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ON CONSTRAINED PROGRAMMING PROBLEMS WITH SINGULAR DESIGNS VIA SUPER CONVERGENT LINE SERIES

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ABSTRACT: *The study extends the Super Convergent Line Series Algorithm to a case where the determinant information matrix of the design is zero by employing the Moore-Penrose inverse approach. The algorithm is tested using a numerical example on a constrained programming problem. The optimal solution obtained by the algorithm compares favorably with the one obtained by an existing Frank-Wolfe method and the value of the optimizer satisfies the given constraint equation.*

KEYWORDS: Algorithm, Direction Vector, Support Points, Optimizer, Information Matrix.

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INTRODUCTION

Super Convergent Line Series (SCLS) is a line search technique that seeks to find an optimizer of a response surface at the optimum of its variance function (Onukogu & Chigbu, 2002). SCLS is also a strong instrument for finding solutions to distinct optimization questions and is based on the concept of Super Convergence that has many points of withdrawal. Generally, it is known that when the design matrix is singular, there is a break in solving optimization problems using the SCLS algorithm. Several authors have employed Super Convergent Line Series to solve optimization problems. Chigbu and Ugbe (2002) used Super Convergent Line Series including partitioning of the response surfaces to obtain solutions of linear and quadratic programming problems. It was confirmed that the number of partitions, P, for which optimal solutions are reached is 2 and 4 for the linear and quadratic programming problems, respectively. Chigbu and Ukaegbu (2007) compared two approaches (precision matrices and mean square error) to obtain average information matrices while solving optimization problems.

Etukudo and Umoren (2008) extended the modified super convergent line series to solve quadratic programming problems, and it was shown that the Modified Super Convergent Lines Series (MSCLS) is simpler and preferable to the modified simplex method (MSM) since the optimizer could be reached in one iteration against the four iterations of modified simplex method (MSM) (Umoren & Etukudo, 2009). Umoren and Etukudo (2010) presented a Modified Super Convergent Line Series for solving unconstrained optimization problems which used the same basics of optimal experimental design and compared favorably with the gradient method.

Ugbe and Chigbu (2014) confirmed that the finest number of partitions is two (P=2) for Linear Programming Problems, four (P=4) for Quadratic Programming Problems, and eight (P=8) for Cubic Programming Problems, for non-overlapping partitioning of the response surface. Ugbe and Akpan (2014) compared the solution of boundary and interior support points for solving quadratic programming problems through super convergent line series. It was confirmed that the support points from the boundary of the response surface produced optimal solutions that compared favorably with the existing solutions of the examples used. Ugbe and Chigbu (2017) obtained the solutions for linear programming problems by the partitioning of the cuboidal response surface through the super convergent line series method; it was confirmed that the number of partitions, P, for which optimal solution is obtained, is two. Etukudo (2017) presented a new approach for solving unconstrained optimization problems with univariate quadratic surfaces. It was shown that an optimizer could be reached in just one iteration compared with existing search techniques.

Related studies on the aforementioned line search algorithm are: Andrei (2008) developed line search algorithms for solving large-scale unconstrained optimization problems such as quasi-Newton methods, truncated Newton and conjugate gradient. Chouzenoux, Moussaoui, and Idier (2009) used a line search algorithm based on the Majorize-Minimum principle; a tangent majorant function is built to approximate a scalar criterion containing a barrier function, which leads to a simple line search ensuring the convergence of several classical descent optimization strategies, including the most classical variants of non-linear conjugate gradient. Zhu, Ruan and Huang (2010) presented the fundamental ideas, concepts and theorems of basic line search algorithms for solving linear programming problems which can be regarded as an extension of the simplex method. The basic line search algorithm can be used to find an optimal solution

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with only one iteration. Gardeux, Chelouah, Siarry and Glover (2011) presented a performance of a one-dimensional search algorithm for solving general high-dimensional optimization problems which uses line search algorithms as subroutines. Umelo-Ibemere (2020) compared three starting points for the generation of D-optimal second-order designs. Na (2021) presented a global convergence of online optimization for nonlinear model predictive control. Na and Anitescu (2022) proposed a super convergence online optimization for model predictive control. Umelo-Ibemere and Chigbu (2023) compared three starting points for the simultaneous construction of D-optimal second-order designs.

