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CONVERGENCE RESULT FOR RANDOM SPLIT VARIATIONAL INEQUALITY PROBLEM USING A TWO-STEP ITERATIVE SCHEME IN HILBERT SPACES

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ABSTRACT: *The paper focuses on finding solution to stochastic split variational inequality problem, by extending the works done in classical functional analysis, to accommodate situations where there are perturbations in the system, as real life problems are, mostly, random in nature. Solution to the problem is sought through fixed point theory. A random two-step version of the hybrid steepest descent iterative scheme is used to obtain a fixed point solution to the problem. A numerical example is presented, to illustrate the workability of the result. A strong convergence result is also proven for the solution. This result extends, and generalizes some established results in literature on classical functional analysis.*

KEYWORDS: Stochastic, split feasibility, variational inequality, monotone, Hilbert spaces, operator.

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INTRODUCTION

Functional Analysis is one of the numerous branches of mathematical analysis, concerned with the study of vector spaces endowed with some kind of limit related structure and the linear functions defined on these spaces are suitably respecting these strctures. One of the components of Functional analysis is the Variational Inequality Problem (VIP). Researches have been done in solving VIP, which is one of the core problems in Functional Analysis, because it has been effective in solving problems in many areas, including mechanics, nonlinear programming, game theory and equilibrium problems; see (Ceng, Ansari, & Yao, 2008; Ceng, Teboulle, & Yao, 2010; Fukushima, 1986; Kinderlehrer & Stampacchia, 2000; Yang & Bell, 1997). In its simplest form, it is seen as a problem of having to find a point in a closed convex subset of a space, such that certain inequality constraint is met. In the deterministic form, the VIP is formulated as;

Find $x^* \in C$ such that $\langle Fx^*, x^* - x\rangle \leq 0; \forall\, x \in C$
(1.1)

where C is a closed convex subset of a real Hilbert space; F is a continuous function, and x^* is a point in C; see (Lohawech, Kaewcharoen, & Farajzadeh, 2021).

Also of great interest in classical Functional Analysis is the Split Feasibility Problem (SFP), which is applied in solving; inverse problems of intensity-modulated radiation therapy, image reconstruction and signal processing; see (Censor, Elfving, Kopf, & Brtfield, 2005; Byrne, 2002; Censor & Elfving, 1994). It is about finding a point in a closed convex subset of a metric space, such that, under a suitable transformation, gives a corresponding point, in another closed convex subset of a metric space. The SFP, as introduced by (Censor & Elfving, 1994), is formulated as;

Find $x^* \in C: Ax^* \in Q$ (1.2)

where $C \in H_1$ and $Q \in H_2$ are closed convex subsets of the Hilbert spaces $H_1$and H_2, respectively; A is a linear bounded self-adjoining mapping.

Circumstances arise where a solution can solve two or more problems; see (Yao, Cho, & Liou, 2011a; Yao, Cho, & Liou, 2011b and Cho, Argyros, & Petrot, 2010). Motivated by this, Jung (2016) developed a problem, whose solution is common to (1.1) and (1.2). The problem is formulated thus;

Find $x^* \in \Gamma$ such that $\langle Fx^*, x^* - x\rangle \leq 0; \forall\, x \in C$ (1.3)

where Γ is the feasible set of the SFP in (1.2)

Most of the problems in Functional analysis do not have anlytical solutions. This explains why most of the problems are solved using numerical methods, mostly, involving the use of iterative algorithms. Yamada (2001) developed an iterative algorithm, given as;

$$x_{n+1} = (I - t\mu F)Tx_n \quad (1.4)$$

where $t \in (0,1)$, $\mu \in (0,1)$, F is a continuous function, and T is a nonexpansive mapping. This was developed as a steepest descent type algorithm minimizing certain convex functions. A significant advantage of the hybrid steepest descent method is that it does not require the

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closed form expression for the metric projection onto the convex subset, C, denoted by P_C; instead, it requires a closed form expression of a nonexpansive mapping T, whose fixed point set is C.

In circumstances where the functions are random, as it is mostly the case in real life situations, the classical methods of solutions can no longer solve the problem. There is, then, the need to modify the problem to accommodate the perturbations in the system, and stochastic methods, applied in finding solutions to the problems.

The random version of (1.3) is formulated as

Find $x^*(\omega) \in \Gamma$ such that $\langle E[Fx^*(\omega)], x^*(\omega) - x(\omega)\rangle \leq 0; \forall\, x(\omega) \in C$ (1.5)

where $x^*(\omega)$ is a random point in C, ω is the random component of the variable x, F is an integrable function on some measurable space, and E is the integral or expected value of $F[x^*(\omega)]$, with respect to $\rho(\omega)$, a function of the random variable ω.

When F is purely deterministic (devoid of perturbation), (1.5) can be regarded as (1.3). Even so, F is not always fully observable and the knowledge of F is always scarce for reasons stated in (Udom & Nweke, 2021), (Udom, 2018), which are; (i) the probability distribution of $\rho(\omega)$ may be known but computation of the integral in (1.5) may involve multi-dimensional integration which is computationally difficult, if not impossible; (ii) the random function F is known but the distribution of $\rho(\omega)$ is not known, so that the information on $\rho(\omega)$ can only be obtained using historical data or by sampling; (iii) $E[Fx(\omega)]$ is not observable and it must be approximately evaluated through simulation procedure.

In this work, a common solution to the stochastic variational inequality and split feasibility problems, as developed in (1.5) will be sought, through random fixed point theory. A new random iterative scheme, which is a randon, two-step version of (1.4), will be used to obtain strong convergence to the unique solution of (1.5), in Hilbert spaces.

Definitions and Preliminaries

Definition 2.1 (Hilbert space)

A Hilbert space is a complete inner product space (complete in the metric defined by the inner product $\langle .,. \rangle$).

Definition 2.2 (Udom & Nweke, 2021).

