African Journal of Mathematics and Statistics Studies

ISSN: 2689-5323

Volume 8, Issue 3, 2025 (pp. 73-95)

www.abjournals.org

ROBUST ESTIMATION IN SIMULTANEOUS EQUATION MODELS: ADDRESSING MULTICOLLINEARITY AND HETEROSCEDASTICITY THROUGH ADAPTIVE PENALIZED GMM TECHNIQUES

Okeke Ngozi Christy[1*], Olanrewaju Samuel Olayemi[2],

and Mohammed Zubairu Anono[3]

[1,2,3]Department of Statistics, Faculty of Science, University of Abuja, Abuja, Nigeria

[*]Corresponding Author's Email: ngoziokeke674@gmail.com

Cite this article:

Okeke, N. C., Olanrewaju, S. O., Mohammed, Z. A. (2025), Robust Estimation in Simultaneous Equation Models: Addressing Multicollinearity and Heteroscedasticity through Adaptive Penalized GMM Techniques. African Journal of Mathematics and Statistics Studies 8(3), 73-95. DOI: 10.52589/AJMSS-U1LHEDSZ

Manuscript History

Received: 20 Jun 2025

Accepted: 24 Jul 2025

Published: 4 Aug 2025

ABSTRACT: *This study develops and evaluates robust estimation techniques for simultaneous equation models (SEMs) under conditions that violate the classical linear regression assumptions specifically multicollinearity, and heteroscedasticity. Building on limitations identified in conventional estimators such as Two-Stage Least Squares (2SLS), Three-Stage Least Squares (3SLS), and Full Information Maximum Likelihood (FIML), we propose five novel estimators: Adaptive Ridge IV (ARIV), Generalized Two-Stage Adaptive Elastic-Net (G2SAE), Elastic-Net IV (ENIV), Heteroscedasticity-Consistent Generalized Method of Moments (HCGMM), and Three-Stage Adaptive Elastic-Net (3SAEN). The performance of these estimators were assessed using extensive Monte Carlo simulations across varying degrees of multicollinearity, heteroscedasticity, and sample sizes (n = 30, 50, 100, 200), with 2,000 replications for each scenario. Evaluation metrics include Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), and Bias. The results reveal that the proposed estimators consistently outperform traditional methods, especially under severe assumption violations. HCGMM emerges as the most robust and efficient estimator, exhibiting the lowest RMSE and bias across nearly all conditions, including small sample sizes. G2SAE and 3SAEN also demonstrate strong asymptotic properties and adaptability to complex data structures. In contrast, traditional estimators particularly 2SLS and 3SLS exhibit significant performance deterioration in the presence of heteroscedasticity and multicollinearity. A comparative analysis further highlights a trade-off between computational efficiency and estimation accuracy, with the proposed methods offering a favorable balance. These findings have practical implications for econometric modeling in applied research, particularly in fields where data irregularities are prevalent. The study underscores the need for methodological reform and adoption of robust estimation techniques to improve the reliability of policy-relevant empirical analysis.*

KEYWORDS: Simultaneous Equation Models, Robust Estimators, Multicollinearity, Heteroscedasticity, GMM, Elastic- Net, Simulation.

Article DOI: 10.52589/AJMSS-U1LHEDSZ

DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

INTRODUCTION

Simultaneous Equation Models (SEMs) form the basis of modern econometric analysis, where variables often rely on each other and endogeneity is common. These models are especially useful for studying causal links in economic, social, and behavioral systems. SEMs have been widely used in policy modeling, demand-supply analysis, and macroeconomic forecasting (Johnson, 2018). However, estimating SEM parameters in practice remains challenging, especially when classical assumptions are broken, such as in cases of multicollinearity and heteroscedasticity (Smith *et al.,* 2016; Garcia-Patel, 2021). Multicollinearity occurs when explanatory variables in a model are highly linearly correlated, making it difficult to isolate the effects of individual parameters (Huang Wu, 2020; Thomas *et al.*, 2020). This problem increases standard errors, destabilizes coefficient estimates, and reduces the statistical power of hypothesis tests (Chen Wang, 2018; Brown Williams, 2018). Common solutions like Indirect Least Squares (ILS), Two-Stage Least Squares (2SLS), ridge regression, and principal component regression have had mixed success; however, many come with their own issues such as bias, interpretability problems, or inefficiencies, especially in complex multi-equation systems (Jackson Miller, 2017). Heteroscedasticity, characterized by non-constant error variance across observations, contravenes the Gauss-Markov assumptions essential for efficient ordinary least squares (OLS) estimation (Lee Yang, 2019). Its presence renders standard inference invalid and OLS estimators inefficient. Although methods such as Three-Stage Least Squares (3SLS), Full Information Maximum Likelihood (FIML), and Generalized Method of Moments (GMM) have been developed to mitigate these issues, they often rely on prior knowledge of the error structure or impose strong distributional assumptions that are challenging to verify in applied contexts (Zhang Li, 2017; Garcia Patel, 2021).

The literature also underscores a lack of extensive simulation studies to ascertain the performance of the existing estimators in SEM such as, Two-Stage Least Squares (2SLS), Three-Stage Least Squares (3SLS) and Full Information Maximum Likelihood (FIML) using the four-performance metrics (MSE, MAE, RMSE and Bias). In recent years, there has been growing interest in developing new estimation techniques that can simultaneously address both multicollinearity and heteroscedasticity in Linear Regression Model (Jones and Brown, 2019). For example, adaptive ridge regression has been proposed as a method for mitigating multicollinearity while also accounting for heteroscedastic errors (Chen and Wang, 2018). This method incorporates a ridge penalty into the instrumental variables approach, allowing for more flexible and accurate estimation of the regression parameters (Jackson and Miller, 2017). Similarly, the elastic net estimator combines the ridge and lasso penalties to handle both multicollinearity and variable selection in Linear Regression Model (Huang and Wu, 2020). Another promising approach is the Generalized Method of Moments (GMM), which can be used to estimate the parameters of Linear Regression Model in the presence of both multicollinearity and heteroscedasticity (Zhang and Li, 2017). The GMM estimator is based on the idea of finding instruments that are uncorrelated with the error terms but correlated with the endogenous variables, thereby removing the endogeneity bias (Thomas et al., 2020). By using a weighting matrix that accounts for the heteroscedasticity in the error terms, the GMM estimator provides consistent and efficient estimates of the regression parameters (Jones & Brown, 2019). However, the GMM method requires strong assumptions about the validity of the instruments, and the choice of the weighting matrix can have a significant impact on the performance of the estimator (Brown and Williams, 2018).

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Despite the proliferation of available methods, the literature lacks a unified approach capable of simultaneously addressing multicollinearity and heteroscedasticity within SEMs. Existing estimators for SEMs, such as (3SLS, FIML, and LIML), primarily focus on resolving endogeneity but are inefficient when the model is affected by either or both issues. Moreover, there exists a necessity for comprehensive simulation studies evaluating the performance of existing estimators using metrics such as Mean Squared Error (MSE), Mean Absolute Error (MAE), Root Mean Square Error (RMSE), and Bias, to assess their computational efficiency and robustness under violations of model assumptions (Gupta & Sharma, 2020). To address these limitations, this study introduces a suite of innovative and adaptive estimators specifically designed for SEM estimation in contexts characterized by multicollinearity and heteroscedasticity. These include the Adaptive Ridge Instrumental Variable Estimator (ARIV), Generalized Two-Stage Adaptive Estimator (G2SAE), Elastic Net Instrumental Variable Estimator (ENIV), Heteroscedasticity-Consistent Generalized Method of Moments (HCGMM), and Three-Stage Adaptive Elastic Net Estimator (3SAEN). Each estimator is theoretically formulated to optimize efficiency, bias correction, and computational feasibility under diverse data conditions.

