other is global asymptotically stable, it is shown that the system is globally asymptotically stable [25].

The following theorem considers a case where the two fractional differential equations in the system are globally asymptotically stable.

Theorem 3.1

Consider the fractional differential equations defined by $D_{c}^{\alpha,\rho}x_1 = f(x_1)$ and $D_{c}^{\alpha,\rho}x_2 = f(x_2)$ which are globally asymptotically stable, then, the system

$$\begin{cases} D_{c}^{\alpha,\rho}x_1 = f(x_1) \\ D_{c}^{\alpha,\rho}x_2 = f(x_2) \end{cases} \qquad (3.1)$$

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is globally asymptotically stable.

Proof

The global asymptotic stability of $D_c^{\alpha,\rho} x_1 = f(x_1)$ and $D_c^{\alpha,\rho} x_2 = f(x_2)$ respectively implies that $\lim_{t\to\infty} \|x_1\| = 0$ and $\lim_{t\to\infty} \|x_2\| = 0$. Hence, the system (3.1) is convergent. Next, we show that the system is stable.

The global asymptotic stability of $D_c^{\alpha,\rho} x_1 = f(x_1)$ $(D_c^{\alpha,\rho} x_2 = f(x_2))$ implies that there exists a class KL function $\beta_1(\beta_2)$ such that for any initial condition ξ_1 (ξ_2), the following inequalities hold respectively

$$\|x_1(t)\| \leq \beta_1(\|\xi_1\|), t^\rho) \leq \beta_1(\|\xi_1\|), 0) < \epsilon$$

and

$$\|x_2(t)\| \leq \beta_2(\|\xi_2\|), t^\rho) \leq \beta_2(\|\xi_2\|), 0) < \epsilon$$

The system is therefore stable. Combining the convergence and stability of the system (3.1), we conclude that the system is asymptotically stable.

The second theorem considers a case where the two fractional differential equations in the system are fractional input stable.

Theorem 3.2

Consider the fractional differential equations defined by $D_c^{\alpha,\rho} x_1 = f_1(x_1, x_3)$ and $D_c^{\alpha,\rho} x_2 = f_2(x_2, x_4)$ which are fractional input stable, then, the system

$$\begin{cases} D_c^{\alpha,\rho} x_1 = f_1(x_1, x_3) \\ D_c^{\alpha,\rho} x_2 = f_2(x_2, x_4) \end{cases} \tag{3.2}$$

is fractional input stable.

Proof

From the fractional input stability of the fractional differential equations defined by

$D_c^{\alpha,\rho} x_1 = f_1(x_1, x_3)$ and $D_c^{\alpha,\rho} x_2 = f_2(x_2, x_4)$ and the converging-input converging-state property, respectively, we obtain the following

$$\lim_{t\to\infty} \|x_1(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|x_2(t)\| = 0.$$

Therefore, solution of the system (3.2) is convergent.

Also, since the fractional differential equation defined by $D_c^{\alpha,\rho} x_1 = f_1(x_1, x_3)$ is fractional input stable, there exists a class KL function β_1 and a K_∞ function γ such that for any initial condition ξ_1 , its solution satisfies

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$$\begin{aligned}\|x_1(x)\| &\le \beta_1(\|\xi_1\|, t^\rho) + \gamma(\|x_3\|_\infty)\\ &\le \beta_1(\|\xi_1\|, 0) + \gamma(\|\beta_3(\|\xi_1\|, 0)\|) < \epsilon\end{aligned}$$

Similarly, from the fractional input stability of the fractional differential equation defined by $D_c^{\alpha,\rho} x_2 = f_2(x_2, x_4)$, there exists a class KL function β_2 and a K_∞ function γ such that for any initial condition ξ_2 , its solution satisfies

$$\begin{aligned}\|x_2(x)\| &\le \beta_2(\|\xi_2\|, t^\rho) + \gamma(\|x_4\|_\infty)\\ &\le \beta_2(\|\xi_2\|, 0) + \gamma(\|\beta_4(\|\xi_2\|, 0)\|) < \epsilon\end{aligned}$$

Combining the convergence and stability of the system, the system is therefore fractional input stable.

APPLICATION AND ANALYSIS

Example 1:

Consider the fractional differential equation defined by

$$\begin{cases} D_c^{\alpha,\rho} x_1 = -2x_1 \\ D_c^{\alpha,\rho} x_2 = -x_2 \end{cases} \tag{4.1}$$

The above fractional equation is described by Caputo fractional derivative.

Using Lyapunov method and the Lyapunov function $V(x) = \frac{x_1^2}{2}$, clearly, $D_c^{\alpha,\rho} x_1 = -2x_1$ is globally asymptotically stable.

Similarly, $D_c^{\alpha,\rho} x_2 = -x_2$ is globally asymptotically stable.

Thus, by Theorem 3.1, the system (4.1) is globally asymptotically stable. This result can be confirmed by comparing with the result obtained by using the classical condition

$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, where $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. The result obtained using the classical condition agrees with the result obtained by the application of Theorem 3.1.

Example 2

Here the case where the two sub-fractional differential equations are fractional input stable is considered.

$$\begin{cases} D_c^{\alpha,\rho} x_1 = -2x_1 + x_2 \\ D_c^{\alpha,\rho} x_2 = -x_2^3 - x_2 + 2x_1 \end{cases} \tag{4.2}$$

Equation (4.2) is described by Caputo fractional derivative.

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The solution of the sub-fractional differential equation $D_c^{\alpha,\rho}x_1 = -2x_1 + x_2$ is given by

$$x_1(t) = \phi E_\alpha(B(\frac{t^\rho}{\rho})^\alpha) + \int_0^t \left(\frac{t^\rho - l^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}(B(\frac{t^\rho}{\rho})^\alpha)x_2(l)\frac{dl}{l^{1-\rho}} \quad (4.3)$$

Equation (4.3) reduces to

$$\|x_1(t)\| \leq D(\|\phi_1\|, t^\rho) + K\|x_2\| \quad (4.4)$$

where $D(\|\phi_1\|, t^\rho) = \phi E_\alpha(B(\frac{t^\rho}{\rho})^\alpha)$ and K is a constant number.

From equation (4.4), clearly, the sub-fractional differential equation $D_c^{\alpha,\rho}x_1 = -2x_1 + x_2$ is fractional input stable.

Next, the second sub-fractional differential equation is examined.

Let $V(x_2) = \frac{x_2^2}{2}$ be a Lyapunov function, the second sub-fractional differential equation becomes

$$D_c^{\alpha,\rho}x_2 = -x_2^3 - x_2 + 2x_1 \leq -(1-\beta)x_2^4$$

where $\beta \in (0,1)$.

Therefore, $D_c^{\alpha,\rho}x_2 = -x_2^3 - x_2 + 2x_1$ is fractional input stable.

Thus, applying Theorem 3.2, the system (4.2) is fractional input stable.

CONCLUSION

Fractional input stability and global asymptotic stability have been discussed extensively. A system of fractional differential equation consisting of two sub-fractional differential equations which are globally asymptotically stable have been studied. Also, a system comprising two sub-fractional equations which are fractional input stable have been discussed. Two theorems have been stated and proved for the two cases. Examples have been given and analysed to show the effectiveness and utilization of the theorems.

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