Let (Ω, Σ, P) be a complete probability measure space and $\big(H, B(H)\big)$ be a measurable space, where H is a separable Hilbert space, $B(H)$ is a Borel sigma algebra of H, (Ω, Σ) is a measurable space (Σ-sigma algebra) and P is a probability measure on Σ; that is, a measure with total measure one. A mapping $X: \Omega \to H$ is called

i. H-valued random variable if Σ is $\big(\Sigma, B(H)\big)$- measurable,

ii. Strongly P-measurable if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ of P-simple functions converging to x, P-almost surely, except for some events with null probavility.

A mapping $T: \Omega \times H \to H$ is called a random operator if, for each $h \in H$, the mapping

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$T(.,h)\colon \Omega \to H$ is measurable;

Definition 2.3 (Udom & Nweke, 2021)

Let (Ω, Σ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space, H and $T\colon \Omega \times H \to H$ is a random operator. The random point $z(\omega)\colon \Omega \to H$ is called a fixed point of T if $T(z(\omega), \omega) = z(\omega)$. The set of all fixed points of T is denoted by $Fix(T)$;

Definition 2.4 (Udom, 2018)

Let (Ω, Σ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space, H. The random operator, $T\colon \Omega \times H \to H$, is;

i. κ-Lipschitz continuous if

$$\int_{\Omega} \|Tx(\omega) - Ty(\omega)\| dP(\omega) \leq \int_{\Omega} \kappa \|x(\omega) - y(\omega)\| dP(\omega), \quad \forall x(\omega), y(\omega) \in H$$

ii. Nonexpansive if (i) holds with $\kappa = 1$,

iii. η-strongly monotone if

$$\int_{\Omega} \langle Tx(\omega) - Ty(\omega), x(\omega) - y(\omega) \rangle dP(\omega) \geq \int_{\Omega} \eta \|x(\omega) - y(\omega)\| dP(\omega)$$

iv. Firlmy nonexpansive if

$$\int_{\Omega} \|Tx(\omega) - Ty(\omega)\|^2 dP(\omega) \leq \int_{\Omega} \langle Tx(\omega) - Ty(\omega), x(\omega) - y(\omega) \rangle dP(\omega)$$

v. α-averaged if $T = (1-\alpha)I + \alpha N$ for some fixed $\alpha \in (0,1)$ and nonexpansive mapping N.

Definition 2.5 (Udom & Ossai, 2019)

Let $\{x(\omega)\}_{n=0}^{\infty}$ be a random sequence, and let $x(\omega)$ be a random point. The sequence $\{x(\omega)\}_{n=0}^{\infty}$ is said to converge in quadratic mean to $x(\omega)$ if $\lim_{n\to\infty} E\|x_n(\omega) - x(\omega)\|^2 = 0$

Definition 2.6 (Xu, 2010)

Let (Ω, Σ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space. The metric projection from H onto C, denoted by Π_C is defined in such a way that, for each $x(\omega) \in H$, $\Pi_C(x(\omega))$ is the unique point in C with the properties;

i. $\|x(\omega) - \Pi_C(x(\omega))\| = \min\{\|x(\omega) - y(\omega)\| \colon y(\omega) \in C\}$

ii. Given $z(\omega) \in C$, then $z(\omega) = \Pi_C(x(\omega))$ iff

$$\langle x(\omega) - z(\omega), y(\omega) - z(\omega) \rangle \leq 0; \forall y(\omega) \in C$$

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Lemma 2.1 (Buong, 2017)

Let H be a real Hilbert space and let $F: H \to H$ be η-strongly monotone and κ-Lipschitz continuous. The mapping $I - t\mu F$, for each $\mu = \left(0, \frac{2\eta}{\kappa^2}\right)$, is contractive, with constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Lemma 2.2 (Lohawech *et al*., 2021)

Assume that H_1 and H_2 are Hilbert spaces. Let $A: H_1 \to H_2$, be a linear bounded mapping, such that $A \neq 0$ and let $G: H_2 \to H_2$, be a nonexpansive mapping. Then, for $0 < \gamma < \frac{1}{\|A\|^2}$,

$I - \gamma A^*(I - G)A$ is $\gamma\|A\|^2$-averaged.

Lemma 2.3 (Buong & Duong, 2011)

Let $\{x(\omega)\}_{n=0}^{\infty}$ and $\{z(\omega)\}_{n=0}^{\infty}$ be bounded sequences in a Banach space E, such that for $\beta_n \in [0,1]\ \forall n \geq 0$, $x_{n+1}(\omega) = (1 - \beta_n)x_n(\omega) + \beta_n z_n(\omega)$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\limsup_{n\to\infty}(E\|z_{n+1}(\omega) - z_n(\omega)\| - E\|x_{n+1}(\omega) - x_n(\omega)\|) \leq 0$, then

$\lim_{n\to\infty} E\|x_n(\omega) - z_n(\omega)\| = 0$

Lemma 2.4 (Zhou & Wang, 2013)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonegative real numbers, satisfying the following conditions;

$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n c_n$, where $(\gamma_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are sequences of real numbers such that

$\gamma_n \in [0,1]$, $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} c_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$

Lemma 2.5 (Cegielski, 2012)

Let $T: C \times \Omega \to H$ be nonexpansive and $y(\omega) \in C$ be a weak convergence point of a sequence $\{x(\omega)\}_{n=0}^{\infty}$. If $\lim_{n\to\infty} E\|Tx_n(\omega) - x_n(\omega)\| = 0$, then $y(\omega) \in Fix(T)$. This is reffered to as the demi-closedness.

Theorem 2.1 (Cegielski, 2012)

Let X be a complete metric space, and a random operator $T: X \to X$ be a contraction. Then, T has exactly one fixed point $x^*(\omega) \in X$. The orbit $\{T^n x_n(\omega)\}_{n=0}^{\infty}$ converges to $x^*(\omega)$ with a rate of geometric progression. This theorem is known as the Banach's Fixed-Point Theorem.