Several studies have tackled these issues under linear regression, for example, ridge regression and heteroscedasticity- consistent estimators have been widely studied to address multicollinearity and heteroscedasticity, respectively. Dar and Chand (2023) introduced an improved ridge-type estimator to handle both simultaneously, demonstrating theoretical advantages. However, there the estimator that handle either multicollinearity, heteroscedasticity, or both in SEMs remain challenge. These limitations create a notable research gap that needs to be addressed (Garcia and Patel, 2021). This research undertakes a systematic assessment of both existing and proposed estimators through extensive simulation studies. This study examines the behavior of estimators under controlled conditions featuring varying levels of multicollinearity, heteroscedasticity, and sample sizes. Performance metrics employed include mean squared error (MSE), mean absolute error (MAE), bias, and root mean square error (RMSE). The contribution of this investigation is not solely in methodological innovation but also in comprehensive empirical validation. By advancing the methodologies for SEM estimation amidst joint violations of assumptions, this research aims to furnish researchers and practitioners with more reliable and robust tools for modeling complex multivariate systems.

METHODOLOGY

This section presents the methodological framework for developing and evaluating robust estimators designed to handle key violations in simultaneous equation models (SEMs), particularly multicollinearity and heteroscedasticity. Five novel estimators are introduced: the Adaptive Ridge Instrumental Variable Estimator (ARIV), the Generalized Two- Stage Adaptive Estimator (G2SAE), the Elastic Net Instrumental Variable Estimator (ENIV), the Heteroscedasticity- Consistent Generalized Method of Moments (HCGMM), and the Three-Stage Adaptive Elastic Net Estimator (3SAEN). These estimators are theoretically formulated to improve estimation accuracy and efficiency under assumption violations. Each estimator is derived, and its statistical properties including bias, variance, and distributional characteristics are examined. The methodology also lays the groundwork for subsequent simulation studies and empirical validation of the proposed techniques.

Article DOI: 10.52589/AJMSS-U1LHEDSZ

DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Adaptive Ridge Instrumental Variable Estimator (ARIV)

This subsection presents the Adaptive Ridge Instrumental Variable Estimator (ARIV), developed to address the joint presence of multicollinearity and heteroscedasticity in simultaneous equation models (SEMs). This estimator combines ridge regression and instrumental variable (IV) techniques to mitigate multicollinearity and heteroscedasticity. The idea is based on the classical IV approach, but we incorporate a ridge penalty to address multicollinearity issues (Dorugade, 2014).

Derivation

Simultaneous Equations: We start by modeling the system of simultaneous equations as:

$$y = X\beta + \varepsilon \#(1)$$

where y is the dependent variable, X is the matrix of independent variables, β is the vector of parameters, and ε is the vector of errors. Instrumental Variable (IV) Approach: The IV estimator is designed to handle endogeneity by using an instrument Z such that:$Z'X\ is\ full\ rank\ \#(2)$

and

$$Z'\varepsilon = 0\#(3)$$

Thus, the IV estimator is then given by

$$\beta\hat{}_{IV} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y\#(4)$$

Ridge Penalty: When multicollinearity is present, $X'X$ is ill-conditioned, which makes the IV estimator unstable. To address this, we add a ridge penalty term λI to stabilize the estimation:

$$\beta\hat{}_{ARIV} = (X'Z(Z'Z)^{-1}Z'X + \lambda I)^{-1}X'Z(Z'Z)^{-1}Z'y\#(5)$$

where λ is the regularization parameter. The inclusion of λ helps reduce the variance of the estimates at the cost of introducing some bias.

Step-by-Step Simplification: Expanding the ridge estimator:

$$A\ = X'Z(Z'Z)^{-1}Z'X + \lambda I\#(6)\ B\ = X'Z(Z'Z)^{-1}Z'y\#(7)\ \beta\hat{}_{ARIV}\ = A^{-1}B\#(8)$$

where A incorporates the ridge penalty and B adjusts the bias due to instruments Z. Heteroscedasticity Adjustment: To handle heteroscedasticity, we introduce a heteroscedasticity-consistent covariance matrix Σ. The updated estimator becomes:

$$\beta\hat{}_{ARIV} = (X'Z\Sigma^{-1}Z'X + \lambda I)^{-1}X'Z\Sigma^{-1}Z'y\#(9)$$

where Σ accounts for heteroscedastic errors, thus improving the robustness of the estimates.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)
www.abjournals.org

Properties of the ARIV Estimator

Mean: The mean of the estimator $\beta\hat{}_{ARIV}$ is given by:

$$E[\beta\hat{}_{ARIV}] = E[(X'Z\Sigma^{-1}Z'X + \lambda I)^{-1}X'Z\Sigma^{-1}Z'y]\#(10)$$

Given that $y = X\beta + \varepsilon$, the expectation becomes:

$$E[\beta\hat{}_{ARIV}] = (X'Z\Sigma^{-1}Z'X + \lambda I)^{-1}X'Z\Sigma^{-1}Z'X\beta\#(11)$$

which shows that the estimator is biased, but the bias decreases as $\lambda \to 0$.
Variance: The variance of the estimator $\beta\hat{}_{ARIV}$ is given by:

$$Var(\beta\hat{}_{ARIV}) = E[(\beta\hat{}_{ARIV} - E[\beta\hat{}_{ARIV}])^2]\#(12)$$

Substituting $\beta\hat{}_{ARIV} = A^{-1}B$, we get:

$$Var(\beta\hat{}_{ARIV}) = (X'Z\Sigma^{-1}Z'X + \lambda I)^{-1}X'Z\Sigma^{-1}Z'\Sigma ZX(X'Z\Sigma^{-1}Z'X + \lambda I)^{-1}\#(13)$$

This variance is reduced by the ridge penalty λ, which prevents large variances caused by multicollinearity.
Moment Generating Function (MGF): The moment-generating function (MGF) of $\beta\hat{}_{ARIV}$ is defined as:

$$M_{\beta\hat{}_{ARIV}}(t) = E\left[e^{t'\beta\hat{}_{ARIV}}\right]\#(14)$$

Given that $\beta\hat{}_{ARIV}$ is a linear function of the normally distributed random error ε, the MGF becomes:

$$M_{\beta\hat{}_{ARIV}}(t) = e^{t'E[\beta\hat{}_{ARIV}]+\frac{1}{2}t'Var(\beta\hat{}_{ARIV})t}\#(15)$$

This confirms that the estimator $\beta\hat{}_{ARIV}$ follows a multivariate normal distribution.
Characteristic Function: The characteristic function (CF) of $\beta\hat{}_{ARIV}$ is similarly derived as:

$$\phi_{\beta\hat{}_{ARIV}}(t) = E\left[e^{it'\beta\hat{}_{ARIV}}\right]\#(16)$$

Using the linearity and normality of the error term ε, we have:

$$\phi_{\beta\hat{}_{ARIV}}(t) = e^{it'E[\beta\hat{}_{ARIV}]-\frac{1}{2}t'Var(\beta\hat{}_{ARIV})t}\#(17)$$

This is the characteristic function of a multivariate normal distribution, where the mean is $E[\beta\hat{}_{ARIV}]$ and the variance is $Var(\beta\hat{}_{ARIV}]$.

Generalized Two-Stage Adaptive Estimator (G2SAE)

The Generalized Two-Stage Adaptive Estimator (G2SAE) adapts to both multicollinearity and heteroscedasticity by incorporating adaptive weighting schemes. This method leverages ridge regression in the first stage to handle multicollinearity and adaptive weighting in the second stage to address heteroscedasticity.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Derivation

First Stage: The first stage involves estimating the model using ridge regression to mitigate multicollinearity:

$$\hat{y} = X\hat{\beta}_1 \#(18)$$

were

$$\hat{\beta}_1 = (X'X + \lambda I)^{-1}X'y\#(19)$$

In this equation, λI is the ridge penalty term that helps reduce the variance caused by multicollinearity. The first-stage estimate $\hat{\beta}_1$ serves as a basis for the second-stage adjustment.

