DYNAMIC RESPONSE OF A NON-UNIFORM RAYLEIGH BEAM UNDER ACCELERATING DISTRIBUTED MASSES ON A BI-PARAMETRIC ELASTIC FOUNDATION

Gyobe A. M.[1], Adeloye T. O.[2*], and Adeoye A. S.[3]

[1]Department of Physics, Nigeria Maritime University, Okerenkoko, P.M.B. 1005, Warri South-West, Delta State, Nigeria.
Email: gyobe.austin@nmu.edu.ng

[2]Department of Mathematical Sciences, Nigeria Maritime University, Okerenkoko, P.M.B. 1005, Warri South-West Delta State, Nigeria.
Email: tolulope.adeloye@nmu.edu.ng

[3]Department of Mathematical Sciences, Achievers University, Owo, Ondo State, Nigeria.
Email: adeoye.as@achievers.edu.ng

[*]Corresponding Author's Email: tolulope.adeloye@nmu.edu.ng

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ABSTRACT: *This study investigates the problem of a non-prismatic Rayleigh beam subjected to moving loads with arbitrarily prescribed velocities. The research addresses a complex structural dynamics problem that has significant implications for various engineering applications, including railway systems, bridge structures, and industrial machinery where moving loads interact with flexible structural elements. The non-uniform Rayleigh beam model incorporates both translational and rotational inertia effects, providing a more accurate representation of real-world structural behaviour compared to classical Euler-Bernoulli beam theory. The beam's non-uniformity is characterised by spatially varying material properties and geometric parameters, which significantly influence the dynamic response patterns. The bi-parametric elastic foundation is modelled using Winkler and Pasternak foundation parameters, accounting for both normal and shear interactions between the beam and its supporting medium. The accelerating distributed masses represent a realistic loading scenario encountered in many practical applications, where the velocity and acceleration of moving loads vary continuously along the beam span. This loading condition introduces complex inertial effects and coupling between the beam's natural vibration modes and the motion characteristics of the distributed masses. Gerlakin's weighted residual method is employed to treat this vibrating system problem, as it was in the preceding section. This technique is first used to transform the fourth-order partial differential equation with singular and variable coefficients governing the motion of this vibrating beam. The resulting system of equations, called Gerlakin's equations, is further simplified using the asymptotic method of Struble to obtain a second-order ordinary differential equation, which is then solved using the Duhamel integration method.*

KEYWORDS: Bi-parametric elastic foundation, Dynamic response, Axially prestressed thick beam, Rotatory inertia, Travelling time.

INTRODUCTION

The dynamic response of structural elements, particularly beams, under various loading conditions is a cornerstone of structural engineering and applied mechanics. The accurate prediction of such responses is crucial for ensuring the safety, reliability, and optimal performance of a wide array of engineering structures, ranging from high-speed railway tracks and bridges to robotic arms and aerospace components. A particularly challenging and practically significant problem arises when these structures are subjected to moving loads, which induce dynamic effects that can be far more severe than those caused by static loads of the same magnitude. The complexity is further amplified when the structural properties are not uniform along their length, when the supporting medium exhibits complex elastic behaviour, and when the moving loads are not constant but rather accelerating.

The study of beams on elastic foundations is fundamental to civil and mechanical engineering, providing models for railway tracks, pipelines, and building foundations. The inclusion of moving loads, especially those with varying velocities and accelerations, reflects realistic operational scenarios and introduces significant inertial forces that must be accurately accounted for. Furthermore, considering non-uniform beam properties and sophisticated foundation models, such as bi-parametric elastic foundations, enhances the fidelity of the theoretical models to real-world structures and their interactions with the supporting soil.

The motivation for this comprehensive investigation stems from the increasing demand for more precise and robust designs in modern engineering. Traditional simplified models often fall short in capturing the nuanced dynamic behaviours observed in complex systems, potentially leading to conservative designs or, more critically, to unforeseen failures. By integrating the complexities of non-uniformity, accelerating distributed masses, and bi-parametric foundations, this review aims to provide a holistic understanding of the coupled dynamic behaviour, identify critical parameters influencing the response, and highlight the current state of research in this specialised domain. This understanding is vital for the development of advanced design methodologies, optimisation strategies, and predictive maintenance protocols for structures operating under dynamic and challenging environments.

The theoretical modelling of beams has evolved significantly over time to capture increasingly complex behaviours. The classical Euler-Bernoulli beam theory, while foundational, assumes that plane sections remain plane and perpendicular to the neutral axis after bending and neglects the effects of shear deformation and rotational inertia. While adequate for slender beams under static or low-frequency dynamic loads, these assumptions become invalid for thicker beams or those subjected to high-frequency vibrations and dynamic loads. This led to the development of more refined beam theories, among which the Rayleigh beam theory stands as a significant improvement.

Euler-Bernoulli beam theory, developed in the 18th century, provides a simplified yet powerful model for analysing the bending behaviour of slender beams. Its governing equation is a fourth-order partial differential equation that relates the beam's transverse deflection to the applied load, material properties, and geometric properties (moment of inertia of the cross-section). However, this theory inherently neglects the effects of shear deformation and the rotational inertia of the beam's cross-section. This means it assumes that the shear strain is zero and that the kinetic energy due to the rotation of the cross-section is negligible (Ogunyebi, 2016). (Adeoye & Awodola, 2018) investigated the dynamic response to moving distributed masses

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of pre-stressed uniform Rayleigh beam resting on variable elastic Pasternak foundation. In this same vein, (Adeoye & Awodola, 2017) studied the influence of rotatory inertial correction factor on the vibration of elastically supported non-uniform Rayleigh beam on variable foundation.

Recognising these limitations, Lord Rayleigh introduced an improvement in 1877 by incorporating the effect of rotational inertia of the cross-sectional area into the beam's kinetic energy formulation (Nguyen, 2017). The Rayleigh beam theory thus accounts for the kinetic energy associated with the rotation of the beam's cross-section as it vibrates, in addition to the kinetic energy from transverse translation. This inclusion makes the Rayleigh beam theory more accurate than the Euler-Bernoulli theory for beams where rotational inertia effects are significant, such as those with larger cross-sectional dimensions or those undergoing higher frequency vibrations (Zhu & Chung, 2019). The governing equation for a Rayleigh beam is a sixth-order partial differential equation, reflecting the added complexity of rotational inertia.

While the Rayleigh beam theory provides a more accurate model for uniform beams, many real-world structures possess non-uniform properties along their length. A non-uniform beam is characterised by variations in its geometric properties and/or material properties along its longitudinal axis (Saurin, 2019). These variations can be continuous or discontinuous (Magnucki *et al.*, 2021; Avcar, 2017; Kumar et al., 2017).The relationship between Density and Porosity of Rocks from Jos and Environs, North Central Nigeria was established using correlation coefficient of the two parameters that always served as guide on engineering design, especially on strength of materials in (Akanbi et al, 2019). In the work of (Teru et al, 2019) , the Vibrational modes of the Tympanic membrane were derived. (Adeoye & Akintomide, 2017) examined the dynamic behavior of Bernoulli-Euler beam with elastically supported boundary conditions under moving distributed masses and resting on constant foundation using generalized Galerkin method.