However, none of the above authors have gone beyond using non-singular designs. Hence, this study extends the SCLS to the case where the design matrix is singular, thereby determining if there is a solution for a singular design matrix and if it compares favorably with other existing techniques. To achieve this, we employed the Moore-Penrose inverse. An illustrative example is presented in section 3 to demonstrate the efficacy of this study.

MATERIALS AND METHODS

Definitions and Notations

The super convergent line series (SCLS) is defined by Onukogu and Chigbu (2002) as:

$$\underline{X} = \overline{\underline{X}} \pm \rho d . \tag{1}$$

which is actually a line equation. Therefore, rewriting (1) in series, we have

$$X_{k+1} = \overline{\underline{X}}_k \pm \rho_k d_k \text{ , } k = 0,1,2,\ldots , \tag{2}$$

where X_{k+1} is the vector of the optimal values

$$\overline{\underline{X}}_k = \sum_{t=1}^{N} W_t^* X_t \text{ , } k = 0,1,2,\ldots \tag{3}$$

is the optimal starting point, where

$$W_t^* > 0; \ \sum_{t=1}^{N} W_t^* = 1 \text{, } \qquad W_t^* = \frac{a_t^{-1}}{\sum_{t=1}^{N} a_t^{-1}} \text{ , } \tag{4}$$

$$a_t = X_t^T X_t \text{, } \quad t = 1,2,\ldots,N \tag{5}$$

$\underline{dd}$ is the direction vector defined as $\underline{dd} = M^{-1}(\xi_N)Z, M^{-1}(\xi_N)Z,$ (6)

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where $\mathbf{Z} = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix}$ is an n-component vector of responses; $z_i z_i$= f($m_i m_i$) is the

ith row of the average information matrix $M(\xi_N M(\xi_N)$, where $M^{-1}(\xi_N) M^{-1}(\xi_N)$ is the inverse of the average information matrix.

ρ is the step length defined as $$\rho = \min\left|\frac{C_i \overline{\underline{X}}_0^* - b_i}{C_i d^*}\right|, \tag{7}$$

where $C_i C_i$ is the vector which represents the parameter of linear inequalities;

$\overline{\underline{X}}_k$ is the starting point and $b_i b_i$ is a scalar of the linear inequalities,

d^* is the normalized direction vector

$\xi_N \xi_N$is an N-point design measure whose support points may or may not have equal weights.

$\tilde{X}$ is the experimental space of the response surface that can be partitioned into segments such that every pair of support points in the group is a subset of $\tilde{X}$.

Note: The operator in (1) is positive "+" for a maximization case and negative "-" for a minimization case.

Super Convergent Line Series Algorithm for Solving Optimization Problem

The sequential steps for the algorithm is as follows by Onukogu and Chigbu (2002) is as follows:

Step 1: Select N support points such that $3m^* \leq N \leq 4m^* m^* \leq N \leq 4m^*$, where $2 \leq m^* \leq 3 \leq m^* \leq 3$ is the number of partitioned groups or segments desired.

Step 2: Compute the vectors $\overline{\underline{X}}_k$, d and obtain the step length ρ

Step 3: Make a move to the point, $X_{k+1} = \overline{\underline{X}}_k \pm \rho_k d_k$.

Step 4: Is $|f(\overline{\underline{X}}_{k+1})$- $f(\overline{\underline{X}}_k)| < \in ? < \in ?$

Yes: Stop,

No: Replace $\overline{\underline{X}}_k$ by $\overline{\underline{X}}_{k+1}$, obtain ρ and return to Step 3.