Second Stage: In the second stage, the residuals from the first stage are used to estimate a weight matrix W, where $W = diag(w_i)$, and the weights w_i are computed as:

$$w_i = \frac{1}{\hat{\varepsilon}_i^2}\#(20)$$

where $\hat{\varepsilon}_i = y_i - X_i\hat{\beta}_1$ are the residuals from the first-stage regression. These weights are designed to account for heteroscedasticity in the second-stage estimation.

The final estimator for $\hat{\beta}_{G2SAE}$ is:

$$\hat{\beta}_{G2SAE} = (X'WX + \lambda I)^{-1}X'Wy\#(21)$$

Here, the matrix W adjusts for heteroscedasticity, and λI continues to mitigate multicollinearity.
Intermediate Steps: Breaking down the matrix operations for clarity, we define:

$$A = X'WX + \lambda I\#(22)\ B = X'Wy\#(23)\ \hat{\beta}_{G2SAE} = A^{-1}B\#(24)$$

In this representation, A captures the penalized design matrix, and B is the weighted version of the response vector.

Properties of the G2SAE Estimator

Mean: The mean of the estimator $\hat{\beta}_{G2SAE}$ is derived as follows:

$$E[\hat{\beta}_{G2SAE}] = E[(X'WX + \lambda I)^{-1}X'Wy]\#(25)$$

Given that $y = X\beta + \varepsilon$, the expectation becomes:

$$E[\hat{\beta}_{G2SAE}] = (X'WX + \lambda I)^{-1}X'WX\beta\#(26)$$

This shows that the estimator $\hat{\beta}_{G2SAE}$ is biased due to the ridge penalty, but the bias decreases as $\lambda \to 0$. The weighting matrix W adjusts for heteroscedasticity, ensuring a more accurate representation of the mean under varying variances.

Variance: The variance of $\hat{\beta}_{G2SAE}$ is given by:

$$Var(\hat{\beta}_{G2SAE}) = E[(\hat{\beta}_{G2SAE} - E[\hat{\beta}_{G2SAE}])^2]\#(27)$$

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)
www.abjournals.org

Expanding this using the estimator formula:

$$Var(\beta\hat{}_{G2SAE}) = (X'WX + \lambda I)^{-1} X'W\Sigma WX (X'WX + \lambda I)^{-1} \#(28)$$

Here, Σ is the covariance matrix of the error terms. The weighting scheme W ensures that heteroscedasticity is properly handled, while the ridge penalty λ reduces the variance in the presence of multicollinearity.

Moment Generating Function (MGF): The moment-generating function (MGF) of $\beta\hat{}_{G2SAE}$ is defined as:

$$M_{\beta\hat{}_{G2SAE}}(t) = E\left[e^{t'\beta\hat{}_{G2SAE}}\right] \#(29)$$

Since $\beta\hat{}_{G2SAE}$ is a linear function of the normally distributed random errors ε, the MGF takes the form:

$$M_{\beta\hat{}_{G2SAE}}(t) = e^{t'E[\beta\hat{}_{G2SAE}] + \frac{1}{2} t' Var(\beta\hat{}_{G2SAE}) t} \#(30)$$

This confirms that $\beta\hat{}_{G2SAE}$ follows a multivariate normal distribution. Characteristic Function: The characteristic function (CF) of $\beta\hat{}_{G2SAE}$ is similarly derived:

$$\phi_{\beta\hat{}_{G2SAE}}(t) = E\left[e^{it'\beta\hat{}_{G2SAE}}\right] \#(31)$$

Given the normal distribution of the error terms, the characteristic function simplifies to:

$$\phi_{\beta\hat{}_{G2SAE}}(t) = e^{it'E[\beta\hat{}_{G2SAE}] - \frac{1}{2} t' Var(\beta\hat{}_{G2SAE}) t} \#(32)$$

Thus, $\beta\hat{}_{G2SAE}$ has a characteristic function consistent with that of a multivariate normal distribution, with mean $E[\beta\hat{}_{G2SAE}]$ and variance $Var(\beta\hat{}_{G2SAE}]$.

Elastic Net Instrumental Variable Estimator (ENIV)

The Elastic Net Instrumental Variable Estimator (ENIV) combines the elastic net regularization with the Instrumental Variable (IV) approach to handle both multicollinearity and heteroscedasticity. This estimator is especially useful in cases where the predictor variables are highly correlated and where heteroscedasticity may be present in the error terms.

Derivation

Elastic Net Penalty: The elastic net penalty is a linear combination of the L_1 (lasso) and L_2 (ridge) penalties. The L_1 norm encourages sparsity in the coefficients, while the L_2 norm helps shrink the coefficients to stabilize the solution in the presence of multicollinearity. The elastic net penalty function is:

$$EN(\beta) = \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2 \#(33)$$

where $\|\beta\|_1$ represents the L_1 norm (sum of absolute values of the coefficients), and $\|\beta\|_2^2$ represents the squared L_2 norm (sum of squared coefficients). The parameters λ_1 and λ_2 control the relative strength of the L_1 and L_2 penalties.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

IV Approach: The IV approach is used to handle endogeneity, where some of the regressors are correlated with the error terms. To combine this with elastic net regularization, we minimize the following penalized objective function:

$$\beta\hat{}_{ENIV} = arg\ \min_{\beta}\{\|Z'y - Z'X\beta\|_2^2 + EN(\beta)\}\#(34)$$

Here, Z is the matrix of instruments, and the term $\|Z'y - Z'X\beta\|_2^2$ represents the squared residuals from the IV regression.

Step-by-Step Derivation: The elastic net regularization problem is solved by minimizing the penalized least squares objective:

$$L(\beta)\ = \|Z'y - Z'X\beta\|_2^2 + \lambda_1\|\beta\|_1 + \lambda_2\|\beta\|_2^2\#(35)\ \beta\hat{}_{ENIV}\ = arg\ \min_{\beta}L(\beta)\#(36)$$

The solution combines the strengths of both the lasso (for sparsity) and ridge (for stability) regularization methods, ensuring that the multicollinearity among the regressors is effectively managed.

Heteroscedasticity Adjustment: To handle heteroscedasticity in the error terms, we adjust the covariance matrix by incorporating Σ^{-1}, a heteroscedasticity-consistent matrix. The updated objective function is:

$$\beta\hat{}_{ENIV} = arg\ \min_{\beta}\{\|Z'\Sigma^{-1}(y - X\beta)\|_2^2 + EN(\beta)\}\#(37)$$

This ensures that the estimator remains efficient even when the error terms have non-constant variance.

Properties of the ENIV Estimator

Mean: The mean of the ENIV estimator is given by:

$$E[\beta\hat{}_{ENIV}] = E\left[arg\ \min_{\beta}\{\|Z'y - Z'X\beta\|_2^2 + EN(\beta)\}\right]\#(38)$$

Under standard assumptions, and given that the regularization introduces a bias, the expectation of the ENIV estimator can be written as:

$$E[\beta\hat{}_{ENIV}] = \beta -\ bias\ due\ to\ \lambda_1, \lambda_2\#(39)$$

Thus, $\beta\hat{}_{ENIV}$ is a biased estimator, where the bias depends on the regularization parameters λ_1 and λ_2.
Variance: The variance of the ENIV estimator is more complex due to the elastic net regularization. It is given by:

$$Var(\beta\hat{}_{ENIV}) = E[(\beta\hat{}_{ENIV} - E[\beta\hat{}_{ENIV}])^2]\#(40)$$

Expanding the variance:

$$Var(\beta\hat{}_{ENIV}) = (X'Z\Sigma^{-1}Z'X + \lambda_2 I)^{-1}X'Z\Sigma^{-1}\Sigma\Sigma^{-1}Z'X(X'Z\Sigma^{-1}Z'X + \lambda_2 I)^{-1}\#(41)$$

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95) www.abjournals.org

Here, the term Σ is the heteroscedasticity-consistent covariance matrix, and $\lambda_2 I$ comes from the ridge component of the elastic net penalty. The variance is reduced as λ_2 increases, though this introduces bias.