The non-uniformity of a beam significantly influences its dynamic response characteristics. Unlike uniform beams, where analytical solutions for natural frequencies and mode shapes are often straightforward, non-uniform beams present a much greater challenge. The spatially varying properties lead to governing differential equations with variable coefficients, which are generally difficult to solve analytically. Consequently, numerical and semi-analytical methods are frequently employed for their analysis (Klein, 1974). The implications of non-uniformity on the dynamic behaviour of structural elements include alterations in natural frequencies and mode shapes due to the varying distribution of mass and stiffness, which affect the beam's vibrational characteristics and can help prevent resonance (Bhat & Ganguli, 2018). Additionally, non-uniformity can lead to localised dynamic responses, causing stress concentrations or excessive deflections in specific regions (Ogunyebi *et al*., 2013), while also making stress wave propagation more complex due to reflections and transmissions at points where material properties change.

Due to the complexity introduced by variable coefficients, various methods have been developed to analyse non-uniform Rayleigh beams. Analytical methods, such as the Frobenius method, perturbation techniques, or integral transforms, can yield exact or approximate solutions for simple variations like linear or exponential tapering (Sarkar *et al*., 2016), but these are limited to specific cases. For more general non-uniformities, numerical methods are indispensable. These include the Finite Element Method (FEM), which discretises the beam to

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approximate complex geometries and material variations (Wang *et al.*, 2022); the Differential Transform Method (DTM), which converts differential equations into algebraic forms for vibration analysis of non-uniform beams on elastic foundations (Olotu *et al*., 2021); the Generalised Differential Quadrature Method (GDQM), which approximates derivatives as weighted sums at discrete points (Xiao, Zhang & Dai, 2024); and the Transfer Matrix Method, which efficiently relates state vectors along multi-span or stepped beams (Yang *et al*., 2021).

The interaction between moving loads and structural elements is a classical problem in structural dynamics, with significant implications for the design and analysis of various engineering systems. While much of the early research focused on loads moving at constant velocities, real-world scenarios often involve loads that accelerate or decelerate, introducing additional complexities and dynamic effects that are not captured by constant velocity models. This section explores the characteristics and challenges associated with accelerating distributed masses. Moving loads can be broadly classified based on their characteristics, such as whether they are concentrated or distributed and whether they move at constant or variable velocities. A concentrated load acts at a single point, while a distributed load is spread over a length or area—for example, the weight of a train distributed across its bogies or the pressure of a fluid moving through a pipe (Daniel, 2020). While early studies often assumed loads moving at constant velocities, many real-world applications involve loads with variable velocities—such as vehicles accelerating on bridges, cranes traversing gantries, or high-speed trains entering or leaving stations—introducing acceleration or deceleration effects that must be considered (Lee, 1995).

The accelerating nature of distributed masses introduces several significant challenges in dynamic analysis. Complex inertial effects arise because an accelerating mass generates inertial forces proportional to its mass and acceleration, which, for distributed masses, act along the contact length, resulting in time-dependent and spatially varying loading that produces coupled equations of motion linking the beam's response to the load's motion (Esen, 2011). As the mass accelerates or decelerates, the magnitude and distribution of the load change with time, making the governing differential equations non-homogeneous with time-varying coefficients that are more challenging to solve than those for constant velocity loads (Adeloye, 2024). Furthermore, accelerating loads can cause greater dynamic amplification than constant velocity loads, potentially resulting in larger deflections, stresses, and vibrations; resonance conditions may occur even below critical speeds because of the changing frequency content of the excitation (Omolofe & Adara, 2020). In some situations, the motion of the load and the deformation of the structure are mutually dependent—such as when a beam's deflection influences the load's acceleration and vice versa—requiring a coupled analysis to account for fluid-structure or vehicle-structure interaction effects (Omolofe & Adara, 2020).

Understanding the dynamic response to accelerating distributed masses is crucial for the design and safety of various engineering structures. In railway systems, for example, high-speed trains accelerating or braking can be modelled as distributed loads acting on tracks considered as beams on elastic foundations, with the train's weight and acceleration significantly affecting track stability and passenger comfort (Ogunlusi & Awodola, 2024). Similarly, bridges subjected to heavy vehicles that accelerate or decelerate—especially long-span bridges—experience pronounced dynamic effects, with dynamic impact factors for accelerating loads often much higher than for static or constant-speed loads (Meher, 2012). In industrial machinery, such as conveyor belts transporting materials, gantry cranes moving heavy loads,

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or equipment where distributed masses are started or stopped, the resulting dynamic forces can influence structural integrity and operational efficiency. Likewise, in aerospace and robotics, the design of aircraft wings or robotic arms carrying distributed payloads must account for rapid acceleration or deceleration during manoeuvres to ensure performance and safety (Yang *et al.*, 2021; Esen, 2011).

Thus, this article focuses on the dynamic response of a simply supported non-uniform thick beam subjected to the influence of partially distributed accelerating masses. The beam is modelled on a non-winkler foundation, which accounts for both Winkler and Pasternak elastic parameters. These parameters represent the stiffness and shear interaction of the foundation, further complicating the dynamic analysis. This study provides a comprehensive insight into the interaction between accelerating masses and non-uniform thick beams supported by bi-parametric foundations, exploring how these factors influence the system's natural frequencies, mode shapes, and dynamic response. The analysis is intended to offer theoretical and practical insights applicable to real-world structural systems.

The remaining paper is organised in the following manner. Section 2 introduces the mathematical formulation of the problem. Then, Section 3 discusses the discretisation procedure. Next, Section 4 provides illustrative examples and a detailed discussion of the results. Finally, a concluding remark is presented in Section 5.

EQUATION OF MOTION

The problem to be investigated herewith is that of a transverse vibration of a simply supported non-uniform Rayleigh beam under compressive axial force and a moving distributed mass. The beam is initially loaded with a load

$$P_0(x,t) = P_f(x,t)[1 - \frac{d^2}{dt^2}[\frac{\theta(x,t)}{g}]] \tag{1}$$

where the continuous moving force $P_f(x,t)$ acting on the beam model is given as

$$P_f(x,t) = \frac{Mg}{x_0}[H(x-(f(t))+x_0) - H(x-(f(t))-x_0)] \tag{2}$$

where H is the Heaviside unit step function with the property,

$$H(x) = \{0\, if \quad x<0, 1\, if \quad x>0, \tag{3}$$

For the limiting condition as $\xi \to 0$ one obtains

$$\delta(x-(f(t))) = \frac{1}{x_0}[H(x-(f(t))+x_0) - H(x-(f(t))-x_0)] \tag{4}$$

where $\delta(x-(f(t)))$ is the Dirac delta function. Furthermore, the operator $\frac{d^2}{dt^2}$ in equation (1) is defined as

$$\frac{d^2}{dt^2}[\cdot] = [\frac{\partial^2}{\partial t^2} + 2\frac{d}{dt}(f(t))\frac{\partial^2}{\partial x \partial t} + (\frac{d}{dt}(f(t)))^2\frac{\partial^2}{\partial x^2} + \frac{d^2}{dt^2}(f(t))\frac{\partial}{\partial x}][\cdot] \tag{5}$$

The first term on the right side of equation 1 represents the centripetal acceleration of the moving mass; the second term is the well-known Coriolis acceleration; the third term is the