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Procedure of the Moore-Penrose Inverse

The Moore-Penrose inverse by Vera and Branko (2014) states that:

For the matrix P$\in C_r^{m*n} \in C_r^{m*n}$, the rank r$< \min\{m,n\} < \min\{m,n\}$, let there be given regular matrices B$\in C_r^{m*m} \in C_r^{m*m}$ and A$\in C_r^{n*n} \in C_r^{n*n}$ such that

BPA = $E_r\ E_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (8)

and let the square matrices B . B* and A*. A have the shape

B . B* $= \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ and A*. A $= \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}\begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$

with appropriate sub matrices $U_1 \in C^{r*r} U_1 \in C^{r*r}$, $U_2 \in C^{r*(m-r)} U_2 \in C^{r*(m-r)}$, $U_3 \in C^{(m-r)*r} U_3 \in C^{(m-r)*r}$, $U_4 \in C^{(m-r)*(m-r)} U_4 \in C^{(m-r)*(m-r)}$, $V_1 \in C^{r*r} V_1 \in C^{r*r}$, , $V_2 \in C^{r*(n-r)} V_2 \in C^{r*(n-r)}$, $V_3 \in C^{(n-r)*r} V_3 \in C^{(n-r)*r}$, $V_4 \in C^{(n-r)*(n-r)} V_4 \in C^{(n-r)*(n-r)}$. Matrices B . B*$\in C^{m*m} \in C^{m*m}$ and A*. A $\in C^{n*n} \in C^{n*n}$ are hermitian. Also, U_4 *and* $V_4 U_4$ *and* V_4 are invertible. The Moore-Penrose inverse is therefore

$$P^+P^+ = A \cdot \begin{bmatrix} I_r & -U_2U_4^{-1} \\ -V_4^{-1}V_3 & V_4^{-1}V_3U_2U_4^{-1} \end{bmatrix}\begin{bmatrix} I_r & -U_2U_4^{-1} \\ -V_4^{-1}V_3 & V_4^{-1}V_3U_2U_4^{-1} \end{bmatrix} . B \quad (9)$$

In the case of m = r, the matrix P is a matrix of full rank by rows; hence, submatrices U_2U_2 and U_4U_4 disappear by dimension. Therefore, the matrix P^+P^+ has the shape

$$P^+ = A \cdot \begin{bmatrix} I_r \\ -V_4^{-1}V_3 \end{bmatrix} P^+ = A \cdot \begin{bmatrix} I_r \\ -V_4^{-1}V_3 \end{bmatrix} . B \quad (10)$$

If n=r, the matrix P^+P^+ has the shape

$\mathrm{P^+ = A \cdot [I_r \quad -U_2U_4^{-1}]P^+ = A \cdot [I_r \quad -U_2U_4^{-1}]}$. B
(11)

If P is regular and square, say m = n = r, the Moore- Penrose inverse P^+P^+

of the matrix P is equal with $P^{-1}P^{-1}$; in that case, the matrix P^+P^+has the shape

$$P^+ = P^{-1} = A.\ B \quad (12)$$

In this case, $P_i^+P_i^+ = M_i^{-1}M_i^{-1} = \begin{bmatrix} y_{i11} & y_{i21} & y_{i31} \\ y_{i12} & y_{i22} & y_{i32} \\ y_{i13} & y_{i23} & y_{i33} \end{bmatrix}\begin{bmatrix} y_{i11} & y_{i21} & y_{i31} \\ y_{i12} & y_{i22} & y_{i32} \\ y_{i13} & y_{i23} & y_{i33} \end{bmatrix}$
(13)

To obtain (9), (10), (11) and (12), the following steps shall be applied:

Step 1: Obtain the rank of the matrices r.

Step 2: For a matrix P$\in C_r^{m*n} \in C_r^{m*n}$, make an expanded matrix $\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}$

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Step 3: Transform $\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}$ by elementary transformations on the columns and rows, in equivalent matrix: $\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix}\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix}$

Step 4: Verify that BPA = E_r E_r=$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, otherwise go to 3.2.

Step 5: Obtain the sub matrices U_2, U_4, V_3, V_4 and determine $-U_2U_4^{-1} - U_2U_4^{-1}$, $-V_4^{-1}V_3 - V_4^{-1}V_3$, ($V_4^{-1}V_3V_4^{-1}V_3$)($U_2U_4^{-1}U_2U_4^{-1}$) from the product B . B^* and A^*. A

Step 6: Obtain the Moore-Penrose inverse of P^+ for either (9), (10) or (11).