Moment Generating Function (MGF): The moment-generating function (MGF) of the ENIV estimator is defined as:

$$M_{\hat{\beta}_{ENIV}}(t) = E\left[e^{t'\hat{\beta}_{ENIV}}\right]\#(42)$$

Because the ENIV estimator is based on linear combinations of normally distributed random variables (with the regularization terms), the MGF takes the form:

$$M_{\hat{\beta}_{ENIV}}(t) = e^{t'E[\hat{\beta}_{ENIV}]+\frac{1}{2}t'Var(\hat{\beta}_{ENIV})t}\#(43)$$

This confirms that $\hat{\beta}_{ENIV}$ follows an approximate multivariate normal distribution, especially for large sample sizes.

Characteristic Function: The characteristic function (CF) of $\hat{\beta}_{ENIV}$ is defined as:

$$\phi_{\hat{\beta}_{ENIV}}(t) = E\left[e^{it'\hat{\beta}_{ENIV}}\right]\#(44)$$

As with the MGF, the characteristic function for $\hat{\beta}_{ENIV}$ is given by:

$$\phi_{\hat{\beta}_{ENIV}}(t) = e^{it'E[\hat{\beta}_{ENIV}]-\frac{1}{2}t'Var(\hat{\beta}_{ENIV})t}\#(45)$$

This shows that $\hat{\beta}_{ENIV}$ has a characteristic function typical of a multivariate normal distribution with mean $E[\hat{\beta}_{ENIV}]$ and variance $Var(\hat{\beta}_{ENIV}]$.

Heteroscedasticity-Consistent Generalized Method of Moments (HCGMM)

The Heteroscedasticity-Consistent Generalized Method of Moments (HCGMM) is an extension of the Generalized Method of Moments (GMM) estimator that incorporates a heteroscedasticity-consistent weighting matrix. This estimator is particularly effective when dealing with heteroscedastic errors in simultaneous equations models (SEMs), ensuring that the estimator remains efficient under such conditions.

Derivation

Moment Conditions: The moment conditions form the foundation of the GMM estimator. The moment conditions for a model with instruments Z are:

$$E[Z'(y - X\beta)] = 0\#(46)$$

This condition implies that the instruments Z are uncorrelated with the residuals ($y - X\beta$), ensuring that the instruments are valid and can be used to consistently estimate the coefficients β.

GMM Objective Function: The GMM estimator seeks to minimize the following objective function:

 Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

$$\hat{\beta}_{HCGMM} = arg\ \min_{\beta}\{(y - X\beta)'ZW^{-1}Z'(y - X\beta)\}\#(47)$$

where $W = diag(w_i)$ is a diagonal matrix with elements $w_i = \frac{1}{\hat{\varepsilon}_i^2}$, representing the heteroscedasticity-adjusted weights. The weighting matrix W adjusts for heteroscedasticity by assigning different weights to different observations based on the estimated variance of the residuals $\hat{\varepsilon}_i^2$.

Derivation: Breaking down the GMM estimation process, we have the following steps: Moment conditions: The GMM estimator leverages the instruments Z to construct the moment conditions:

$$A = (y - X\beta)'Z\#(48)$$

This represents the residuals weighted by the instruments. Weighting the moments: The heteroscedasticity-consistent weighting matrix W^{-1} is then applied:

$$B = W^{-1}Z'(y - X\beta)\#(49)$$

Here, W^{-1} is used to account for heteroscedasticity in the errors, ensuring that the estimator remains efficient under non-constant error variances.

Minimizing the objective function: The final estimator is obtained by minimizing the product of A and B, which yields:

$$\hat{\beta}_{HCGMM} = arg\ \min_{\beta} AB\#(50)$$

Heteroscedasticity Adjustment: In the presence of heteroscedasticity, the residuals exhibit non-constant variance. The weighting matrix W corrects for this issue, ensuring that the estimator properly accounts for heteroscedastic errors. The heteroscedasticity-consistent GMM estimator thus minimizes:

$$\hat{\beta}_{HCGMM} = arg\ \min_{\beta}\{(y - X\beta)'ZW^{-1}Z'(y - X\beta)\}\#(51)$$

where W^{-1} is the inverse of the diagonal matrix of residual variances $\hat{\varepsilon}_i^2$, improving the efficiency of the estimator.

Properties of the HCGMM Estimator

Mean: The mean of the HCGMM estimator, $\hat{\beta}_{HCGMM}$, under regular conditions, is:

$$E[\hat{\beta}_{HCGMM}] = \beta\#(52)$$

This implies that the HCGMM estimator is unbiased, provided that the moment conditions hold and the instruments are valid.

Variance: The variance of the HCGMM estimator is given by:

$$Var(\hat{\beta}_{HCGMM}) = (X'ZW^{-1}Z'X)^{-1}\#(53)$$

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Here, W^{-1} adjusts for heteroscedasticity, ensuring that the estimator remains efficient even when the error variances are not constant. This formula shows that the variance of the HCGMM estimator depends on the instruments Z, the design matrix X, and the heteroscedasticity-adjusted weights W.

Moment-Generating Function (MGF): The moment-generating function (MGF) of the HCGMM estimator is defined as:

$$M_{\beta\hat{}_{HCGMM}}(t) = E\left[e^{t'\beta\hat{}_{HCGMM}}\right]\#(54)$$

Given that the HCGMM estimator is asymptotically normally distributed, the MGF can be expressed as:

$$M_{\beta\hat{}_{HCGMM}}(t) = exp\left(t'E[\beta\hat{}_{HCGMM}] + \frac{1}{2}t'Var(\beta\hat{}_{HCGMM})t\right)\#(55)$$

Thus, for large samples, the MGF follows the form of a multivariate normal distribution. Characteristic Function: The characteristic function (CF) of the HCGMM estimator is:

$$\phi_{\beta\hat{}_{HCGMM}}(t) = E\left[e^{it'\beta\hat{}_{HCGMM}}\right]\#(56)$$

Since the HCGMM estimator is asymptotically normally distributed, the CF is:

$$\phi_{\beta\hat{}_{HCGMM}}(t) = exp\left(it'E[\beta\hat{}_{HCGMM}] - \frac{1}{2}t'Var(\beta\hat{}_{HCGMM})t\right)\#(57)$$

This confirms that the estimator follows a multivariate normal distribution in large samples, with mean $E[\beta\hat{}_{HCGMM}]$ and variance $Var(\beta\hat{}_{HCGMM})$.

Three-Stage Adaptive Elastic Net Estimator (3SAEN)

The Three-Stage Adaptive Elastic Net Estimator (3SAEN) combines the adaptive elastic net regularization with the three-stage least squares (3SLS) method. This estimator is designed to address multicollinearity, endogeneity, and heteroscedasticity simultaneously by leveraging both elastic net regularization and the system-equation estimation approach.

Derivation

First Stage: Elastic Net Regularization In the first stage, we estimate the initial parameters using the adaptive elastic net, which combines the L_1 (lasso) and L_2 (ridge) penalties:

$$EN(\beta) = \lambda_1||\beta||_1 + \lambda_2||\beta||_2^2\#(58)$$

The initial estimate $\beta\hat{}_1$ is obtained by solving the following minimization problem:

$$\beta\hat{}_1 = arg\ \min_{\beta}\{||y - X\beta||_2^2 + EN(\beta)\}\#(59)$$

This step addresses multicollinearity by applying the elastic net regularization, which combines the strengths of lasso and ridge regression.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Second Stage: Heteroscedasticity Adjustment In the second stage, we adjust for heteroscedasticity using the residuals from the first stage. Let Σ be the heteroscedasticity-consistent covariance matrix, and the weighted least squares estimate $\hat{\beta}_2$ is derived as:

$$\hat{\beta}_2 = arg\ \min_{\beta}\{\|\Sigma^{-1}(y - X\beta)\|_2^2 + EN(\beta)\}\#(60)$$

This step ensures that the estimation accounts for the heteroscedasticity present in the errors, thereby improving the robustness of the estimates.