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acceleration component in the vertical direction when the moving load speed is not a constant; and the last term is the support beam acceleration at the point of contact with the moving mass. The distance covered by the load on the same structure at any given instance of time is given as

$$(f(t)) = x_0 + ct + \frac{1}{2}at^2 \tag{6}$$

where x_0 is the point of application of the force $P_0 = Mg$ at the instance $t = 0$, c is the initial velocity and a the constant acceleration of motion. The flexural motion of a non-uniform Rayleigh beam supported by an elastic foundation and traversed by uniform partially distributed masses M is considered in this section. The mass M is assumed to hit the beam at time $t = 0$ and travel across it with a non-uniform velocity such that the motion of the contact point of the moving load is described by the function $(f(t))$ which is the equation of the trajectory of the moving load P_0. The fourth-order partial differential equation of motion, assuming uniform cross-section and neglecting damping is given as

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2\theta(x,t)}{\partial x^2}] - N\frac{\partial^2\theta(x,t)}{\partial x^2} + \mu(x)\frac{\partial^2\theta(x,t)}{\partial t^2} - R_0\frac{\partial}{\partial x}[\mu(x)\frac{\partial^3\theta(x,t)}{\partial x\partial t^2}] + \theta(x,t) = P_0(x,t) \tag{7}$$

where $EI(x)$ is the variable flexural rigidity of the beam, N is the axial force, $\mu(x)$ is the variable mass per unit length of the beam, R_0 is the rotatory inertia correction factor, and $\theta_f(x,t)$ is the foundation reaction. The relationship between the foundation reaction and the lateral deflection $\theta(x,t)$ is given as

$$\theta(x,t) = K\theta(x,t) - G\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{8}$$

where K is the foundation stiffness and G is the shear modulus. Also, we have

boundary Conditions: $\theta(0,t) = \theta(L,t) = 0, \quad \frac{\partial^2\theta(0,t)}{\partial x^2} = 0 = \frac{\partial^2\theta(L,t)}{\partial x^2}$

Initial Conditions:

$$\theta(x,0) = 0 = \frac{\partial\theta(x,0)}{\partial t} \tag{9}$$

Furthermore, for the variable moment of inertial $I(x)$ and variable mass per unit length $\mu(x)$ of the beam, we adopt the example in Omolofe and Adedowole (2017) and thus, $I(x)$ and $\mu(x)$ are respectively given as

$$\mu(x) = \mu_0(1 + sin\frac{\pi x}{L}), \qquad I(x) = I_0(1 + sin\frac{\pi x}{L})^3 \tag{10}$$

Equation (7) in view of equations (1), (2) and (5) becomes

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2\theta(x,t)}{\partial x^2}\right] - N\frac{\partial^2\theta(x,t)}{\partial x^2} + \mu(x)\frac{\partial^2\theta(x,t)}{\partial t^2} - R_0\frac{\partial}{\partial x}\left[\mu(x)\frac{\partial^3\theta(x,t)}{\partial x\partial t^2}\right] + K\theta(x,t) - G\frac{\partial^2\theta(x,t)}{\partial x^2} = \frac{Mg}{x_0}\left[H\big(x - \big(f(t)\big) + x_0\big) - H\big(x - \big(f(t)\big) - x_0\big)\right] + -\frac{M}{x_0}[H(x -$$

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$$(f(t)) + x_0) - H(x - (f(t)) - x_0)] \times [\frac{\partial^2\theta(x,t)}{\partial t^2} + 2\frac{d}{dt}(f(t))\frac{\partial^2\theta(x,t)}{\partial x\partial t} + (\frac{d}{dt}(f(t)))^2\frac{\partial^2\theta(x,t)}{\partial x^2} + \frac{d^2}{dt^2}(f(t))\frac{\partial\theta(x,t)}{\partial x}] \quad (11)$$

Equation (11) represents the flexural motions of a structurally prestressed non-prismatic Rayleigh beam with a simple support end condition and under prestressed axial force and accelerating loads. The beam parameters do not vary over the span length of the beam. Nonetheless, most often, many practical problems in dynamics of structures under fast-travelling loads involve structural members whose physical properties vary over the span length l. Consequently, the governing partial differential equation contains variable coefficients. Thus, an exact solution to the dynamical problem involving such structural members is no longer feasible. In what follows, an approximate analytical solution to the beam-mass problem in equation (11) is sought. Evidently, equation (11) is not amenable to the commonly used analytical method. Thus, an approximate analytical method of solution is adopted.

Discretization Procedure

To treat (11), consider

$$\varphi(x,t) = \sum_{k=1}^{\infty} Y_k(t)U_k(x) \quad (12)$$

To solve problems involving mechanical vibrations, the Weighted Residual Method [?], one of the approximate methods best suited for solving diverse problems in the dynamics of structures, a versatile technique, will be adopted.

$$U_k(x) = sin\frac{\iota_k x}{l} + \Upsilon_{1m}cos\frac{\iota_k x}{l} + \Upsilon_{2m}sinh\frac{\iota_k x}{l} + \Upsilon_{3m}cosh\frac{\iota_k x}{l} \quad (13)$$

The mode frequency ι_k and the constants Υ_{1m}, Υ_{2m}, and Υ_{3m} characterise the $k-th$ normal mode of vibration of a uniform beam function. These constants can be obtained by substituting 9 with the pinned-pinned boundary condition. It is important to highlight that in the case of a pinned-pinned beam, $\Upsilon_{1m} = \Upsilon_{2m} = \Upsilon_{3m} = 0$ and $\iota_k = k\pi$. Furthermore, the following property of Heaviside function

$$\frac{1}{x_0}\int_0^L [H(x - (f(t)) + x_0) - H(x - (f(t)) - x_0)]dx = (f(t)) + x_0^2 f''\frac{(f(t))}{3!} + x_0^2 f^{iv}\frac{(f(t))}{5!} \quad (14)$$

will be useful The technique for solving the problem requires that ?? is expressed as

$$\Omega(x,t) = \sum_{k=1}^{\infty} Y_k(t)U_k(x) \quad (15)$$

substituting equation (12) into equation (11) and taking into account equation (14) and equation (15), yields

$$\sum_{k=1}^{\infty} \{\mu_0[(U_k(x) + U_k(x)sin\frac{\pi x}{L}) - R_0(U_k''(x) + U_k'(x)\frac{\pi}{L}cos\frac{\pi x}{L} + U_k''(x)sin\frac{\pi x}{L})]\ddot{Y}_k(t) + \frac{EI_0}{4}[\left(10 + 15sin\frac{\pi x}{L} - 6cos\frac{2\pi x}{L} - sin\frac{3\pi x}{L}\right)U_k^{iv}(x) + (30\frac{\pi}{L}cos\frac{\pi x}{L} +$$