We shall apply the above steps for the different groups. And for the sake of clarity, we shall use P in place of M.

RESULTS AND DISCUSSIONS (ILLUSTRATIVE EXAMPLE)

An Illustrative Example: Constrained Programming Problem (Hillier & Lieberman, 1967, 6th ed. pp 623-624)

Maximize $Z = 3x_1 + 4x_2 - x_1^3 - x_2^2$

Subject to: $x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Step 1: We arbitrarily select N support points such that:

$$3m^* \le N \le 4m^* m^* \le N \le 4m^*$$

where $2 \le m^* \le 3 \le m^* \le 3$ is the number of segmented groups desired. Selecting

$m^* = 2 \quad m^* = 2$ we have $6 \le N \le 8 \le N \le 8$. Therefore, by arbitrarily choosing 6 support points such that it does not contravene the constraint equations, the first design matrix is:

$$X = \begin{bmatrix} 1 & 0 & 0.2 \\ 1 & 0.2 & 0.4 \\ 1 & 0.4 & 0.6 \\ 1 & 0.2 & 0 \\ 1 & 0.4 & 0.2 \\ 1 & 0.6 & 0.4 \end{bmatrix}$$

Step 2: Partitioning XX into 2 non overlapping groups such that the design matrix is singular, we have:

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$$X_1 = \begin{bmatrix} 1 & 0 & 0.2 \\ 1 & 0.2 & 0.4 \\ 1 & 0.4 & 0.6 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 1 & 0.2 & 0 \\ 1 & 0.4 & 0.2 \\ 1 & 0.6 & 0.4 \end{bmatrix}$$

The information matrices for the two groups respectively are:

$$M_1 = X_1^T X_1 = \begin{bmatrix} 3 & 0.6 & 1.2 \\ 0.6 & 0.2 & 0.32 \\ 1.2 & 0.32 & 0.56 \end{bmatrix}, \qquad M_2 = X_2^T X_2 = \begin{bmatrix} 3 & 1.2 & 0.6 \\ 1.2 & 0.56 & 0.32 \\ 0.6 & 0.32 & 0.2 \end{bmatrix}$$

Step 3: Compute the inverses of M_1 and M_2 using the Moore-Penrose inverse procedure.

Case 1: For

$$M_1 = \begin{bmatrix} 3 & 0.6 & 1.2 \\ 0.6 & 0.2 & 0.32 \\ 1.2 & 0.32 & 0.56 \end{bmatrix},$$

Step 3.1: The rank r of $M_1 = P = 2$

Step 3.2: We make an expanded matrix $\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0.6 & 1.2 & 1 & 0 & 0 \\ 0.6 & 0.2 & 0.32 & 0 & 1 & 0 \\ 1.2 & 0.32 & 0.56 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

Step 3.3: Transforming the expanded matrix by elementary row and column operation in equivalent matrix, we have that

$$\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{10}{3} & \frac{-5}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{-5}{3} \\ 1 & \frac{-3}{2} & \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{15}{2} & \frac{15}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-15}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{10}{3} & \frac{-5}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{-5}{3} \\ 1 & \frac{-3}{2} & \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{15}{2} & \frac{15}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-15}{2} & 0 & 0 & 0 \end{bmatrix}$$

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The regular matrices of

$$B=\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{10}{3} & \frac{-5}{3} \\ \frac{1}{3} & \frac{5}{3} & \frac{-5}{3} \end{bmatrix} and \;\; A=\begin{bmatrix} 1 & \frac{-3}{2} & \frac{3}{2} \\ 0 & \frac{15}{2} & \frac{15}{2} \\ 0 & 0 & \frac{-15}{2} \end{bmatrix}$$

Step 3.4: It is verified that

$$BPA=E_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. By making product of the matrices

$$B.B^*=\begin{bmatrix} \frac{1}{9} & 0 & \frac{1}{9} \\ 0 & \frac{125}{9} & \frac{25}{3} \\ \frac{1}{9} & \frac{25}{3} & \frac{17}{3} \end{bmatrix} \text{ and } \quad A^*.A=\begin{bmatrix} 1 & \frac{-3}{2} & \frac{3}{2} \\ \frac{-3}{2} & \frac{117}{2} & 54 \\ \frac{3}{2} & 54 & \frac{459}{4} \end{bmatrix}$$