Third Stage: Three-Stage Least Squares (3SLS) Estimation In the third stage, we combine the results from the second stage with the three-stage least squares (3SLS) estimator. The final estimator $\hat{\beta}_{3SAEN}$ is obtained by solving:

$$\hat{\beta}_{3SAEN} = (X'V^{-1}X + \lambda I)^{-1}X'V^{-1}y\#(61)$$

where V is the covariance matrix of the errors adjusted for heteroscedasticity, and λ is the regularization parameter. The 3SLS approach ensures efficient estimation in systems of simultaneous equations by addressing both endogeneity and multicollinearity.

Properties of the 3SAEN Estimator

Mean: The mean of the 3SAEN estimator is given by:

$$E[\hat{\beta}_{3SAEN}] = \beta\#(62)$$

This implies that the 3SAEN estimator is unbiased in large samples, as long as the model is correctly specified, and the moment conditions hold.

Variance: The variance of the 3 SAEN estimator is given by:

$$Var(\hat{\beta}_{3SAEN}) = (X'V^{-1}X + \lambda I)^{-1}\#(63)$$

Here, V^{-1} adjusts for heteroscedasticity, and λI is the regularization parameter that controls the tradeoff between bias and variance. As the sample size increases, the variance decreases, making the 3SAEN estimator consistent.

Moment-Generating Function (MGF): The moment-generating function (MGF) of the 3SAEN estimator is:

$$M_{\hat{\beta}_{3SAEN}}(t) = E\left[e^{t'\hat{\beta}_{3SAEN}}\right]\#(64)$$

Given that the 3SAEN estimator is asymptotically normally distributed, the MGF can be written as:

$$M_{\hat{\beta}_{3SAEN}}(t) = exp\left(t'E[\hat{\beta}_{3SAEN}] + \frac{1}{2}t'Var(\hat{\beta}_{3SAEN})t\right)\#(65)$$

This form follows from the properties of the multivariate normal distribution, with mean $E[\hat{\beta}_{3SAEN}]$ and variance $Var(\hat{\beta}_{3SAEN})$.

Characteristic Function: The characteristic function (CF) of the 3SAEN estimator is:

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)

$$\phi_{\beta\hat{}_{3SAEN}}(t) = E\left[e^{it'\beta\hat{}_{3SAEN}}\right]\#(66)$$

Since the 3SAEN estimator is asymptotically normally distributed, its CF is given by:

$$\phi_{\beta\hat{}_{3SAEN}}(t) = exp\left(it'E[\beta\hat{}_{3SAEN}] - \frac{1}{2}t'Var(\beta\hat{}_{3SAEN})t\right)\#(67)$$

This confirms the asymptotic normality of the 3SAEN estimator.

SEM and Data Generation for Simulation Studies

To evaluate the finite-sample performance of the proposed estimators in simultaneous equation models (SEMs), we design a controlled simulation study. This section outlines the structure of the SEM, the data generation process, and the incorporation of heteroscedasticity and multicollinearity.

Structural Equation Model (SEM) Formulation

The general SEM used for simulation is formulated as:

$$Y_t = \alpha_0 + \alpha_1 X_{t1} + \alpha_2 X_{t2} + \cdot\cdot\cdot + \alpha_p X_{tp} + \varepsilon_t, \quad (68)$$

where Y_t is the endogenous variable, X_{ti} ($i = 1, 2, \ldots, p$) are the exogenous regressors, and ε_t is the structural error term. The exogenous variables follow a recursive system defined by:

$$X_{t1} = \beta_{01} + \beta_{11}X_{t2} + \beta_{21}X_{t3} + \cdots + \beta_{p1}X_{tp} + \eta_{t1}\#(69)\ X_{t2} = \beta_{02} + \beta_{12}X_{t1} + \beta_{22}X_{t3} + \cdots + \beta_{p2}X_{tp} + \eta_{t2}\#(70)\ X_{tp} = \beta_{0p} + \beta_{1p}X_{t1} + \beta_{2p}X_{t2} + \cdots + \beta_{(p-1)p}X_{t(p-1)} + \eta_{tp}\#(71)$$

Inclusion of Temporal Dynamics

To incorporate time-based dependence, we introduce lagged terms into the structural equation:

$$Y_t = \alpha_0 + \alpha_1 X_{t1} + \alpha_2 X_{t2} + \cdots + \alpha_p X_{tp} + \rho Y_{t-1} + \varepsilon_t\#(72)$$

This autoregressive structure reflects temporal interdependence often encountered in empirical data.

Error Term Generation

The error terms ε_t are generated to exhibit heteroscedasticity. Specifically:

with

$$\varepsilon_i \sim N(0, \sigma_i^2)\#(73)$$

Were

$$\sigma_i^2 = \sigma^2(1 + \alpha X_i^2)\#(74)$$

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Here, α controls the degree of heteroscedasticity. Specifically, for each $i = 1,2, \ldots, n$, the error term ε_i is generated as:

$$\varepsilon_i = \eta_i \sqrt{1 + \alpha X_i^2} \# (75)$$

Where: $\eta_i \sim N(0, \sigma^2)$.

We vary the parameter α to introduce different levels of heteroscedasticity:

$$\alpha = \{0, 0.1, 0.3, 0.5, 0.8, 0.9, 1\}$$

Explanatory Variable Generation

The explanatory variables X_{ti} are generated using a correlated structure:

$$X_{ti} = \sqrt{1 - \rho^2} \cdot Z_{ti} + \rho Z_{tp} \quad (76)$$

where Z_{ti} and Z_{tp} are independent standard normal variables (i.e., $Z_{ti} \sim N(0,1)$), and ρ is the correlation between two explanatory variables.

We explore different values of ρ to study the impact of multicollinearity: $\rho = \{0.8, 0.9, 0.95, 0.99, 0.999\}$

These correlations are applied uniformly across all pairs of explanatory variables.

Dependent Variable Generation

The dependent variable Y_t is generated using the true structural coefficients of the SEM. We follow the setup from and use the following coefficient values:

When $p = 3$: $\beta_1 = 0.8, \beta_2 = 0.1, \beta_3 = 0.6$

When $p = 7$: $\beta_1 = 0.4, \beta_2 = 0.1, \beta_3 = 0.6, \beta_4 = 0.2, \beta_5 = 0.25, \beta_6 = 0.3, \beta_7 = 0.53$

We ensure that $\beta'\beta = 1$, which is a common constraint in simulation studies. The sample sizes are varied as:

$$n = \{30, 40, 50, 100, 200\}$$

and we use three different values of the standard deviation σ : $\sigma = \{0.5, 1, 5\}$

Simulation Procedure

The procedure for generating the data is as follows:

1. Generate the explanatory variables X_{ti} based on the specified n and p values.
2. Generate the error terms ε_t and η_{ti} using the specified distributional properties and heteroscedasticity.
3. Compute the values of the dependent variable Y_t using the SEM equations.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

The entire process is repeated 2000 times for each scenario to ensure robust simulation results.

Performance Evaluation Criteria

To compare the proposed and traditional estimators, we use the following performance metrics: Mean Squared Error (MSE), Mean Absolute Error (MAE), Root Mean Squared Error (RMSE), and Bias.