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$$24\frac{\pi}{L}sin\frac{2\pi x}{L} - 6\frac{\pi}{L}cos\frac{3\pi x}{L})U_k'''(x) + (24(\frac{\pi}{L})^2cos\frac{2\pi x}{L} - 15(\frac{\pi}{L})^2sin\frac{\pi x}{L} + 9(\frac{\pi}{L})^2sin\frac{3\pi x}{L})U_k''(x) - NU_k''(x) + KU_k(x) - GU_k''(x)]Y_k(t)\} = [\frac{Mg}{U_0}[U_k(f(t)) + \frac{x_0^2}{6}U_k''(f(t))] - \frac{M}{U_0}\sum_{k=1}^{\infty}\{[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) - \frac{M}{U_0}\sum_{k=1}^{\infty}\{[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\ddot{Y}_k(t) + 2(c+at)[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t))$$

$$+((f(t)))'(f(t))U_n'(f(t)))]\dot{Y}_k(t) + (c+at)^2[U_k''(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k^{iv}(f(t))U_n(f(t)) + U_k''(f(t))U_n''(f(t)) + ((f(t)))'''(f(t))U_n'(f(t)))]Y_k(t) + a[U_k'(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k'''(f(t))U_n(f(t)) + U_k'(f(t))U_n''(f(t)) + ((f(t)))''(f(t))U_n'(f(t)))]Y_k(t)]U_k(x)\} = 0 \qquad (16)$$

The method of solution requires that the LHS of equation (16) be orthogonal to the function $U_n(x)$. Thus, multiplying equation (16) by $U_n(x)$ and integrating from 0 to l, we have

$$\int_0^L \{\sum_{k=1}^{\infty}\{\mu_0[(U_k(x) + U_k(x)sin\frac{\pi x}{L}) - R_0(U_k''(x) + U_k'(x)\frac{\pi}{L}cos\frac{\pi x}{L} + U_k''(x)sin\frac{\pi x}{L})]\ddot{Y}_k(t) + \frac{EI_0}{4}[(10 + 15sin\frac{\pi x}{L} - 6cos\frac{2\pi x}{L} - sin\frac{3\pi x}{L})U_k^{iv}(x) + (30\frac{\pi}{L}cos\frac{\pi x}{L} + 24\frac{\pi}{L}sin\frac{2\pi x}{L} - 6\frac{\pi}{L}cos\frac{3\pi x}{L})U_k'''(x) + (24(\frac{\pi}{L})^2cos\frac{2\pi x}{L} - 15(\frac{\pi}{L})^2sin\frac{\pi x}{L} + 9(\frac{\pi}{L})^2sin\frac{3\pi x}{L})U_k''(x) - NU_k''(x) + KU_k(x) - GU_k''(x)]Y_k(t)\} = [\frac{Mg}{U_0}[U_k(f(t)) + \frac{x_0^2}{6}U_k''(f(t))] - \frac{M}{U_0}\sum_{k=1}^{\infty}\{[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\ddot{Y}_k(t) + 2(c+at)[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\dot{Y}_k(t) + (c+at)^2[U_k''(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k^{iv}(f(t))U_n(f(t)) + U_k''(f(t))U_n''(f(t)) + ((f(t)))'''(f(t))U_n'(f(t)))]Y_k(t) + a[U_k'(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k'''(f(t))U_n(f(t)) + U_k'(f(t))U_n''(f(t)) + ((f(t)))''(f(t))U_n'(f(t)))]Y_k(t)]U_k(x)\}U_n(x)\}dx = 0 \qquad (17)$$

After some simplifications and rearrangements, equation (17) can be written as

$$\sum_{k=1}^{\infty}\left\{\mu_0\left[(D_1 + D_2) - R_0\left(D_3 + \frac{\pi}{L}D_4 + D_5\right)\right]\ddot{Y}_k(t) + \frac{EI_0}{4}[(D_6 + D_7 + D_8) - ND_3 + KD_1 - GD_3]Y_k(t)\right\} + +M\sum_{k=1}^{\infty}\{[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\ddot{Y}_k(t) + 2(c+at)[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\dot{Y}_k(t) + (c+at)^2[U_k''(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k^{iv}(f(t))U_n(f(t)) + U_k''(f(t))U_n''(f(t)) + ((f(t)))'''(f(t))U_n'(f(t)))]Y_k(t) +$$

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$a[U_k'(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k'''(f(t))U_n(f(t)) + U_k'(f(t))U_n''(f(t)) + ((f(t)))''(f(t))U_n'(f(t)))]Y_k(t)]\} = Mg[U_k(f(t)) + \frac{x_0^2}{6}U_k''(f(t))]$ (18)

where

$$D_1 = \int_0^L U_k(x)U_n(x)dx\,; \quad D_2 = \int_0^L U_k(x)U_n(x)sin\frac{\pi x}{L}dx \;;$$

$$D_3 = \int_0^L U_k''(x)U_n(x)dx;$$

$$D_4 = \int_0^L U_k'(x)U_n(x)cos\frac{\pi x}{L}dx\,; \quad D_5 = \int_0^L U_k''(x)U_n(x)sin\frac{\pi x}{L}dx$$

$$D_6 = \int_0^L (10 + 15sin\frac{\pi x}{L} - 6cos\frac{2\pi x}{L} - sin\frac{3\pi x}{L})U_k^{iv}(x)U_n(x)dx$$

$$D_7 = \int_0^L (30\frac{\pi}{L}cos\frac{\pi x}{L} + 24\frac{\pi}{L}sin\frac{2\pi x}{L} - 6\frac{\pi}{L}cos\frac{3\pi x}{L})U_k^{iii}(x)U_n(x)dx$$

$$D_8 = \int_0^L (24(\frac{\pi}{L})^2cos\frac{2\pi x}{L} - 15(\frac{\pi}{L})^2sin\frac{\pi x}{L} + 9(\frac{\pi}{L})^2sin\frac{3\pi x}{L})U_k^{ii}(x)U_n(x)dx$$

equation (11) in view of (15) becomes

$\ddot{Y}_k(t) + \alpha_{ss}^2Y_k(t) + \Psi\sum_{k=1}^{\infty}\Delta^*[[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\ddot{Y}_k(t) + 2(c + at)[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\dot{Y}_k(t) + (c+at)^2[U_k''(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k^{iv}(f(t))U_n(f(t)) + U_k''(f(t))U_n''(f(t)) + ((f(t)))'''(f(t))U_n'(f(t)))]Y_k(t) + a[U_k'(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k'''(f(t))U_n(f(t)) + U_k'(f(t))U_n''(f(t)) + ((f(t)))''(f(t))U_n'(f(t)))]Y_k(t)] = \frac{Mg}{D_9\mu_0}[U_k(f(t)) + \frac{x_0^2}{6}U_k''(f(t))]$ (19)

where

$$\Delta^* = \frac{L}{D_9} \qquad and \qquad \alpha_{ss}^2 = \frac{EI_0(D_6+D_7+D_8)}{4D_9\mu_0} - \frac{ND_3-KD_1+GD_3}{D_9} \tag{20}$$

Equation (20) represents the transformed governing equation of a pinned-pinned beam resting on an elastic foundation and subjected to moving distributed masses. In what follows, we shall discuss two special cases of equation (20).

