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GUMBEL-EXPONENTIAL DISTRIBUTION: ITS PROPERTIES AND APPLICATION

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ABSTRACT: *This paper presents a mixture distribution of a new modelling tool, which is termed Gumbel-Exponential (GEXP) distribution. The distribution allows us to capture some real characteristics of data and it is an important tool for understanding the phenomenon. The various statistical properties of this distribution were fully explored and discussed. These include: the mean, variance, moments, mode, reliability function and hazard function. The worth of the mixing distribution has been demonstrated by applying it to real life data.*

KEYWORDS: Gumbel Distribution; Exponential Distribution; Reliability Function; Hazard Function; Maximum Likelihood Estimation.

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INTRODUCTION

Statistical analysis of real life problems has drawn great attention over the years as results obtained from these studies have been used efficiently to resolve a wide range of problems and enhanced policy formulation. The quality of procedure used in statistical analysis depends on the assumed probability model or distribution because this considerable effort has been expended in the large development classes of standard probability distribution along with relevant statistical methodology.

The use and application of mixing distributions in research related to reliability, biomedicine, ecology and several other areas are of tremendous practical importance in mathematics, probability and statistics. These distributions arise naturally as a result of observations generated from a stochastic process and recorded with some function.

The foundational concepts of these distributions have been employed in a wide variety of research. The need to enhance greater flexibility of various statistical distributions to many real life phenomena has led to the development of several new families of distributions like the generalized gamma distribution (Stacy, 1962), generalized normal distribution (Box & Tiao, 1962) and generalized logistic distribution (Prentice, 1976). Also, Johnson *et al.* (1995) summarized three types of extreme value distributions: Type I, Type II and Type III, with Type I also known as the Gumbel distribution, Type II the Frechet distribution and Type III the Weibull distribution. Some others include: the Beta-Gumbel (Nadarajah & Kotz, 2004), the Exponentiated Gumbel (Nadarajah, 2006), the generalized Gumbel (Cooray, 2010), Weibull-Pareto Distribution (Alzaatreh *et al.,* 2013; Alzaatreh *et al.,* 2014) and Multivariate Generalized Poisson (Famoye, 2015).

The application of some of these distributions includes the work of Mudholkar *et al.* (1995) which emphasized the usefulness of the Exponentiated Weibull Distribution compared to the 2-parameter Weibull distribution by studying applications on bus motor failure data. In his proposed Exponentiated Gumbel distribution, Nadarajah (2006) illustrated the application of the distribution by using Orlando, Florida rainfall data. Ewemoje and Ewemooje (2011) showed that Log-Pearson Type III distribution is the most suitable distribution for modeling on-site annual maximum flood flow for Ona River under Ogun-Oshun river basin. Ewemooje (2014) also modelled road traffic fatalities in Nigeria with Weibull distribution in order to curb rising traffic injuries and fatalities. Therefore, we propose mixture distribution of the classical Gumbel and Exponential distribution, which is termed Gumbel-Exponential (GEXP) distribution, with various reliability properties of this distribution fully explored, as well as a comparison of the performance of the proposed distribution with the individual distribution in modelling life data.

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MATERIAL AND METHODS

The Gumbel Distribution

A continuous random variable X is said to follow the Gumbel distribution if it has the probability density function (PDF) and cumulative distribution function (CDF) respectively.

$$f(x) = \frac{1}{\alpha} exp\ exp - \left(\frac{x-\varepsilon}{\alpha}\right) exp\left[- exp\ exp\ - \left(\frac{x-\varepsilon}{\alpha}\right)\right] \quad (1)$$

$$F(x) = \quad (2)$$

$-\infty < x < \infty,\ \alpha > 0, -\infty < \varepsilon < \infty$

where α and ε are scale and location parameters respectively (Johnson *et al.,* 1995).

If we define $Z = \frac{x-\varepsilon}{\alpha}$ and substitute into (1) and (2), then we would obtain the standard Gumbel distribution given by its PDF and CDF respectively by:

$$f(x) = exp - z\ exp[-exp - z] \quad (3)$$

$$F(x) = exp\ exp\ [-exp - z] \quad (4)$$

where z is the standard Gumbel random variable.