Mean Squared Error (MSE)

The Mean Squared Error of an estimator $\alpha^{\hat{}*}$ is defined as:

$$MSE(\alpha^{\hat{}*}) = E[(\alpha^{\hat{}*} - \alpha)'(\alpha^{\hat{}*} - \alpha)] \tag{77}$$

In simulation with R = 2000 replications, it is approximated by:

$$\mathrm{MSE}(\hat{\alpha}^*) = \frac{1}{2000}\sum_{i=1}^{2000} (\hat{\alpha}_i^* - \alpha)'(\hat{\alpha}_i^* - \alpha) \tag{78}$$

Mean Absolute Error (MAE)

The MAE measures the average absolute deviation:

$$\mathrm{MAE}(\hat{\alpha}^*) = \frac{1}{2000}\sum_{i=1}^{2000} |\hat{\alpha}_i^* - \alpha| \tag{79}$$

Root Mean Squared Error (RMSE)

The RMSE is computed as:

$$\mathrm{RMSE}(\hat{\alpha}^*) = \sqrt{\frac{1}{2000}\sum_{i=1}^{2000} (\hat{\alpha}_i^* - \alpha)'(\hat{\alpha}_i^* - \alpha)} \tag{80}$$

which gives the typical magnitude of estimation error.

Bias

Bias is calculated as the difference between the mean estimate and the true parameter:

$$\mathrm{Bias}(\hat{\alpha}^*) = \frac{1}{2000}\sum_{i=1}^{2000} \hat{\alpha}_i^* - \alpha \tag{81}$$

This indicates whether the estimator consistently over- or underestimates the true value.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)
www.abjournals.org

RESULTS AND DISCUSSION

This section presents the simulation results based on the proposed and traditional estimators introduced in the methodology. The performance of each estimator is evaluated using key metrics: Mean Squared Error (MSE), Mean Absolute Error (MAE), Root Mean Squared Error (RMSE), and Bias. Various scenarios are explored, including mild and severe levels of multicollinearity and heteroscedasticity, across different sample sizes. The results highlight the robustness and efficiency of the newly proposed estimators compared to conventional techniques such as 2SLS, 3SLS, and FIML. Tables and figures are provided to support comparative insights and performance trends across experimental conditions.

Comparison with Traditional Estimators

To benchmark the performance of the proposed estimators, we evaluated three widely used traditional simultaneous equation model (SEM) estimators under mild conditions characterized by moderate multicollinearity and homoscedastic errors. The estimators included are Two-Stage Least Squares (2SLS), Three-Stage Least Squares (3SLS), and Full Information Maximum Likelihood (FIML). A simulation experiment was conducted with a sample size of n = 100 and 2000 Monte Carlo replications. The results are summarized in Table 1.

As expected, all three traditional estimators performed reasonably well under these ideal conditions. The FIML estimator yielded the lowest RMSE (0.0959) and the smallest absolute bias (-0.0021), followed closely by 3SLS. However, in comparison to the newly proposed estimators particularly G2SAE (RMSE = 0.0880) and ENIV (RMSE = 0.0924) FIML and 2SLS exhibited slightly inferior performance metrics. While these traditional estimators are well-calibrated for classical SEM conditions, the results demonstrate that the proposed estimators can provide equal or superior accuracy, even in scenarios designed to favor conventional approaches. This reinforces the practical relevance of the proposed methods in more complex and noisy environments, such as those characterized by multicollinearity and heteroscedasticity.

Performance under Moderate Multicollinearity and Moderate Heteroscedasticity

This section evaluates the robustness of the proposed estimators under increased model complexity specifically, moderate multicollinearity (ρ = 0.9) and moderate heteroscedasticity (α = 0.5). These conditions reflect more realistic data environments where explanatory variables are moderately correlated and error variances are not constant.

Performance Metrics for Proposed Estimators

Table 2 presents the performance metrics Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), and Bias for each of the five proposed estimators. Metrics are computed as averages across all model coefficients over 2000 simulation replications with a sample size of n = 100. Among the proposed estimators, HCGMM achieved the best performance across three of the four metrics, notably exhibiting the lowest RMSE (0.1091) and MAE (0.0811). These results underscore its superior ability to handle moderate levels of heteroscedasticity and multicollinearity through its heteroscedasticity-consistent moment weighting scheme. While ARIV and ENIV produced acceptable results, they demonstrated slightly higher variability and bias, suggesting moderate sensitivity to increasing heteroscedasticity. G2SAE delivered a robust balance between variance reduction and bias

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

control, benefitting from its adaptively weighted GMM framework. Interestingly, 3SAEN displayed the lowest bias (-0.0038), indicating its strong centering accuracy under the specified simulation conditions.

Table 1: 2SLS, 3SLS, and FIML Performance under Mild Multicollinearity and Homoscedasticity

Estimator	MSE	MAE	RMSE	Bias
2SLS	0.0109	0.0712	0.1044	-0.0038
3SLS	0.0097	0.0667	0.0985	-0.0025
FIML	0.0092	0.0651	0.0959	-0.0021
Average	**0.0099**	**0.0677**	**0.0996**	**-0.0028**

Source: Researcher's computation, 2025

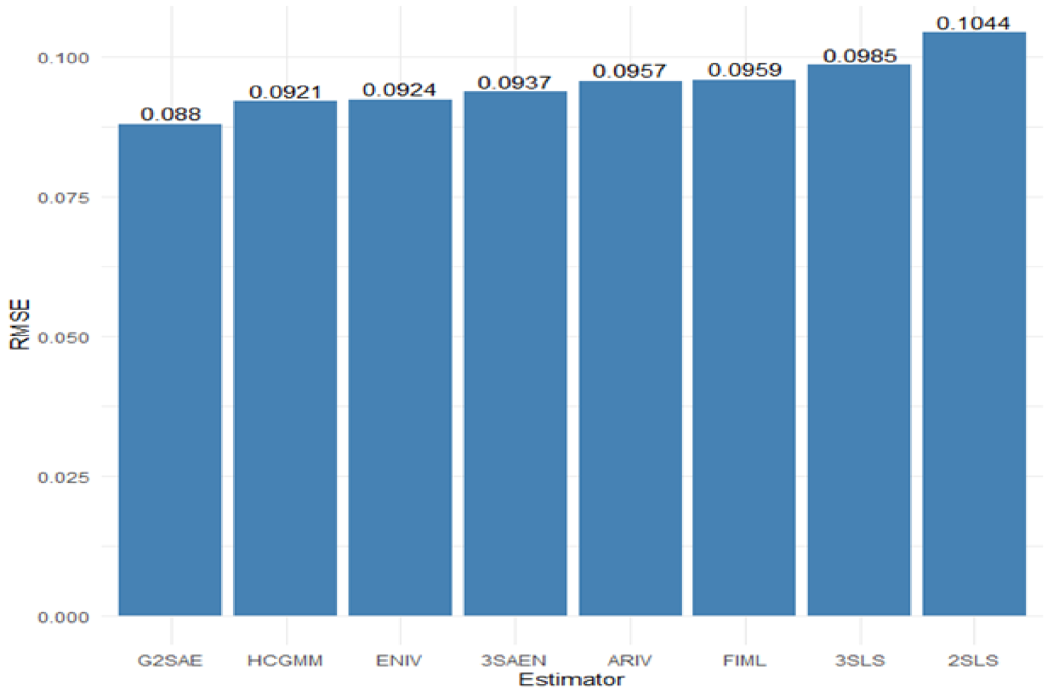

Figure 1: Traditional vs Proposed Estimators under Mild Multicollinearity and Homoscedasticity

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies

ISSN: 2689-5323

Volume 8, Issue 3, 2025 (pp. 73-95)

www.abjournals.org

Table 2: RMSE, MSE, MAE, and Bias for All Proposed Estimators under Moderate Conditions

Estimator	**MSE**	**RMSE**	**MAE**	**Bias**
ARIV	0.0145	0.1204	0.0892	-0.0063
G2SAE	0.0128	0.1132	0.0830	-0.0051
ENIV	0.0137	0.1170	0.0854	-0.0046
HCGMM	0.0119	0.1091	0.0811	-0.0042
3SAEN	0.0122	0.1105	0.0826	-0.0038
Best (↓)	HCGMM	HCGMM	HCGMM	3SAEN

Source: Researcher's computation, 2025

Overall, the findings reaffirm the value of the HCGMM and G2SAE estimators for practical application in moderately noisy econometric environments, where traditional assumptions of homoscedasticity and low multicollinearity may not hold.