The Non-Prismatic Rayleigh Beam Traversed by Moving Force

The system's approximate model is obtained when the inertia effect of the moving distributed mass is significantly smaller than the effect of their weight, allowing the inertia forces to be completely neglected (i.e Setting $\Psi = 0$) in equation (20); one obtains the equation of motion

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of the simply supported non-uniform Rayleigh beam subjected to a distributed moving force. Thus, equation (20), after some rearrangements reduces to

$$\ddot{Y}_k(t) + \alpha_{ss}^2 Y_k(t) = P_{ss}[sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{x_0^2}{6}\{-(\frac{m\pi(c+at)}{L})^2 sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{m\pi a}{L} cos\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\}] \quad (21)$$

equation (4.83) when solved in conjunction with the initial conditions, one obtains

$$\theta(x,t) = \sum_{m=1}^{\infty} \frac{P_{ss}}{\alpha_{mf}}\{\frac{c_1+c_2 i}{8}\sqrt{\frac{\pi}{a\omega}}[(i+1)erf(\frac{(1-i)(\omega at + c\omega - \alpha_{mf})}{2\sqrt{a\omega}}) - erf(\frac{(1-i)(c\omega-\alpha_{mf})}{2\sqrt{a\omega}})(cosm_1 - isinm_1) + (i-1)(erf(\frac{(1+i)(\omega at + c\omega - \alpha_{mf})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega-\alpha_{mf})}{2\sqrt{a\omega}}))(cosm_2 - isinm_2)] - \frac{(c_1-c_2 i)(1-i)}{8}\sqrt{\frac{\pi}{a\omega}}[erf(\frac{(1+i)(\omega at + c\omega - \alpha_{mf})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega-\alpha_{mf})}{2\sqrt{a\omega}})(cosm_3 - isinm_3) - (erf(\frac{(1+i)(\omega at + c\omega + \alpha_{mf})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega+\alpha_{mf})}{2\sqrt{a\omega}}))(cosm_4 + isinm_4)]] \times [sin\frac{m\pi x}{L}] \quad (22)$$

which represents the transverse displacement response of a pinned-pinned non-uniform Rayleigh beam resting on elastic foundation to a force moving with varying velocities.

Analytical Method in Moving Mass Problem

In this section, the effect of the mass of the moving system is considered. In this case, $\Psi \neq 0$, and it is termed the moving mass problem. In this case, the solution to the entire equation (19) is required. In doing this, the governing equation (19) is rearranged to take the form

$$\ddot{Y}_m(t) + \alpha_{mm}^2 Y_k(t) + \Psi\sum_{m=1}^{\infty} \Delta^*[[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\ddot{Y}_m(t) + 2(c + at)[U_k(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k(f(t))U_n''(f(t)) + U_k''(f(t))U_n(f(t)) + ((f(t)))'(f(t))U_n'(f(t)))]\dot{Y}_m(t) + (c+at)^2[U_k''(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k^{iv}(f(t))U_n(f(t)) + U_k''(f(t))U_n''(f(t)) + ((f(t)))'''(f(t))U_n'(f(t)))]Y_m(t) + a[U_k'(f(t))U_n(f(t)) + \frac{x_0^2}{6}(U_k'''(f(t))U_n(f(t)) + U_k'(f(t))U_n''(f(t)) + ((f(t)))''(f(t))U_n'(f(t)))]Y_m(t)] = \frac{Mg}{D_9\mu_0}[sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{x_0^2}{6}\{-(\frac{m\pi(c+at)}{L})^2 sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{m\pi a}{L} cos\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\}] \quad (23)$$

As in the previous case, an exact analytical solution to the above equation is not possible. Thus, following arguments similar to those in the last sections, equation (23) is simplified to take the form

$$\ddot{Y}_m(t) + \alpha_{mm}^2 Y_k(t) = \frac{Mg}{D_9\mu_0}[sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{x_0^2}{6}\{-(\frac{m\pi(c+at)}{L})^2 sin\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2) + \frac{m\pi a}{L} cos\frac{m\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\}] \quad (24)$$

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where

$$\alpha_{mm} = \alpha_{mf}\{1 - \frac{\Gamma\Delta^*}{2}\{\varrho_1 - \frac{c^2\varrho_2 + a\varrho_3}{\alpha_{mf}^2}\}\} \quad (25)$$

is the modified natural frequency corresponding to the frequency of the free system? Equation (24) when solved in conjunction with the initial conditions, one obtains

$$\theta(x,t) = \sum_{m=1}^{\infty} \frac{P_{ss}}{\alpha_{mm}}\{\frac{c_1+c_2 i}{8}\sqrt{\frac{\pi}{a\omega}}[(i+1)erf(\frac{(1-i)(\omega at + c\omega - \alpha_{mm})}{2\sqrt{a\omega}}) - erf(\frac{(1-i)(c\omega-\alpha_{mm})}{2\sqrt{a\omega}})(cosm_1 - isinm_1) + (i-1)(erf(\frac{(1+i)(\omega at + c\omega - \alpha_{mm})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega-\alpha_{mm})}{2\sqrt{a\omega}}))(cosm_2 - isinm_2)] - \frac{(c_1-c_2 i)(1-i)}{8}\sqrt{\frac{\pi}{a\omega}}[erf(\frac{(1+i)(\omega at + c\omega - \alpha_{mm})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega-\alpha_{mm})}{2\sqrt{a\omega}})(cosm_3 - isinm_3) - (erf(\frac{(1+i)(\omega at + c\omega + \alpha_{mm})}{2\sqrt{a\omega}}) - erf(\frac{(1+i)(c\omega+\alpha_{mm})}{2\sqrt{a\omega}}))(cosm_4 + isinm_4)]] \times [sin\frac{\lambda_m x}{L}] \quad (26)$$

which represents the traversed displacement response to a distributed moving mass with non-Uniform velocity of a simply supported non-Uniform Rayleigh beam resting on an elastic foundation.

ANALYSIS OF THE RESULTS AND DISCUSSION

This illustration is done by considering a non-homogenous beam of modulus of elasticity $E = 3.1 \times 10^{10} N/m^2$, the moment of inertial $J = 2.87698 \times 10^-3m^4$, the beam span $l = 50m$ and the mass per unit length of the beam $\mu = 2758.291kg/m$. The values of foundation moduli are varied between $0N/m^3$ and $40000N/m^3$, and the values of axial force N are varied between $0N$ and $2.0 \times 10^8 N$. The beam is subjected to distributed masses with uniformly accelerated, decelerated or uniform velocity motion. The mass is assumed to strike the beam at $\chi = 0$ and $t = 0$.

Simply Supported End Condition

As an example in this section a non-uniform beam of modulus of elasticity $E = 3.1 \times 10^{10} N/m^2$, the moment of inertial $I = 2{,}87698 \times 10^-3m^4$, the beam span $L = 50m$ and the mass per unit length of the beam $\mu = 2758.291kg/m$ is considered. The values of foundation moduli vary between $0N/m^3$ and $40000N/m^3$, the values of axial force N is varied between $0N$ and 2.0×10^8.

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Effect of varying N

Figure 1 displays the axial force $N = 0{,}400000{,}800000$. Figure 1 shows the dynamic deflection profiles of a simply supported non-prismatic Rayleigh beam under accelerating forces for the motion stage. Higher values of the axial force generally lead to an increase in the response amplitudes. The compressive axial force destabilises the beam and makes it bend very easily. Thus, making it less stable and far more susceptible to sideways deflection.