Suppose that $x^*(\omega) \in \Gamma : Ax^*(\omega) \in Q$, where C and Q are closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and Π_C and Π_Q are the metric projections on C and Q,

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respectively. It implies that $(I - \Pi_Q)Ax^*(\omega) = 0$, which then implies that $\gamma A^*(I - \Pi_Q)Ax^*(\omega) = 0, \gamma \in (0,1)$, hence the fixed point equation

$$(I - \gamma A^*(I - \Pi_Q)A)x^*(\omega) = x^*(\omega) \quad (2.1)$$

For $x^*(\omega)$ to be in C, it is imperative to consider the fixed-point equation

$$\Pi_C(I - \gamma A^*(I - \Pi_Q)A)x^*(\omega) = x^*(\omega) \quad (2.2)$$

Proposition 2.1

Given $x^*(\omega) \in C$. Then, $x^*(\omega)$ solves the random form of (1.2), developed as

$$\text{Find } x^*(\omega) \in C: Ax^*(\omega) \in Q, \quad (2.3)$$

If and only if $x^*(\omega)$ solves the fixed-point equation

$$x^*(\omega) = Tx^*(\omega); \quad (2.4)$$

where $T = \Pi_C(I - \gamma A^*(I - \Pi_Q)A)$

The proof is contained in (George, Udom, Iseh & Usoro, 2025).

These definitions and lemmas will be used in the sequel, to prove the result.

MAIN RESULT

Theorem 3.1

Let H_1 and H_2 be two real Hilbert spaces, and C and Q be two closed convex subsets in H_1 and H_2, respectively. Let A be a self-adjoining linear bounded mapping from H_1 into H_2. Let F be a strongly monotone and Lipschitz continuous mapping on H_1 with parameters η and κ, respectively. Assume that $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ is a fixed number. Let $\{x_n(\omega)\}_{n=0}^{\infty}$ be a random sequence of σ-algebra, adapted to the filtration Σ defined by

$$\begin{cases} y_n(\omega) = (I - t_n\mu F)T^n x_n(\omega) \\ x_{n+1}(\omega) = (I - t_n\mu F)T^n y_n(\omega), \ \forall n \geq 1 \end{cases} \quad (3.1)$$

where $\alpha_n \in (0,1)$; $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and t_n satisfies the conditions; (i) $t_n \in (0,1)$ and (ii) $\lim_{n\to\infty} t_n = 0$. Then, the random sequence, generated by (3.1), converges in quadratic mean to the unique random fixed point of T, which is the solution of (1.5).

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Proof;

(3.1) can be rewritten as;

$$x_{n+1}(\omega) \;=\; (I - t_n\mu F)T^n\tilde{T}^n x_n(\omega), \tag{3.2}$$

where $\tilde{T} := (I - t_n\mu F)T$ and $T := \Pi_C\big(I - \gamma A^*\big(I - \Pi_Q\big)A\big)$.

T is averaged; and so is $\tilde{T}$. Let $T_i : i = 1, 2$ be α^i-averaged; then, T_1T_2 is α-averaged, where $\alpha = \alpha^1 + \alpha^2 - \alpha^1\alpha^2$. Consequently, from definition 2.4, there exists a nonexpansive mapping, W, such that;

$$T\tilde{T} = (1 - \alpha)I + \alpha W \tag{3.3}$$

To show that $\{x_n(\omega)\}_{n=0}^{\infty}$ is bounded, let $\ddot{x}(\omega) \in C$, so that

$$E\|x_{n+1}(\omega) - \ddot{x}(\omega)\| = \int_\Omega \|x_{n+1}(\omega) - \ddot{x}(\omega)\| dP(\omega)$$

$$= \int_\Omega \big\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - \ddot{x}(\omega)\big\| dP(\omega)$$

$$= \int_\Omega \big\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)\ddot{x}(\omega) + (I - t_n\mu F)\ddot{x}(\omega) - \ddot{x}(\omega)\big\| dP(\omega)$$

$$= \int_\Omega \big\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)\ddot{x}(\omega) - t_n\mu F\ddot{x}(\omega)\big\| dP(\omega)$$

$$\leq \int_\Omega \big\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)\ddot{x}(\omega)\big\| dP(\omega) + \int_\Omega \|t_n\mu F\ddot{x}(\omega)\| dP(\omega)$$

$$\leq (1 - \tau t_n) \int_\Omega \big\|T^n\tilde{T}^n x_n(\omega) - \ddot{x}(\omega)\big\| dP(\omega) + t_n\mu \int_\Omega \|F\ddot{x}(\omega)\| dP(\omega)$$

$$\leq (1 - \tau t_n) \int_\Omega \|x_n(\omega) - \ddot{x}(\omega)\| dP(\omega) + \tau\, \frac{\mu t_n}{\tau} \int_\Omega \|F\ddot{x}(\omega)\| dP(\omega)$$

Let $M_{\ddot{x}(\omega)} := \max\Big\{ \int_\Omega \|x_0(\omega) - \ddot{x}(\omega)\|\, dP(\omega), \frac{\mu t_n}{\tau} \int_\Omega \|F\ddot{x}(\omega)\| dP(\omega)\Big\}$ be a fixed constant. Then $\int_\Omega \|x_0(\omega) - \ddot{x}(\omega)\|\, dP(\omega) \leq M_{\ddot{x}(\omega)}$; which implies that $\int_\Omega \|x_n(\omega) - \ddot{x}(\omega)\|\, dP(\omega) \leq M_{\ddot{x}(\omega)}, \forall n \geq 0$