Step 3.5: We obtain the sub matrices $U_2=\begin{bmatrix} \frac{1}{9} \\ \frac{25}{3} \end{bmatrix}$, $U_4=\begin{bmatrix} \frac{17}{3} \end{bmatrix}$, $V_3=\begin{bmatrix} \frac{3}{2} & 54 \end{bmatrix}$, $V_4=\begin{bmatrix} \frac{459}{4} \end{bmatrix}$ of which we can determine sub matrices $-U_2U_4^{-1}=\begin{bmatrix} \frac{-1}{51} \\ \frac{-25}{17} \end{bmatrix}$, $-V_4^{-1}V_3=\begin{bmatrix} \frac{-2}{153} & \frac{-24}{51} \end{bmatrix}$, $(V_4^{-1}V_3)(U_2U_4^{-1})=\begin{bmatrix} \frac{-5402}{7803} \end{bmatrix}$

Step 3.6: The Moore-Penrose inverse of M_1 by (9) is

$$M_1^{-1}=P^+=A\cdot\begin{bmatrix} 1 & 0 & \frac{-1}{51} \\ 0 & 1 & \frac{-25}{17} \\ \frac{-2}{153} & \frac{-24}{51} & \frac{-5402}{7803} \end{bmatrix}\cdot B=\begin{bmatrix} 1.4017 & -1.9784 & -1.6981 \\ -1.9784 & 3.5067 & 3.1110 \\ -1.6981 & 3.1110 & 2.7714 \end{bmatrix}$$

Case 2: For

$$M_2=\begin{bmatrix} 3 & 1.2 & 0.6 \\ 1.2 & 0.56 & 0.32 \\ 0.6 & 0.32 & 0.2 \end{bmatrix},$$

Step 3.1.1: The rank r of $M_2=P=2$

Step 3.2.2: We make an expanded matrix $\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix}=$

$$\begin{bmatrix} 3 & 1.2 & 0.6 & 1 & 0 & 0 \\ 1.2 & 0.56 & 0.32 & 0 & 1 & 0 \\ 0.6 & 0.32 & 0.2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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Step 3.3.3: Transforming the expanded matrix by elementary row and column operation in equivalent matrix, we have that

$$\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{-5}{3} & \frac{5}{3} \\ 1 & 3 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{-15}{2} & \frac{-15}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{15}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P & I_m \\ I_n & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} E_r & B \\ A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{-5}{3} & \frac{5}{3} \\ 1 & 3 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{-15}{2} & \frac{-15}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{15}{2} & 0 & 0 & 0 \end{bmatrix}$$

The regular matrices of $B = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{-5}{3} & 0 \\ \frac{1}{3} & \frac{-5}{3} & \frac{5}{3} \end{bmatrix}$ *and* $A = \begin{bmatrix} 1 & 3 & \frac{3}{2} \\ 0 & \frac{-15}{2} & \frac{-15}{2} \\ 0 & 0 & \frac{15}{2} \end{bmatrix}$

Step 3.4.4: It is verified that $BPA = E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. By making product of the matrices

$B.B^* = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{29}{9} & 3 \\ \frac{1}{9} & 3 & \frac{17}{3} \end{bmatrix}$ and $A^*.A = \begin{bmatrix} 1 & 3 & \frac{3}{2} \\ 3 & \frac{261}{4} & \frac{243}{4} \\ \frac{3}{2} & \frac{243}{4} & \frac{459}{4} \end{bmatrix}$