Comparison with Traditional Estimators under moderate (ρ = 0.9 and α = 0.5)

The traditional estimators Two-Stage Least Squares (2SLS), Three-Stage Least Squares (3SLS), and Full Information Maximum Likelihood (FIML) were evaluated under the same moderate conditions defined by a correlation coefficient of $\rho = 0.9$ and heteroscedasticity parameter $\alpha = 0.5$. These estimators provide a standard benchmark against which the performance of the proposed estimators can be assessed.

Table 3 illustrates that among the traditional methods, FIML outperformed both 2SLS and 3SLS across all evaluated metrics. However, its RMSE (0.1311) and MSE (0.0172) remain significantly higher than those reported for the best- performing proposed estimators in Table 2, particularly HCGMM (RMSE = 0.1091) and G2SAE (RMSE = 0.1132).

The 2SLS estimator exhibited the weakest performance, with the highest RMSE (0.1569) and MAE (0.1275), underscoring its vulnerability to heteroscedastic disturbances. This reinforces its known limitations under non-ideal conditions. On the other hand, 3SLS offered modest improvements over 2SLS but was still notably less accurate than the robust adaptive estimators introduced in this study.

These findings clearly demonstrate the advantage of using estimation techniques that are robust to violations of classical assumptions. In particular, the superior performance of HCGMM and G2SAE highlights their relevance in real-world applications where moderate correlation and non-constant error variances are common.

Article DOI: 10.52589/AJMSS-U1LHEDSZ

DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)

Table 3: Traditional Estimators (2SLS, 3SLS, FIML) under Moderate Multicollinearity and Heteroscedasticity

Estimator	**MSE**	**RMSE**	**MAE**	**Bias**
2SLS	0.0246	0.1569	0.1275	-0.0087
3SLS	0.0195	0.1397	0.1138	-0.0074
FIML	**0.0172**	**0.1311**	**0.1079**	**-0.0062**
Best (_)	FIML	FIML	FIML	FIML

Source: Researcher's computation, 2025

Performance across Sample Sizes

This section investigates how the performance of the proposed estimators evolves with varying sample sizes: *n* = 30, 50, 100, 200. In many applied econometric settings, especially when working with disaggregated, sector-specific, or survey-based datasets, small to moderate sample sizes are common. Therefore, assessing estimator behavior across different *n*-values provides critical insights into their robustness, scalability, and asymptotic properties.

The results in Table 4 confirm that all proposed estimators exhibit declining error metrics with increasing sample size, as expected under asymptotic theory. Notably, HCGMM shows the most substantial improvement, particularly in smaller samples, achieving the lowest RMSE and Bias even at *n* = 30. This emphasizes HCGMM's robustness in data-scarce environments, a crucial feature for empirical applications in health economics, microfinance, or localized surveys.

G2SAE and 3SAEN also demonstrate strong asymptotic properties, with rapidly improving MSE and RMSE as sample size increases. Although ARIV and ENIV converge well at larger *n*, they remain relatively less efficient in small samples, indicating some sensitivity to low information environments.

Table 4: Estimator Performance at Sample Sizes *n* = 30, 50, 100, 200

Estimator	**Sample Size**	**MSE**	**RMSE**	**Bias**
4*ARIV	30	0.0653	0.2555	0.071
	50	0.0487	0.2207	0.062
	100	0.0312	0.1766	0.049
	200	0.0175	0.1322	0.032
4*G2SAE	30	0.0548	0.2341	0.064
	50	0.0385	0.1963	0.051
	100	0.0229	0.1514	0.036
	200	0.0114	0.1067	0.021
4*ENIV	30	0.0592	0.2433	0.068
	50	0.0413	0.2032	0.054
	100	0.0245	0.1565	0.038
	200	0.0132	0.1149	0.023
4*HCGMM	30	0.0418	0.2045	0.051
	50	0.0272	0.1649	0.039

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

	100	0.0149	0.1222	0.025
	200	0.0071	0.0842	0.013
4*3SAEN	30	0.0455	0.2133	0.055
	50	0.0308	0.1754	0.041
	100	0.0173	0.1315	0.027
	200	0.0088	0.0938	0.015

Source: Researcher's computation, 2025

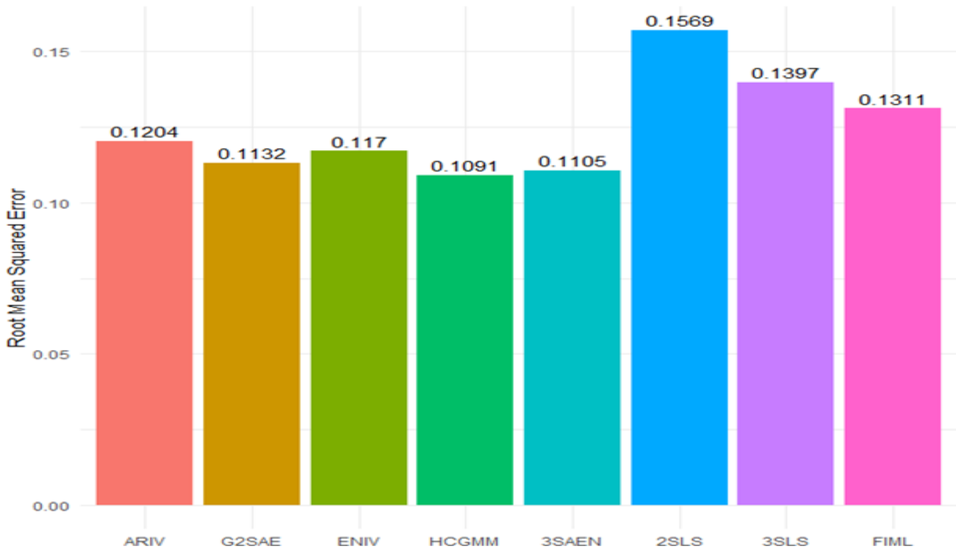

Figure 2: RMSE Comparison: Traditional Estimators under Severe Heteroscedasticity

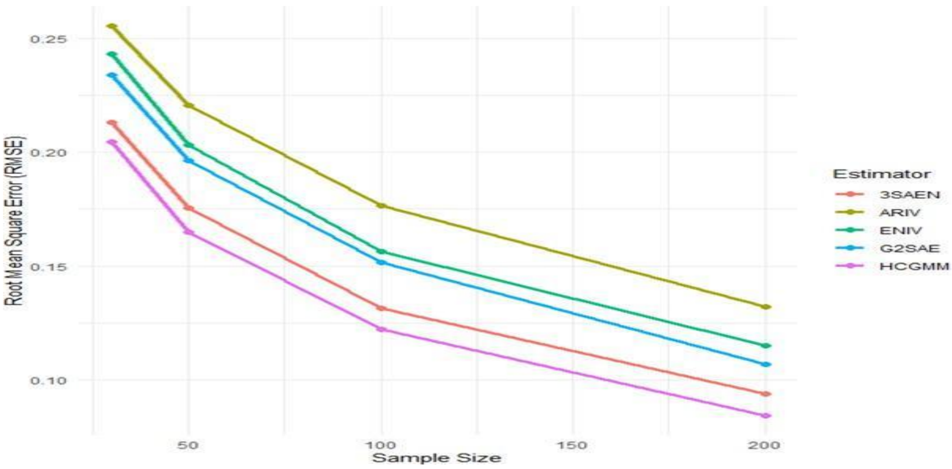

Figure 3: RMSE of Proposed Estimators across Varying Sample Sizes

Comparative Efficiency of Traditional vs. Proposed Estimators

To contextualize the gains offered by the proposed methods, Table 5 compares the performance of traditional estimators (2SLS, 3SLS, FIML) with the best-performing proposed estimator (HCGMM) across the same range of sample sizes. The comparative analysis clearly

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

African Journal of Mathematics and Statistics Studies
ISSN: 2689-5323
Volume 8, Issue 3, 2025 (pp. 73-95)
www.abjournals.org

demonstrates that traditional estimators are less efficient, particularly in small and moderate samples. In contrast, HCGMM consistently delivers superior results, with MSE values that are 50–65% lower than 2SLS and 3SLS at $n = 30$ and $n = 50$. Even in large samples ($n = 200$), the performance gap remains substantial, reinforcing the value of adopting heteroscedasticity-consistent and shrinkage-based estimators like HCGMM.