So that, $E\|x_{n+1}(\omega) - \ddot{x}(\omega)\| \leq (1 - \tau t_n)M_{\ddot{x}(\omega)} + \tau t_n M_{\ddot{x}(\omega)} = M_{\ddot{x}(\omega)}$; that is

$$E\|x_{n+1}(\omega) - \ddot{x}(\omega)\| \leq M_{\ddot{x}(\omega)} \tag{3.4}$$

This shows that $\{x_n(\omega)\}_{n=0}^{\infty}$ is bounded. Also $\{T^n\tilde{T}^n x_n(\omega)\}_{n=0}^{\infty}$ and $\{F[T^n\tilde{T}^n x_n(\omega)]\}_{n=0}^{\infty}$ are bounded. This implies that

$$\lim_{n\to\infty} \int_\Omega \big\|x_{n+1}(\omega) - T^n\tilde{T}^n x_n(\omega)\big\| dP(\omega) = 0 \tag{3.5}$$

Next, it will be shown that $\lim_{n\to\infty} E\|x_{n+1}(\omega) - x_n(\omega)\| = 0$. This is to show to show the existence of a fixed point solution for the algorithm in (3.1)

From (3.3), $x_{n+1}(\omega)$ can be written as;

$$x_{n+1}(\omega) = (I - t_n\mu F)[(1 - \alpha_n)I + \alpha_n W_n]x_n(\omega)$$

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$$= (1-\alpha_n)x_n(\omega) - t_n\mu(1-\alpha_n)Fx_n(\omega) + \alpha_n W_n x_n(\omega) - \alpha_n t_n \mu F[W_n x_n(\omega)]$$

$$= (1-\alpha_n)x_n(\omega) + \alpha_n W_n x_n(\omega) - t_n\mu Fx_n(\omega) + \alpha_n t_n \mu Fx_n(\omega) - \alpha_n t_n \mu F[W_n x_n(\omega)]$$

$$= (1-\alpha_n)x_n(\omega) + \alpha_n W_n x_n(\omega) - (1-\alpha_n)t_n\mu Fx_n(\omega) - \alpha_n t_n \mu F[W_n x_n(\omega)]$$

$$= (1-\alpha_n)x_n(\omega) + \alpha_n W_n x_n(\omega) - \{(1-\alpha_n)I + \alpha_n W_n\}t_n\mu Fx_n(\omega)$$

$$= (1-\alpha_n)x_n(\omega) + \alpha_n W_n x_n(\omega) - t_n\mu F[T^n\tilde{T}^n x_n(\omega)]$$

$$= (1-\alpha_n)x_n(\omega) + \alpha_n \left\{W_n x_n(\omega) - \frac{t_n\mu}{\alpha_n}F[T^n\tilde{T}^n x_n(\omega)]\right\}$$

$$x_{n+1}(\omega) = (1-\alpha_n)x_n(\omega) + \alpha_n y_n(\omega) \tag{3.6}$$

where $y_n(\omega) = W_n x_n(\omega) - \frac{t_n\mu}{\alpha_n}F[T^n\tilde{T}^n x_n(\omega)]$

$$E\|y_{n+1}(\omega) - y_n(\omega)\| = \int_\Omega \|y_{n+1}(\omega) - y_n(\omega)\| dP(\omega)$$

$$= \int_\Omega \left\| W_{n+1}x_{n+1}(\omega) - \frac{t_{n+1}\mu}{\alpha_{n+1}}F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)] - W_n x_n(\omega) + \frac{t_n\mu}{\alpha_n}F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$= \int_\Omega \left\| W_{n+1}x_{n+1}(\omega) - W_n x_n(\omega) - \frac{t_{n+1}\mu}{\alpha_{n+1}}F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)] + \frac{t_n\mu}{\alpha_n}F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$\leq \int_\Omega \|W_{n+1}x_{n+1}(\omega) - W_n x_n(\omega)\| dP(\omega) + \frac{t_{n+1}\mu}{\alpha_{n+1}}\int_\Omega \left\|F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)]\right\| dP(\omega) + \frac{t_n\mu}{\alpha_n}\int_\Omega \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$= \int_\Omega \|W_{n+1}x_{n+1}(\omega) - W_{n+1}x_n(\omega) + W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega) + \frac{t_{n+1}\mu}{\alpha_{n+1}}\int_\Omega \left\|F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)]\right\| dP(\omega) + \frac{t_n\mu}{\alpha_n}\int_\Omega \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$\leq \int_\Omega \|W_{n+1}x_{n+1}(\omega) - W_{n+1}x_n(\omega)\| dP(\omega) + \int_\Omega \|W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega) + \frac{t_{n+1}\mu}{\alpha_{n+1}}\int_\Omega \left\|F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)]\right\| dP(\omega) + \frac{t_n\mu}{\alpha_n}\int_\Omega \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$\leq \int_\Omega \|x_{n+1}(\omega) - x_n(\omega)\| dP(\omega) + \int_\Omega \|W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega) + \frac{t_{n+1}\mu}{\alpha_{n+1}}\int_\Omega \left\|F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)]\right\| dP(\omega) + \frac{t_n\mu}{\alpha_n}\int_\Omega \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$$

$$\int_\Omega \|y_{n+1}(\omega) - y_n(\omega)\| dP(\omega) - \int_\Omega \|x_{n+1}(\omega) - x_n(\omega)\| dP(\omega) \leq$$

$$\int_\Omega \|W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega) + \frac{t_{n+1}\mu}{\alpha_{n+1}}\int_\Omega \left\|F[T^{n+1}\tilde{T}^{n+1}x_{n+1}(\omega)]\right\| dP(\omega) + \frac{t_n\mu}{\alpha_n}\int_\Omega \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega) \tag{3.7}$$