Step 3.5: We obtain the sub matrices $U_2 = \begin{bmatrix} \frac{1}{9} \\ 3 \end{bmatrix}$, $U_4 = \begin{bmatrix} \frac{17}{3} \end{bmatrix}$, $V_3 = \begin{bmatrix} \frac{3}{2} & \frac{243}{4} \end{bmatrix}$, $V_4 = \begin{bmatrix} \frac{459}{4} \end{bmatrix}$ of which we can determine sub matrices $-U_2U_4^{-1} = \begin{bmatrix} \frac{-1}{51} \\ \frac{-9}{17} \end{bmatrix}$, $-V_4^{-1}V_3 = \begin{bmatrix} \frac{-2}{153} & \frac{-9}{17} \end{bmatrix}$, $(V_4^{-1}V_3)(U_2U_4^{-1}) = \begin{bmatrix} \frac{2189}{7803} \end{bmatrix}$

Step 3.6: The Moore-Penrose inverse of M_2 by (9) is

$$M_2^{-1} = P^+ = A \cdot \begin{bmatrix} 1 & 0 & \frac{-1}{51} \\ 0 & 1 & \frac{-9}{17} \\ \frac{-2}{153} & \frac{-9}{17} & \frac{2189}{7803} \end{bmatrix} \cdot B = \begin{bmatrix} 1.4017 & -1.6981 & -1.9784 \\ -1.6981 & 2.7714 & 3.1110 \\ -1.9784 & 3.1110 & 3.5067 \end{bmatrix}$$

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Step 4: The matrices of the interaction effect of the variables for the groups are:

$$X_{11} = \begin{bmatrix} 0 & 0.04 \\ 0.008 & 0.16 \\ 0.064 & 0.36 \end{bmatrix}$$

$$X_{21} = \begin{bmatrix} 0.08 & 0 \\ 0.064 & 0.04 \\ 0.216 & 0.16 \end{bmatrix}.$$

and the vector of interaction parameters obtained from f(x) is:

$$g = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The interaction vectors for the groups are given by:

$$I_1 = M_1^{-1} X_1^T X_{11} g = \begin{bmatrix} 0.0758 \\ -0.4876 \\ -0.4724 \end{bmatrix}$$

$$I_2 = M_2^{-1} X_2^T X_{21} g = \begin{bmatrix} 0.0346 \\ -0.3665 \\ -0.3735 \end{bmatrix}$$

and the matrices of mean square error of the groups are:

$$\overline{M}_1 = M_1^{-1} + I_1 I_1^T = \begin{bmatrix} 1.4075 & -2.0154 & -1.7339 \\ -2.0154 & 3.7444 & 3.3413 \\ -1.7339 & 3.3413 & 2.9945 \end{bmatrix}$$

$$\overline{M}_2 = M_2^{-1} + I_2 I_2^T = \begin{bmatrix} 1.4029 & -1.7108 & -1.9913 \\ -1.7108 & 2.9057 & 3.2479 \\ -1.9913 & 3.2479 & 3.6461 \end{bmatrix}$$

Step 5: The matrices of coefficient of convex combination from $\overline{M}_1^{-1}$ and $\overline{M}_2^{-1}$ are:

$H_1 = diag\{0.5008 \quad 0.5631 \quad 0.4509\}$ and $H_2 = diag\{0.4992 \quad 0.4369 \quad 0.5491\}$

And by normalizing H_1 and H_2, such that $H_1^* H_1^{*T} + H_2^* H_2^{*T} = 1$, we have

$H_1^* = diag\{0.7082 \quad 0.7901 \quad 0.6346\}$

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$$H_2^* = diag\{0.7060 \quad 0.6130 \quad 0.7728\}$$

$$M(\xi_N) = H_1^* M_1 H_1^{*T} + H_2^* M_2 H_2^{*T} = \begin{bmatrix} 2.9999 & 0.8551 & 0.8667 \\ 0.8551 & 0.3353 & 0.3120 \\ 0.8667 & 0.3120 & 0.3450 \end{bmatrix}$$

Step 6: From f(x), obtain the response vector: $Z = \begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \end{bmatrix}$

$Z_0 = f(0.8551, \quad 0.8667) = 4.6757$, $Z_1 = f(0.3353, \quad 0.3120) = 2.1038$, $Z_2 = f(0.3120, \quad 0.3450) = 2.1883$,

It follows that $Z = \begin{bmatrix} 4.6757 \\ 2.1038 \\ 2.1883 \end{bmatrix}$. We obtain the direction vector:

$$d = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = M(\xi_N)^{-1} Z = \begin{bmatrix} -1.3634 \\ 4.2117 \\ 5.9592 \end{bmatrix}$$