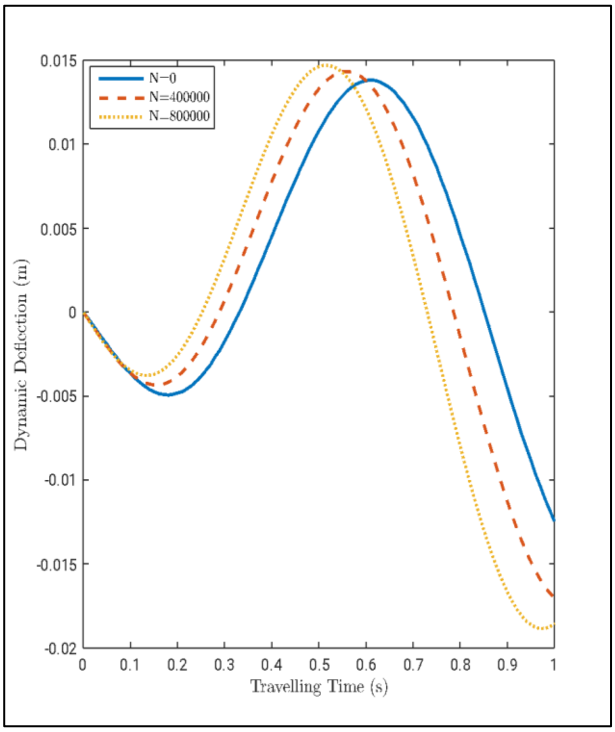

Fig. 1: Effects of axial force N on the dynamic response of simply supported, non-uniform Rayleigh beam for moving force

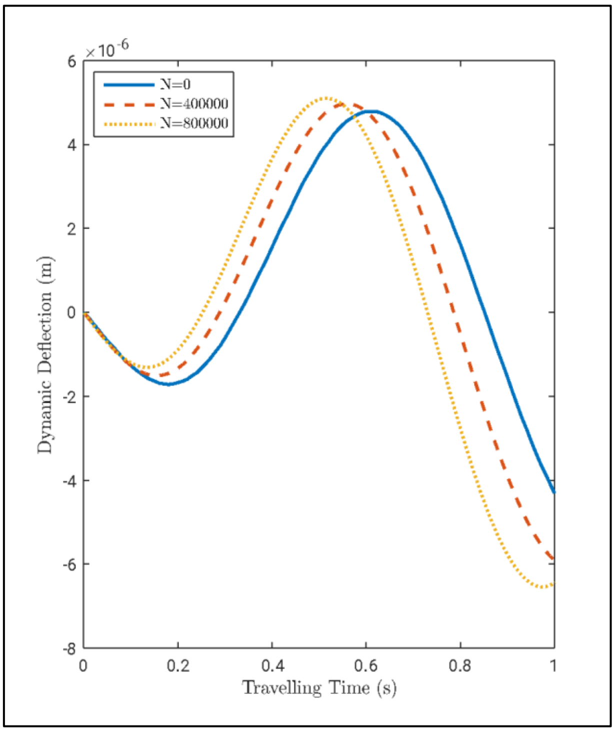

Fig. 2: Effects of axial force N on the dynamic response of simply supported Supported non-uniform Rayleigh beam for moving force

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Effect of varying $\boldsymbol{K}$

Figures 3 and 4 show the dynamic response of the simply supported beam under travelling accelerating forces and masses for various values of foundation modulus $K = 0, 40{,}000, 80{,}000$. It has been observed that higher values of foundation modulus result in a reduction in the dynamic amplitude of the dynamic response of the vibrating beam. The same behaviour characterises the deflection of the same beam under the action of distributed masses moving at variable velocity for various values of foundation modulus. Energy from the travelling load increases the foundation stiffness, thereby leaving less energy to cause deflection on the beam.

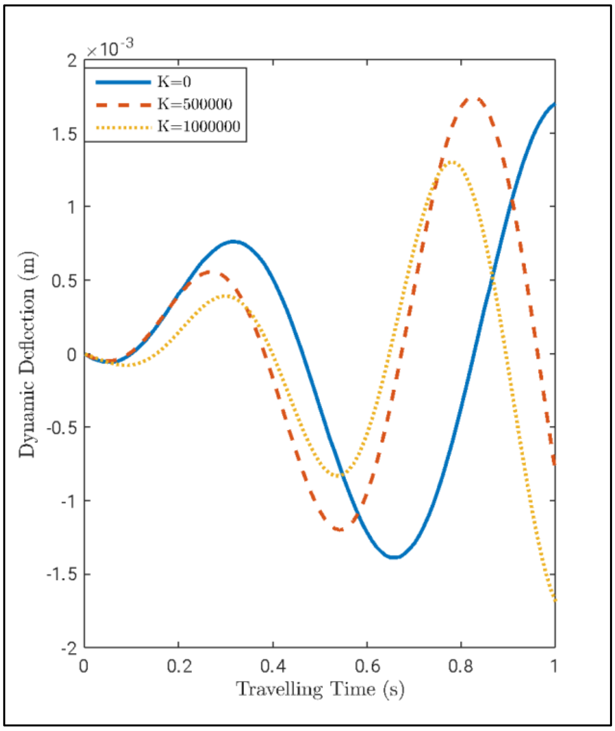

Fig. 3: Effects of foundation modulus K on the dynamic coefficient of pinned-pinned Supported Rayleigh beam for moving force

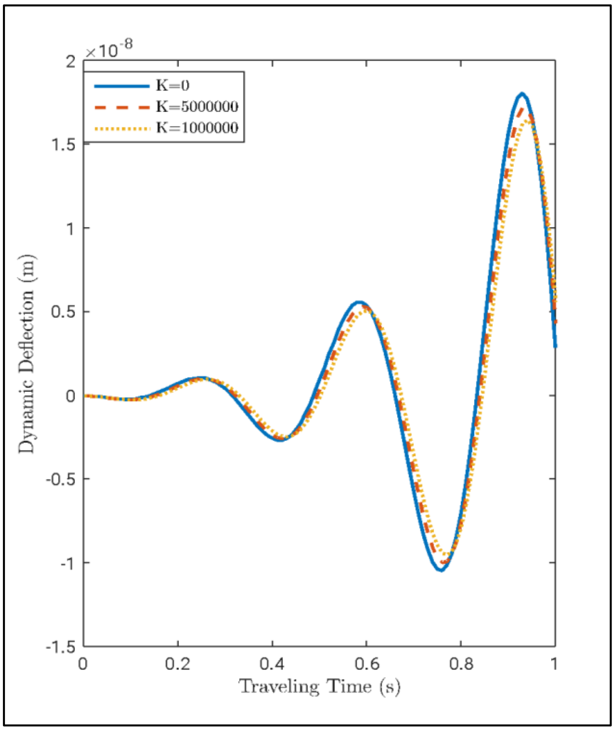

Fig. 4: Effects of foundation modulus K on the dynamic coefficient of pinned-pinned Supported Rayleigh beam for moving force

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Effect of varying G

Figures 5 and 6 illustrate the influence of varying the shear modulus G on the dynamic deflection of a simply supported non-uniform beam subjected to travelling accelerating forces and masses. The results show that the shear modulus G significantly affects the dynamic deflection of the moving load. Higher values of shear modulus G reduce the dynamic deflection of the vibrating beam. This is because when the shear modulus is increased, the beam becomes rigid against shearing; this directly leads to small shear deformation.