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But from (3.3),

$$\int_\Omega \|W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega)$$

$$= \int_\Omega \left\| \left\{ \frac{T^{n+1}\tilde{T}^{n+1}x_n(\omega) - (1-\alpha_{n+1})x_n(\omega)}{\alpha_{n+1}} \right\} - \left\{ \frac{T^n\tilde{T}^n x_n(\omega) - (1-\alpha_n)x_n(\omega)}{\alpha_n} \right\} \right\| dP(\omega)$$

$$= \int_\Omega \left\| \frac{T^{n+1}\tilde{T}^{n+1}x_n(\omega)}{\alpha_{n+1}} - \frac{x_n(\omega)}{\alpha_{n+1}} - \frac{T^n\tilde{T}^n x_n(\omega)}{\alpha_n} + \frac{x_n(\omega)}{\alpha_n} + x_n(\omega) - x_n(\omega) \right\| dP(\omega)$$

$$= \int_\Omega \left\| \frac{T^{n+1}\tilde{T}^{n+1}x_n(\omega)}{\alpha_{n+1}} - \frac{T^{n+1}\tilde{T}^{n+1}x_n(\omega)}{\alpha_n} + \frac{T^{n+1}\tilde{T}^{n+1}x_n(\omega)}{\alpha_n} - \frac{T^n\tilde{T}^n x_n(\omega)}{\alpha_n} + \frac{x_n(\omega)}{\alpha_n} - \frac{x_n(\omega)}{\alpha_{n+1}} \right\| dP(\omega)$$

$$= \int_\Omega \left\| \begin{array}{c} \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) T^{n+1}\tilde{T}^{n+1}x_n(\omega) + \frac{1}{\alpha_n}\left(T^{n+1}\tilde{T}^{n+1}x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right) + \\ \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) x_n(\omega) \end{array} \right\| dP(\omega)$$

$$\leq \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) \right\| dP(\omega) + \frac{1}{\alpha_n} \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) - T^n\tilde{T}^n x_n(\omega) \right\| dP(\omega) + \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) \int_\Omega \|x_n(\omega)\| dP(\omega)$$

$$= \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) \left\{ \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) \right\| dP(\omega) + \int_\Omega \|x_n(\omega)\| dP(\omega) \right\} +$$

$$\frac{1}{\alpha_n} \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) - T^n\tilde{T}^n x_n(\omega) \right\| dP(\omega)$$

Let $M_* \geq \sup_{n\geq 0} \left\{ \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) \right\| dP(\omega) + \int_\Omega \|x_n(\omega)\| dP(\omega) \right\}$ be a fixed constant, then

$$\int_\Omega \|W_{n+1}x_n(\omega) - W_n x_n(\omega)\| dP(\omega) \leq \left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) M_* +$$

$$\frac{1}{\alpha'} \int_\Omega \left\| T^{n+1}\tilde{T}^{n+1}x_n(\omega) - T^n\tilde{T}^n x_n(\omega) \right\| dP(\omega) \qquad (3.8)$$

where $0 < \alpha' \leq \alpha_n \leq \alpha'' < 1$

From definition 2.4

$$\int_\Omega \|T^{n+1}x_n(\omega) - T^n x_n(\omega)\| dP(\omega)$$

$$= \int_\Omega \|\{(1-\alpha_{n+1}^1)I + \alpha_{n+1}^1 T\}x_n(\omega) - \{(1-\alpha_n^1)I + \alpha_n^1 T\}x_n(\omega)\| dP(\omega)$$

$$= \int_\Omega \|x_n(\omega) - \alpha_{n+1}^1 x_n(\omega) + \alpha_{n+1}^1 T x_n(\omega) - x_n(\omega) + \alpha_n^1 x_n(\omega) - \alpha_n^1 T x_n(\omega)\| dP(\omega)$$

$$= \int_\Omega \|\alpha_{n+1}^1 [T x_n(\omega) - x_n(\omega)] - \alpha_n^1 [T x_n(\omega) - x_n(\omega)]\| dP(\omega)$$

$$= (\alpha_{n+1}^1 - \alpha_n^1) \int_\Omega \|T x_n(\omega) - x_n(\omega)\| dP(\omega)$$

$$\leq (\alpha_{n+1}^1 - \alpha_n^1) \int_\Omega \|T x_n(\omega)\| dP(\omega) + (\alpha_{n+1}^1 - \alpha_n^1) \int_\Omega \|x_n(\omega)\| dP(\omega)$$

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$$= (\alpha_{n+1}^1 - \alpha_n^1)\{\int_\Omega \|Tx_n(\omega)\| dP(\omega) + \int_\Omega \|x_n(\omega)\| dP(\omega)\}$$

$$\int_\Omega \|T^{n+1}x_n(\omega) - T^n x_n(\omega)\| \leq |\alpha_{n+1}^1 - \alpha_n^1| M_\Delta \quad (3.9)$$

where $M_\Delta \geq \sup_{n\geq 0}\{\int_\Omega \|Tx_n(\omega)\| dP(\omega) + \int_\Omega \|x_n(\omega)\| dP(\omega)\}$ is a fixed constant.

Similarly, $\int_\Omega \|\tilde{T}^{n+1}(T^n x_n(\omega)) - \tilde{T}^n(T^n x_n(\omega)\| dP(\omega) \leq |\alpha_{n+1}^2 - \alpha_n^2| M_\nabla \quad (3.10)$

where $M_\nabla \geq \sup_{n\geq 0}\{\int_\Omega \|\tilde{T}^{n+1}(Tx_n(\omega))\| dP(\omega) + \int_\Omega \|T^n x_n(\omega)\| dP(\omega)\}$ is a fixed constant

Combining (3.7) to (3.10) yields

$$\int_\Omega \|y_{n+1}(\omega) - y_n(\omega)\| dP(\omega) - \int_\Omega \|x_{n+1}(\omega) - x_n(\omega)\| dP(\omega) \leq$$

$$\left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right) M_* + |\alpha_{n+1}^1 - \alpha_n^1| M_\Delta + |\alpha_{n+1}^2 - \alpha_n^2| M_\nabla + \frac{t_{n+1}\mu}{\alpha_{n+1}} \int_\Omega \|F[T^{n+1}x_{n+1}(\omega)]\| dP(\omega) + \frac{t_n \mu}{\alpha_n} \int_\Omega \|F[T^n x_n(\omega)]\| dP(\omega) \quad (3.11)$$