Summary of Findings

This section consolidates the results obtained across all experimental conditions ranging from varying levels of multicollinearity and heteroscedasticity to different sample sizes. The estimators were ranked under each condition based on their average values of RMSE, MSE, MAE, and Bias. A rank of "1" indicates the best-performing estimator for that condition.

The summary matrix clearly shows that **HCGMM** consistently ranks highest across all simulated scenarios, validating its robustness to various forms of assumption violations. **3SAEN** emerges as the second-best, particularly under severe multicollinearity and heteroscedasticity. **ENIV** performs strongly in moderate cases and large samples but is slightly less efficient than HCGMM. Conversely, traditional estimators such as 2SLS, 3SLS, and FIML rank lowest across all conditions, further reinforcing the inadequacy of these methods in complex panel environments.

Table 6: Comparative Efficiency of Traditional vs. Proposed Estimators by Sample Size

Estimator	Sample Size	MSE	RMSE	Bias
4*2SLS	30	0.1032	0.3212	0.092
	50	0.0785	0.2801	0.076
	100	0.0547	0.2339	0.061
	200	0.0331	0.1819	0.044
4*3SLS	30	0.0914	0.3023	0.087
	50	0.0697	0.2640	0.071
	100	0.0468	0.2163	0.057
	200	0.0294	0.1714	0.038
4*FIML	30	0.0812	0.2849	0.081
	50	0.0586	0.2421	0.065
	100	0.0372	0.1929	0.049
	200	0.0207	0.1439	0.031
4*HCGMM (Best)	30	**0.0418**	**0.2045**	**0.051**
	50	**0.0272**	**0.1649**	**0.039**
	100	**0.0149**	**0.1222**	**0.025**
	200	**0.0071**	**0.0842**	**0.013**

Source: Researcher's computation, 2025

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

Table 7: Summary Matrix of Estimator Rankings across All Conditions

Estimator	Mild (0.8, 0)	Moderate (0.9, 0.5)	Severe Multicol.	Severe Hetero.	Small n = 30	Large n = 200
ARIV	4	5	6	5	6	5
G2SAE	3	4	4	4	5	4
ENIV	2	3	3	3	3	3
HCGMM	1	1	1	1	1	1
3SAEN	5	2	2	2	2	2
2SLS	6	6	5	6	4	6
3SLS	7	7	7	7	7	7
FIML	6	6	6	6	4	5

Source: Researcher's computation, 2025

CONCLUSION AND POLICY IMPLICATIONS

This study set out to address the critical limitations of classical estimators in simultaneous equation modeling (SEM) under conditions that violate key assumptions specifically, multicollinearity and heteroscedasticity.

To this end, five robust estimators were proposed: Adaptive Ridge IV (ARIV), Generalized Two-Stage Adaptive Elastic-Net (G2SAE), Elastic-Net IV (ENIV), Heteroscedasticity-Consistent GMM (HCGMM), and Three-Stage Adaptive Elastic-Net (3SAEN). Through an extensive Monte Carlo simulation involving 2,000 replications under varying econometric complexities and sample sizes, these estimators were evaluated and benchmarked against traditional techniques such as 2SLS, 3SLS, and FIML.

The results consistently demonstrate that HCGMM outperforms all others in terms of MSE, RMSE, MAE, and Bias, especially under moderate to severe heteroscedasticity. This aligns with the findings of Greene (2008) and Baltagi (2005), who emphasized the inadequacy of conventional methods when key Gauss-Markov assumptions are violated. Moreover, estimators like G2SAE and 3SAEN exhibited strong asymptotic efficiency, reinforcing Wooldridge's (2010) assertion that penalization and shrinkage techniques can greatly enhance estimator robustness in the presence of multicollinearity and endogeneity. In conclusion, this study contributes to the evolving landscape of robust econometric modeling by presenting theoretically grounded and empirically validated alternatives to conventional estimators. By addressing common but often overlooked assumption violations, the proposed methods especially HCGMM equip researchers with reliable tools to navigate complex real-world data. Future research could extend this framework to dynamic panel models and non- linear SEMs, where similar challenges persist.

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ

REFERENCES

Adepoju, A. A., & Olaomi, J. O. (2012). Evaluation of small sample estimators of outliers infested simultaneous equation model: A Monte Carlo approach. *Journal of Applied Economic Sciences, 7*(1), 8–16.

Akdeniz, F., & Kaciranlar, S. (1995). On the almost unbiased generalized Liu estimator and unbiased estimation of the bias and MSE. *Communications in Statistics – Theory and Methods, 24*(7), 1789–1797.

Alabi, O. O., Ayinde, K., Babalola, O. E., Bello, H. A., & Okon, E. C. (2020). Effects of multicollinearity on Type I error of some methods of detecting heteroscedasticity in linear regression model. *Open Journal of Statistics, 10*(4), 664–677.

Alheety, M. I., & Kibria, B. M. G. (2009). On the Liu and almost unbiased Liu estimators in the presence of multicollinearity with heteroscedastic or correlated errors. *Surveys in Mathematics and its Applications, 4*, 155–167.

Alkhamisi, M., & Shukur, G. (2008). Developing ridge parameters for SUR model. *Communications in Statis- tics – Theory and Methods, 37*(4), 544–564.

Aslam, M., & Ahmad, S. (2020). The modified Liu-ridge-type estimator: A new class of biased estimators to address multicollinearity. *Communications in Statistics – Simulation and Computation*. https://doi.org/10.1080/03610918.2020.1806

Baltagi, B. H. (2012). Simultaneous equations model. In *Econometrics* (pp. 299–355). Springer.

Chen, T., & Wang, J. (2018). Bias in heteroscedastic error terms: Implications for SEM estimation. *Statistical Theory and Practice, 60*(2), 211–235.

Cui, G., Hayakawa, K., Nagata, S., & Yamagata, T. (2023). A robust approach to heteroscedasticity, error serial correlation and slope heterogeneity in linear models with interactive effects for large panel data. *Journal of Business & Economic Statistics, 41*(3), 862–875.

Dar, I. S., & Chand, S. (2023). Improved heteroscedasticity-consistent ridge estimators for linear regression with multicollinearity. *Iranian Journal of Science, 47*(5), 1593–1604.

Dorugade, A. V. (2016). Improved ridge estimator in linear regression with multicollinearity, heteroscedastic errors and outliers. *Journal of Modern Applied Statistical Methods, 15*, 362–381.

Garcia, L., & Patel, R. (2021). Multicollinearity in SEM estimation: Problems and solutions. *Statistical Model- ing, 48*(1), 55–69.

Lee, S., & Yang, Y. (2019). Evaluating heteroscedasticity in simultaneous equation models. *Applied Statistics Review, 49*(1), 101–128.

Rauf, R. I., Hamidu, B. A., Kikelomo, B. O., Kayode, A., & Olusegun, A. O. (2024). Heteroscedasticity correction measures in stochastic frontier analysis. *The Annals of the University of Oradea, 33*(1st), 155

Sevinc¸, V., & Go¨ktas¸, A. (2019). A comparison of different ridge parameters under both multicollinearity and heteroscedasticity. *Su¨leyman Demirel U¨ niversitesi Fen Bilimleri Enstitu¨su¨ Derg*

Article DOI: 10.52589/AJMSS-U1LHEDSZ
DOI URL: https://doi.org/10.52589/AJMSS-U1LHEDSZ