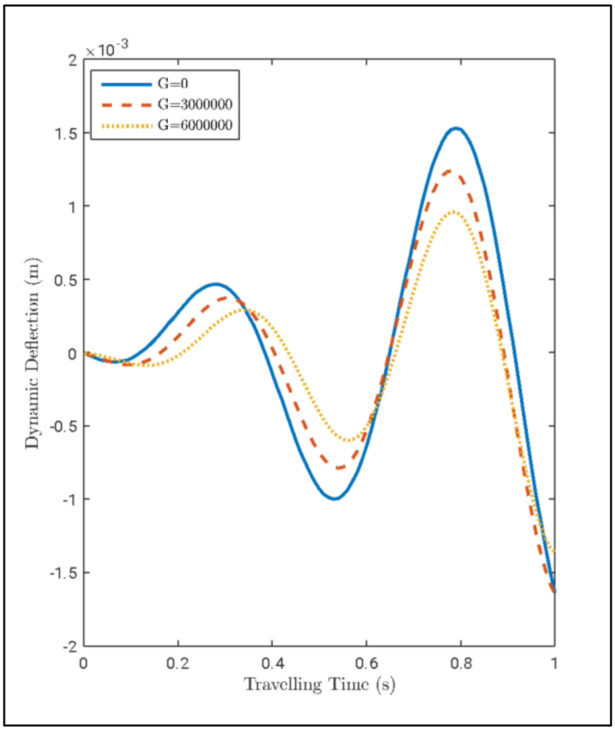

Fig. 5: Effects of shear modulus G on the dynamic deflection of a simply supported non-uniform Rayleigh beam for moving force

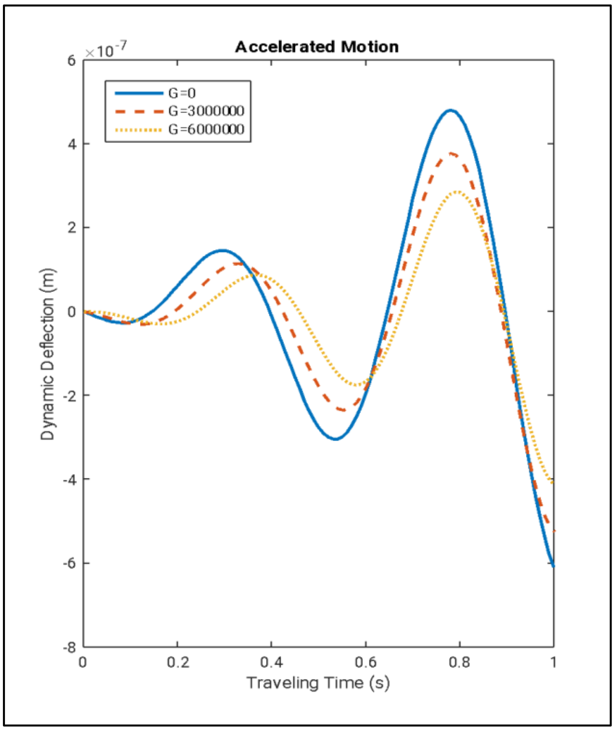

Fig. 6: Effects of shear modulus G on the dynamic deflection of a simply Supported non-uniform Rayleigh beam for moving Mass

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Effect of varying R_0

The figures labelled as 5 and 6 demonstrate the dynamic deflection curves sensitive to the rotatory inertial factor R_0. The non-uniform Rayleigh beam with higher rotatory inertia exhibits a smaller amplitude of vibration. The inertia of the beam is increased, and this makes it harder for the force to move and rotate the segments in the beam, resulting in less overall dynamic deflection of the beam.

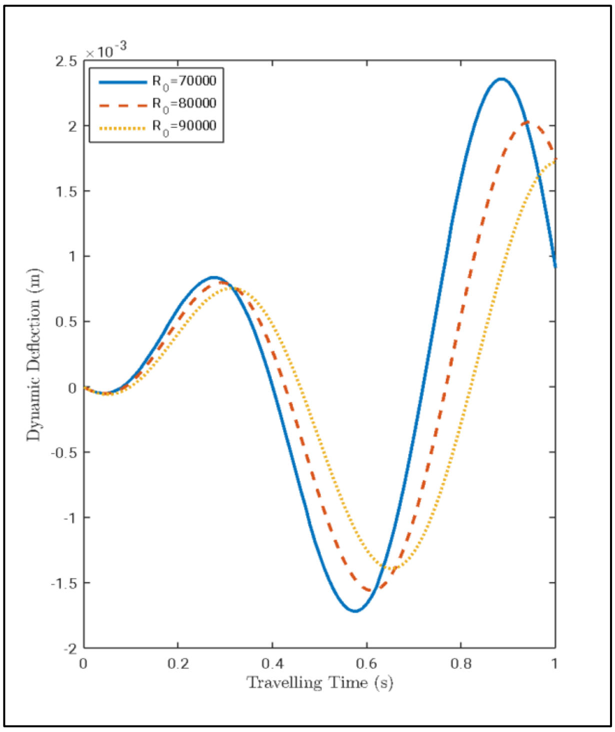

Fig. 7: Impact of rotatory inertia correction factor R_0 on the dynamic deflections of a simply Supported Rayleigh beam for moving force

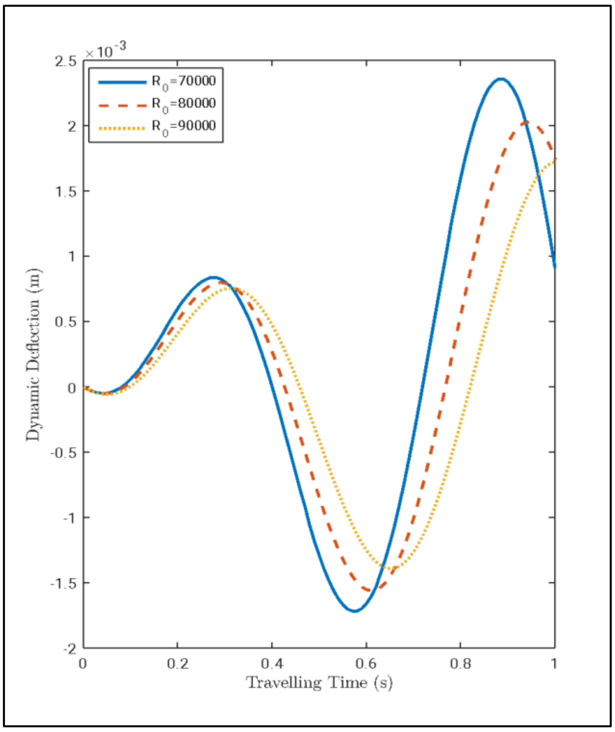

Fig. 8: Impact of rotatory inertia correction factor R_0 on the dynamic deflections of a simply Supported Rayleigh beam for moving mass

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Effect of varying *l*

Figures 9 and 10 depict the dynamic deflection of a simply supported non-uniform Rayleigh beam experiencing an accelerating force and mass with different span lengths. The figures illustrate that the deflection amplitude increases as the value l increases. When the length of the beam is increased, this automatically reduces the stiffness and makes it more susceptible to bending. The deflection will always increase, and it will do so significantly.