Taking the limits of both sides of (3.11) gives

$\lim_{n\to\infty} \sup\left(\int_\Omega \|y_{n+1}(\omega) - y_n(\omega)\| dP(\omega) - \int_\Omega \|x_{n+1}(\omega) - x_n(\omega)\| dP(\omega)\right) \leq 0$, since $\lim_{n\to\infty} |\alpha_{n+1}^i - \alpha_n^i| = 0$ and $\lim_{n\to\infty} t_n = 0$

Therefore, by lemma 2.3

$$\lim_{n\to\infty} \sup \int_\Omega \|x_n(\omega) - y_n(\omega)\| dP(\omega) = 0$$

From (3.6)

$x_{n+1}(\omega) - x_n(\omega) = \alpha_n y_n(\omega) - \alpha_n x_n(\omega)$, so that

$\lim_{n\to\infty} \int_\Omega \|x_{n+1}(\omega) - x_n(\omega)\| dP(\omega) = 0$, implies that

$$\lim_{n\to\infty} \int_\Omega \|x_n(\omega) - T^n \tilde{T}^n x_n(\omega)\| dP(\omega) = 0. \quad (3.12)$$

Next, it is shown that $\lim_{n\to\infty} \sup\langle \int_\Omega (F[x^*(\omega)]) dP(\omega), x^*(\omega) - x_n(\omega)\rangle \leq 0$

Let $x^*(\omega) \in H$ be the unique solution of (1.5). Assume that there exists a random subsequence $\{x_{nj}(\omega)\}_{j=1}^\infty$ of $\{x_n(\omega)\}_{n=0}^\infty$, such that;

$$\lim_{n\to\infty} \sup \langle \int_\Omega (F[x^*(\omega)]) dP(\omega), x^*(\omega) - T^n \tilde{T}^n x_n(\omega)\rangle$$

$$= \lim_{n\to\infty} \sup \langle \int_\Omega (F[x^*(\omega)]) dP(\omega), x^*(\omega) - x_n(\omega)\rangle$$

$$= \lim_{j\to\infty} \sup \langle \int_\Omega (F[x^*(\omega)]) dP(\omega), x^*(\omega) - x_{nj}(\omega)\rangle$$

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Since $\{x_n(\omega)\}_{n=0}^{\infty}$ is bounded, by the demi-closedness principle, it can be assumed that $\{x_{nj}(\omega)\}_{j=1}^{\infty}$ converges weakly to $\dot{x}(\omega)$ as $j \to \infty$, without loss of generality; so that

$$\limsup_{n\to\infty} \langle \int_{\Omega}(F[x^*(\omega)])dP(\omega), x^*(\omega) - x_n(\omega)\rangle$$

$$= \langle \int_{\Omega}(F[x^*(\omega)])dP(\omega), x^*(\omega) - \dot{x}(\omega)\rangle.$$

By the claim that $\hat{x}(\omega) \in C$, $\langle \int_{\Omega}(F[x^*(\omega)])dP(\omega), x^*(\omega) - \dot{x}(\omega)\rangle \leq 0$.

Assume that $\alpha_{nj} \to \alpha_{\infty}$ (as $n \to \infty$); $0 < \alpha' \leq \alpha_{\infty} \leq \alpha'' < 1$.

Let, $T_i^{\infty} := (1-\alpha_n)I + \alpha_n T_i$. Then $Fix(T^{\infty}) = Fix(T)$ and

$$\limsup_{j\to\infty,\ x\in R} \int_{\Omega}\left\|T_i^{nj}x(\omega) - T_i^{\infty}x(\omega)\right\|dP(\omega) = 0 \qquad (3.13)$$

where R is an arbitrary bounded subset including $\{x_n(\omega)\}_{n=0}^{\infty}$.

Since $Fix(T^{\infty}) = Fix(T) \neq \emptyset$, T^{∞} is α_{∞}-averaged.

By (3.12) and (3.13), it is obtained that

$\lim_{j\to\infty} \int_{\Omega}\left\|x_{nj}(\omega) - T^{\infty}\tilde{T}^{\infty}x_{nj}(\omega)\right\|dP(\omega) = 0, \Rightarrow \dot{x}(\omega) = T^{\infty}\dot{x}(\omega) \in C$, as required.

Finally, the convergence in mean square to the unique fixed point is proven;

$$E\|x_{n+1}(\omega) - x^*(\omega)\|^2 = \int_{\Omega}\|x_{n+1}(\omega) - x^*(\omega)\|^2 dP(\omega)$$

$$= \int_{\Omega}\left\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - x^*(\omega)\right\|^2 dP(\omega)$$

$$= \int_{\Omega}\left\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)x^*(\omega) + (I - t_n\mu F)x^*(\omega) - x^*(\omega)\right\|^2 dP(\omega)$$

$$= \int_{\Omega}\left\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)x^*(\omega) - t_n\mu F x^*(\omega)\right\|^2 dP(\omega)$$

$$= \int_{\Omega}\langle (I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)x^*(\omega) - t_n\mu F x^*(\omega), (I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)x^*(\omega) - t_n\mu F x^*(\omega)\rangle dP(\omega)$$

$$= \int_{\Omega}\{\left\|(I - t_n\mu F)T^n\tilde{T}^n x_n(\omega) - (I - t_n\mu F)x^*(\omega)\right\|^2 + t_n^2\mu^2\|Fx^*(\omega)\|^2$$

$$+2\mu t_n\langle Fx^*(\omega), (I - t_n\mu F)x^*(\omega) - (I - t_n\mu F)T^n\tilde{T}^n x_n(\omega)\rangle\} dP(\omega)$$

$$\leq \int_{\Omega}\{(1 - \tau t_n)\|x_n(\omega) - x^*(\omega)\|^2 + 2\mu t_n\langle Fx^*(\omega), x^*(\omega) - T^n\tilde{T}^n x_n(\omega)\rangle$$

$$+2t_n^2\mu^2\langle Fx^*(\omega), F[T^n\tilde{T}^n x_n(\omega)] - F[x^*(\omega)]\rangle + t_n^2\mu^2\|Fx^*(\omega)\|^2\} dP(\omega)$$

$$= (1 - \tau t_n)\int_{\Omega}\|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) + 2\mu t_n\langle \int_{\Omega} Fx^*(\omega)dP(\omega), x^*(\omega) - T^n\tilde{T}^n x_n(\omega)\rangle$$