This is normalized to give $d^* = \begin{bmatrix} d_1^* \\ d_2^* \end{bmatrix} = \begin{bmatrix} 0.5771 \\ 0.8166 \end{bmatrix}$

Step 7: The optimal starting point is $\underline{\overline{X}}_0^* = \sum_{t=1}^{N} w_t^* x_t$; $w_t^* = \frac{a_t^{-1}}{\sum_{t=1}^{N} a_t^{-1}}$, $a_t = x_t^T x_t$, t = 1, 2, …, N

Also, since $w_t^* = \frac{a_t^{-1}}{\sum_{t=1}^{N} a_t^{-1}}, t = 1.2,...,6$

$$\underline{\overline{X}}_0^* = \sum_{t=1}^{6} w_t^* x_t == \begin{bmatrix} 1.0000 \\ 0.2752 \\ 0.2752 \end{bmatrix}$$

Step 8: Step length is $\rho_1 = \min \left| \frac{C_i \underline{\overline{X}}_0^* - b_i}{C_i d} \right| = 0.3226$

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Step 9: $X_1^* = \underline{\overline{X}}_0^* + \rho d^* = \begin{bmatrix} 0.2752 \\ 0.2752 \end{bmatrix} + 0.3226 \begin{bmatrix} 0.5771 \\ 0.8166 \end{bmatrix} = \begin{bmatrix} 0.4614 \\ 0.5386 \end{bmatrix}$.

Therefore, $f(X_1^*) = 3..2275$ and $f(\underline{\overline{X}}_0^*) = 1.8298$

Since $[f(X_1^*) - f(\underline{\overline{X}}_0^*)] = 1.3977$ is large, we make a second move by replacing: $\underline{\overline{X}}_0^* = \begin{bmatrix} 0.2752 \\ 0.2752 \end{bmatrix}$ by $X_1^* = \begin{bmatrix} 0.4614 \\ 0.5386 \end{bmatrix}$

The new step length $\rho_2 = 0$. Therefore, an optimizer had been located at the first iteration and hence, $X_1^* = \begin{bmatrix} 0.4614 \\ 0.5386 \end{bmatrix}$, $Z \cong 3.2$ (in one iteration). This compares favorably with the existing solution obtained by Frank-Wolfe method (Hillier & Lieberman, 1967) as Z = 3.19, (in 3 iterations) and the x-values obtained satisfied the constraint equation.

Table 1: Summary of the Research Technique with Existing Technique

S/N	Technique	Objective function	No. of Iterations	Value of Optimizer	Value of Objective function
1	Frank-Wolf method	Max Z= $3x_1 + 4x_2 - x_1^3 - x_2^2$	3	(0.3333, 0.6667)	3.19
2	SCLSSD	Max Z= $3x_1 + 4x_2 - x_1^3 - x_2^2$	1	(0.46, 0.54)	3.15

CONCLUSION

Based on the summary presented in Table 1, the solution using Super Convergent Line Series with Singular Designs (SCLSSD) compares favorably with other existing techniques (i.e., Frank-Wolf method). From Table 1, we considered the constrained programming problem with objective function as *Max* $Z = 3x_1 + 4x_2 - x_1^3 - x_2^2$, the value of optimizer obtained was $(0.46 \quad 0.54)$ and the value of the objective function was 3.15 which compares favorably to the existing technique of the Frank-Wolfe method having $(0.33 \quad 0.67)$ as the values of the optimizer and 3.19 as the value of objective function. Hence, even with singular design matrices, the optimization of a response function is reached in just one move using Super Convergent Line Series (SCLS).

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REFERENCES

Andrei, N. Performance Profiles of Line Search Algorithm for Unconstrained Optimization: research institute for informatics. Centre for advanced modeling and optimization, ICI Technical report. 2008; 21: 12-18.