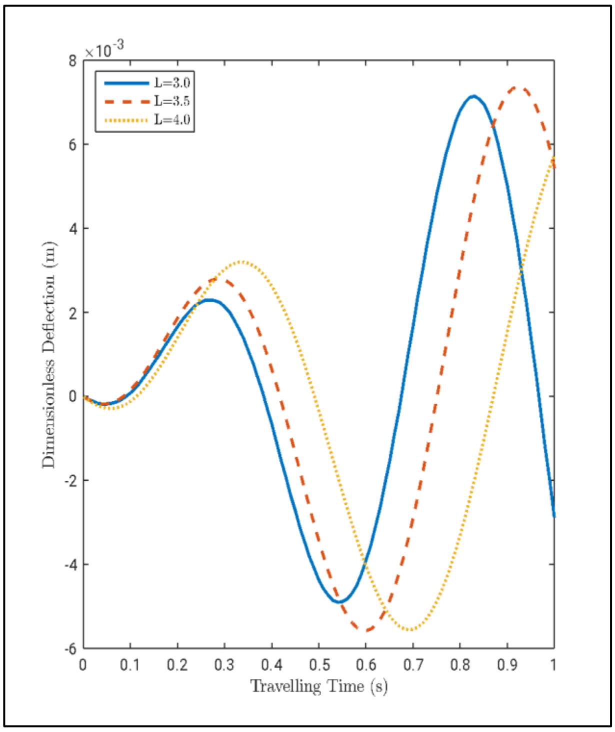

Fig. 9: Effects of span length L on the dynamic deflection of a simply Supported non-uniform Rayleigh beam for moving force

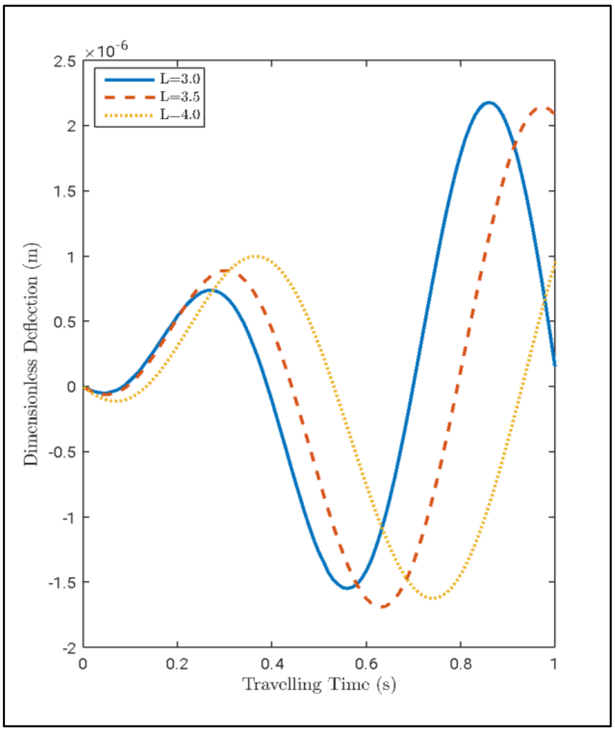

Fig. 10: Effects of span length L on the dynamic deflection of a simply Supported non-uniform Rayleigh beam for moving mass

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Effect of Comparing the Dynamic Responses of the Moving Force and the Moving Mass

The dynamic deflection of the simply supported non-uniform Rayleigh beam for moving force and moving mass systems is compared in 11. The figure demonstrates that the deflection is more remarkable for the moving force system than for the moving mass system.

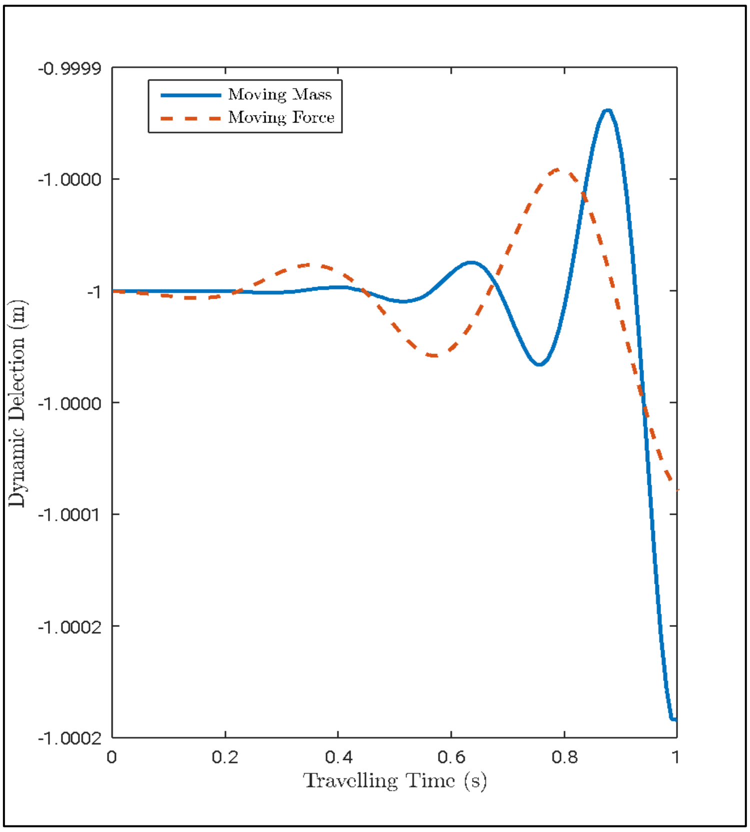

Fig. 11: Comparison of the dynamic deflection of a simply supported beam for the moving force and the moving mass

CONCLUSION

This study evaluates the exact analytical solution of a simply supported non-prismatic Rayleigh beam traversed by moving loads. It is established that the non-uniform Rayleigh beam model offers a more accurate representation of real-world structures by accounting for both translational and rotational inertia and by incorporating spatially varying material and geometric properties. This non-uniformity profoundly influences the beam's dynamic characteristics, including its natural frequencies and mode shapes, often leading to localised responses that uniform beam models cannot capture. The analysis of accelerating distributed masses highlights the critical importance of considering the time-varying nature of moving loads. Unlike constant velocity loads, accelerating masses introduce complex inertial forces and dynamic amplification effects that can significantly alter the structural response, making their accurate representation crucial for safety and performance. Real-world applications, from high-speed rail to industrial machinery, underscore the practical relevance of this loading

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scenario. Furthermore, bi-parametric elastic foundations, particularly the Winkler-Pasternak model, have demonstrated their superiority over simpler one-parameter models in capturing the realistic interaction between the beam and its supporting soil. By incorporating both normal and shear stiffness, these models provide a more accurate prediction of load distribution and beam deflection, which is vital for reliable foundation design. Based on these results, calculations involving the dynamic response induced by moving loads are obtained and expressed in graphical form. The research investigates how essential structural factors influence the magnitude of dynamic deflections in beam-load systems. It computes dynamic responses and determines how beam parameters affect the dynamic deflections of the beam. It's observed that when axial force N and subgrade moduli K maintain constant values, the rotatory inertia correction factor R_0 rises as the dynamic deflection of the beam decreases.

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