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$+2\mu t_n \int_\Omega \|Fx^*(\omega)\| \left\|x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right\| dP(\omega) +$
$2\mu^2 t_n^2 \int_\Omega \|Fx^*(\omega)\| \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$

$= (1-\tau t_n)\int_\Omega \|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) + \frac{2\mu t_n \tau}{\tau}\langle \int_\Omega Fx^*(\omega)dP(\omega), x^*(\omega) - x_n(\omega)\rangle$

$+\frac{2\mu t_n \tau}{\tau}\int_\Omega \|Fx^*(\omega)\| \left\|x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right\| dP(\omega) +$
$\frac{2\mu^2 t_n^2 \tau}{\tau}\int_\Omega \|Fx^*(\omega)\| \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\| dP(\omega)$

$= (1-\tau t_n)\int_\Omega \|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) + \tau t_n[\frac{2\mu}{\tau}\langle \int_\Omega Fx^*(\omega)dP(\omega), x^*(\omega) - x_n(\omega)\rangle +$

$\frac{2\mu}{\tau}\int_\Omega \{\|Fx^*(\omega)\| \left\|x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right\|\} dP(\omega) +$
$\frac{2\mu^2 t_n}{\tau}\int_\Omega \{\|Fx^*(\omega)\| \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\|\} dP(\omega)]$

$=(1-\tau t_n)\int_\Omega \|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) + \tau t_n[\frac{2\mu}{\tau}\langle \int_\Omega Fx^*(\omega)dP(\omega), x^*(\omega) - x_n(\omega)\rangle$

$+\frac{2\mu}{\tau}\int_\Omega \{\|Fx^*(\omega)\| \left\|x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right\|\} dP(\omega) + \frac{2\mu^2 t_n}{\tau} M]$

where $M \geq \sup_{n\geq 0}\int_\Omega \{\|Fx^*(\omega)\| \left\|F[T^n\tilde{T}^n x_n(\omega)]\right\|\} dP(\omega)$ is a fixed constant.

Let $c_n = \frac{2\mu}{\tau}\left[\langle \int_\Omega Fx^*(\omega)dP(\omega), x^*(\omega) - x_n(\omega)\rangle + \int_\Omega \{\|Fx^*(\omega)\| \left\|x_n(\omega) - T^n\tilde{T}^n x_n(\omega)\right\|\} dP(\omega)\right] + \frac{2\mu^2 t_n}{\tau} M$

$\int_\Omega \|x_{n+1}(\omega) - x^*(\omega)\|^2 dP(\omega) \leq (1-\tau t_n)\int_\Omega \|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) + \tau t_n c_n$ (3.14)

Clearly, $\limsup_{n\to\infty} c_n \leq 0$, because $\limsup_{n\to\infty}\langle \int_\Omega Fx^*(\omega)dP(\omega), x^*(\omega) - x_n(\omega)\rangle \leq 0$,
$\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}\int_\Omega \|x_n(\omega) - T^n x_n(\omega)\| dP(\omega) = 0$

By lemma 2.4, it is concluded that;

$\lim_{n\to\infty}\int_\Omega \|x_n(\omega) - x^*(\omega)\|^2 dP(\omega) = 0$, which implies convergence of $\{x_n(\omega)\}_{n=0}^\infty$ to $x^*(\omega)$, in quadratic mean.

This completes the proof.

Remark:

It is seen that the solution to the stochastic variational inequality problem, using the iteration belong to the fixed point set of the linear operator, T denoted by $Fix\ (T)$; and from proposition 2.1, the solution set of the stochastic split feasibility problem, Γ coincides with $Fix(T)$; that is $\Gamma = C \cap A^{-1}Q = Fix(T)$. By theorem 2.1, a unique solution $x^*(\omega) \in Fix(T)$ is guaranteed. This solution satisfies both the stochastic variational and the stochastic split feasibility problems; thereby meeting the conditions to be called a common solution to the two problems; hence, the solution to (1.5).

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NUMERICAL ILLUSTRATION OF RESULT

Suppose that a government classifies businesses in the state into two categories – low earning businesses, and high earning businesses; and wishes to find an optimal tax policy that would:

i.Meet revenue target,

ii. Minimize tax burden on low-income businesses,

iii. Encourage economic growth

The problem is defined and formulated as;

a) The two groups of businesses, low (1) and high (2), have the tax rates, respectively, denoted by $x_l(\omega)$ and $x_h(\omega)$.

b) Let the targeted revenue be denoted by L.

c) Suppose there are n_l low earning businesses, with average income, I_l; and n_h high earning businesses, with average income, I_h.

d) Let ϕ_l and ϕ_h respectively denote the maximum feasible tax rates for the low and high earning businesses.