Chigbu, P. E. and Ugbe, T. A. On the Segmentation of the Response Surfaces for Super Convergent Line Series' Optimal Solutions of Constrained Linear and Quadratic Programming Problems. Global Journal of Mathematical Sciences. 2002; 1(1and 2): 27-34

Chigbu, P. E. and Ukaegbu, E. C. On the Precision and Mean Square Error Matrices Approaches in Obtaining the Average Informatics Matrices Via the Super Convergent Linear Series. Journal of Nigerian Statistical Association. 2007; 19: 14-18.

Chouzenoux, E., Moussaoui, E. S. and Idier, J. A Majorize-Minimize Line Search Algorithm for Barrier Function Optimization. 17th European SignalProcessing Conference, Glasgow. 2009: 1379-1383.

Etukudo, I. A. and Umoren, M. U. A modified super convergent live series algorithm for solving linear programming problems. Journal of Mathematical sciences. 2008; 19(1): 73-88.

Etukudo, I. A. Optimal designs for solving unconstrained optimization problems with univariate quadratic Surfaces. American Journal of Computational and Applied Mathematics. 2017; 7(2): 32-36.

Gardeux, V., Chelouah, R., Siarry, P. and Glover, F. EM323: A Line Search Algorithm for Solving High Dimensional Continuous Non-Linear Optimization Problems. Soft Computing. 2011; 15: 2275-2285.

Gupta, P. K. and Hira, D. S. Problems in Operation Research (Principles and Solution): 1ST Edition; 1991.

Hillier, F.S. and Lieberman, G.J. Introduction to operations Research. McGraw Hill Inc: 6thedition; 1967.

Na, S. Global Convergence of online Optimization for nonlinear model Predictive Control. Advances in Neural Information Processing Systems. 2021; 34:12441-12453.

Na, S. and Anitescu, M. Super convergence of online Optimization for Model Predictive Control. IEEE Transactions on Automatic Control. 2022; 68(3): 1383-1398.

Onukogu, I.B. Foundations of Optimal Exploration of Response surface. Nsukka, Nigeria: Ephrata Press; 1997.

Onukogu, I. B. and Chigbu, P. E.. Super Convergent Line Series in Optimal Design of Experiments and Mathematical Programming. Publishing Company, Nsukka, Nigeria: AP Express; 2002.

Ugbe, T.A. and Akpan, S. S. On Optimal Solution of Boundary and Interior Support Points for Solving Quadratic Programming Problems Through Super Convergent Line Series. Journal of Computation and Modelling.2014; 4(2): 79-92.

Ugbe, T.A. and Chigbu, P .E. On Non-Overlapping Segmentation of the Response Surfaces for Solving Constrained Programming Problems through Super Convergent Line Series. Communications in Statistics-Theory and Methods. 2014; 4: 306-320.

Ugbe, T. A. and Chigbu, P. E. On Non-overlapping Segmentation and Solutions of Three-Dimensional Linear Programming Problems through the Super Convergent Lines. American Journal of Operation Research,2017; 7(3): 225-238.

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Umelo-Ibemere, N. C. A note on Comparison of starting points for the generation of D-optimal second– order designs.African Journal of Applied Statistics, 2020;7(2): 1001-1008.

Umelo-Ibemere, N. C. and Chigbu, P. E. An Algorithm for the Simultaneous Construction of D-optimal experimental designs. Communication in Statistics-Simulation and Computation, 2023; 52(9): 4156-4173.

Umoren, M. U. and Etukudo, I. A. A Modified Super Convergent Line Series Algorithm for Solving Quadratic Programming Problems. Journal of Mathematical Sciences, International Centre for Advance Studies, West Bengal. 2009; 20(1): 55-66.

Umoren, M. U. and Etukudo, I. A. A Modified Super Convergent Line Series Algorithm fo Solving Unconstrained Optimization Problems. Journal of Modern Mathematics and Statistics. 2010; 4(4): 115-122.

Vera, M. J. and Branko, M. Block representations of generalized inverses of matrices. The Symposium on Mathematics and Applications.2014; Faculty of Mathematics, University of Belgarde, V(1)

Zhu, S., Ruan, G. and Huang, X. Some Fundamental Issues of Basic Line Search Algorithm for Linear Programming Problems. Optimization. 2010; 59: 1283-1295.