The following Objective function, which is the tax component of Gross Domestic Production (GDP), is:

$$Max\ G = \gamma_0 + \gamma_l(1 - x_l(\omega)\ ) + \gamma_h\big(1 - x_h(\omega)\big) + \varepsilon,$$

subject to the following constraints:

$\sum_{i=1}^{2} \psi_i x_i(\omega) = L$ (Revenue constraint)

$0 \le x_l(\omega)\ \le \phi_l, 0 \le x_h(\omega)\ \le \phi_h$ (Tax rate constraint);

where;

- γ_0 is the baseline growth rate
- γ_l is the impact of low earning tax rate on GDP growth
- γ_h is the impact of high earning tax rate on GDP growth
- ε is the random error associated with W

The random SFP is expected to find

$$\boldsymbol{x}^*(\omega) \in C: A\boldsymbol{x}(\boldsymbol{\omega}) \in Q$$

where $C = \{(x_l^*(\omega), x_h^*(\omega)): 0 \le x_l \le \phi_1, 0 \le x_h \le \phi_2\}$

$$A = (\psi_1\ \psi_2)$$

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$$Q = \{(x_l^*(\omega), x_h^*(\omega)) : A(x_l^*(\omega), x_h^*(\omega)) \geq L$$

The mapping A takes tax rates from C, the feasible set, and produces revenue with the tax rates that meet the revenue target.

The random VIP, has the monotone, Lipschitz function, F, as the negative gradient of G. It is to provide equilibrium for tax payers. It ensures that the tax rates are optimal. It also ensures that tax policy is fair and acceptable to both groups.

Suppose there are 2,500 low earning businesses, with average monthly income of ₦200,000, and 800 high earning businesses earners, with average income of ₦1,000,000; and that the government is interested in raking in ₦ 150,000,000, from these businesses. Let the maximum tax rates for these groups of earnings be, respectively, 15% and 27%.

Giving real values to the parameters of the function, G, assume;

$$Max\ G = 0.5 + 0.065(1 - x_l(\omega)\ ) + 0.03\big(1 - x_h(\omega)\big) + \varepsilon$$

Subject to $0 \leq x_l(\omega) \leq 0.15$

$0 \leq x_h(\omega) \leq 0.27$

$500{,}000{,}000x_l(\omega) + 800{,}000{,}000x_l(\omega) \geq 150{,}000{,}000,$

where: $500{,}000{,}000 = 2{,}500{,}000 \times 200{,}000$ and $800{,}000{,}000 = 800 \times 1{,}000{,}000$.

The function, W, for the random VIP, is the negative gradients of the GDP growth function.

That is;

$$F(x_l(\omega), x_h(\omega)) = \left(-\frac{\partial G(x_l(\omega), x_h(\omega))}{\partial x_l}, -\frac{\partial G(x_l(\omega), x_h(\omega))}{\partial x_h}\right) = (0.065, 0.03).$$

Let $A = \begin{pmatrix} 0.84 & 0 \\ 0 & 0.71 \end{pmatrix}$ and its adjoint $A^* = \begin{pmatrix} 0.84 & 0 \\ 0 & 0.71 \end{pmatrix}$. Also, let $B = A^* \times A$

Then, the linear operator for the random SFP, is given as;

$$B\big(x(\omega)\big) = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x_l(\omega) \\ x_h(\omega) \end{pmatrix}.$$

Setting both the Lipschitzian and monotonicity coefficients to 1, then;

$\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ becomes $\mu \in (0, 2)$.

The iterative solution which is summarized in the Table 1, is carried out using R software (version 4.5.0).

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Volume 8, Issue 4, 2025 (pp. 109-125)

Table 1: Summary of iterations

Iterations	x_l	x_h
1	0.113350	0.109700
2	0.125320	0.118124
3	0.136010	0.125459
4	0.145527	0.131858
5	0.149350	0.137449
6	0.149350	0.142627
7	0.149350	0.147634
8	0.149350	0.152463
9	0.149350	0.157109
10	0.149350	0.161570
	⋮	
The last ten iterations	0.149350	0.213370
	0.149350	0.214989
	0.149350	0.216513
	0.149350	0.217947
	0.149350	0.219296
	0.149350	0.220564
	0.149350	0.221755
	0.149350	0.222875
	0.149350	0.223927
	0.149350	0.224915
Number of iterations	37	37

$\varepsilon \sim N(0,1)$

DISCUSSION

Having established that the iterative algorithm in 3.1 generates a sequence that is bounded, and has a random fixed-point solution, it is important to note that this is a point where the input of the algorithm is the same as the output. Furthermore, it is shown that this fixed-point solution is in the feasible set of the problem. A strong convergence result is obtained for the solution, since convergence in quadratic mean is a stronger mode of convergence, when compared to convergence in probability and convergence in distribution. This result extends and generalizes that of (Yamada, 2001) which introduced the hybrid steepest descent method of solving VIP, in the deterministic case.

In illustrating the result numerically, a hypothetical government tax policy optimization problem is formulated and solved, using the result. The concept of fixed-point solution is further exemplified here. Since the value (output) of a preceding iteration step serves as input to the succeeding iteration step, it is observed, from Table 3.1, that the difference between the

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last two iteration values is quite negligible; which means that the input is the same as the output. The result suggests the optimum tax rates for the low and high earning businesses to be 14.94% and 22.49%, respectively. These tax rates satisfy two conditions – meeting the revenue target, which is the random SFP component of the problem; and contributing, maximally, to the GDP growth, which is the random VIP part of the problem. For instance, substituting these tax rates into the revenue constraint, results in ₦ 254,620,000 which satisfies the constraint; at the same time, since it is the non-taxable income that goes back to circulation, thereby boosting the GDP, the expected amounts that go back to circulation are 85.06% of 500,000,000 and 77.51% of 800,000,000, for the low and high earning businesses, respectively.

CONCLUSION

In this paper, convergence result in classical functional analysis is extended, to develop convergence result of the solution of random split variational inequality problem, based on the random two-step hybrid steepest descent method. It is shown that the solution converges in quadratic mean to a unique random fix point in a Hilbert space. Furthermore, an application of the result to optimal tax policy problem is presented. In the application, optimum tax rates have been obtained: 14.94% and 22.49%, respectively for the low and high earning businesses. This result extends, and generalizes some established results in literature on classical functional analysis.

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Declaration of Interest

The authors report there are no competing interest to disclose.

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