Moments of the Gumbel Distribution

The moment generating function (mgf) of a continuous random variable X is defined by:

$$M_X(t) = E[exp\ exp\ tx\ ] = txf(x)dx \quad (5)$$

$-\infty$

for $|t| < 1$, where $f(x)$ is the PDF of the given distribution. For the standard Gumbel distribution, the mgf is given by:

$$M_Z(t) = E[exp\ exp\ tx\ ] = tz\ exp\ exp\ - z\ exp\ exp\ [- exp\ exp\ - z\ ]\ dz \quad (6)$$

If we let

$$y = exp\ exp\ - z \implies when\ z = \infty, y = 0, when\ z = -\infty, y = \infty, dy = -exp - zdz.$$

Thus,

$$M_Z(t) = \int_{-\infty}^{0} y^{-}t\ exp\ exp\ - y(-dy)$$

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$$= \int_{-\infty}^{\infty} (exp\, exp\ -\ z\,)^{-} texp - z\, exp[-exp - z]\, dz$$

$$M_Z(t) = \Gamma(1\ -\ t) \tag{7}$$

Since any random Gumbel variable *X* with location and scale parameter ε and α respectively can be expressed as a linear transformation of a standard Gumbel variable *Z*, i.e.,

$Z = \frac{x-\varepsilon}{\alpha}$, it implies that

$$X\ =\ \varepsilon\ +\ \alpha Z,$$

Thus, the mgf of the Gumbel random variable *X* is given by:

$$M_X(t) = M_{\varepsilon+\alpha Z}(t)\ =\ E[exp\, exp\ (\varepsilon\ +\ \alpha Z)t\ ]$$

From (7) $M_{\alpha Z}(t)\ =\ E[exp\, exp\ \alpha zt\ ]\ =\ \gamma(1-\alpha t)$. Thus, the mgf of the Gumbel random variable *X* is given by:

$$M_x(t) = exp\, exp\ \varepsilon t\Gamma(1\ -\ \alpha t) \tag{8}$$

The r^th moment of the random variable *X* can be obtained by differentiating $M_X(t)$ r-times and evaluating the derivative at *t* = 0. Taking the logarithm of (8) gives the cumulant generating function $\phi(t)$ given by:

$$\phi(t)\ =\ logM_X(t)\ =\ log(exp\, exp\ \varepsilon t\gamma(1\ -\ \alpha t\ )$$

$$\phi(t)\ =\ \varepsilon t\ +\ log\gamma(1\ -\ \alpha t)$$

On differentiating $\phi(t)$ once and twice and setting *t* = 0, we obtain the mean and variance respectively.

$$\phi'(0)\ =\ \varepsilon\ -\ \alpha\psi(1)\ =\ \varepsilon\ +\ \alpha\gamma$$

It follows that the mean is given by:

$$\mu x = \ \varepsilon + 0.5772\alpha \tag{9}$$

Hence, the variance is given by:

$$\sigma_X^2 = \frac{\alpha^2\pi^2}{6} \tag{10}$$

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Method of Moment Estimates of the Gumbel Parameters

Given the expression for the mean and variance of the random Gumbel variable in (9) and (10), it implies that:

$$\hat{\varepsilon} = \underline{X} - 0.5772\hat{\alpha} \quad (11)$$

$$\hat{\alpha} = \frac{\sqrt{6}}{\pi} S_x \quad (12)$$

Maximum Likelihood Estimation of the Gumbel Parameters

The likelihood function of the Gumbel distribution for a random independent $sample\ x_1, x_2, \ldots, x_n$ of size n is:

$$L = \prod_{i=1}^{n} \left[\frac{1}{\alpha} exp\ exp\ - (\frac{x_i - \varepsilon}{\alpha} exp\ exp\ \left[- exp\ exp\ - \left(\frac{x_i - \varepsilon}{\alpha} \right) \right] \right] \quad (13)$$

Taking the natural logarithm of the likelihood function, we obtain the log-likelihood function given by:

$$\varepsilon = \alpha \left[In(n) - In \sum_{i=1} n\ exp\ exp\ - \left(\frac{x_i}{\alpha} \right) \right]$$

$$\underline{X} = \alpha + \frac{\sum_{i=1}^{n} x_i\ exp\ exp\ - \frac{x_i}{\alpha}}{\sum_{i=1}^{n} x_i\ exp\ exp\ - \frac{x_i}{\alpha}}$$

The hazard function of a probability distribution is defined as:

$$H(x) = \frac{f(x)}{1 - F(x)}$$

For the Gumbel distribution, the hazard function is given by:

$$H(x) = \frac{\frac{1}{\alpha} exp\ exp\ - \left(\frac{x - \varepsilon}{\alpha} \right) exp\ exp\ \left[- exp\ exp\ - \left(\frac{x - \varepsilon}{\alpha} \right) \right]}{1 - exp\ exp\ \left[- exp\ exp\ - \left(\frac{x - \varepsilon}{\alpha} \right) \right]} \quad (14)$$

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The reliability function of a probability distribution is defined as:

$$R(x) = 1 - F(x)$$

$$R(x) = 1 - exp\,exp\left[- exp\,exp - (\frac{x-\varepsilon)}{\alpha})\right] \quad (15)$$

Exponential Distribution

A random variable X is said to follow the exponential distribution if it has the probability density function (PDF) and cumulative distribution function (CDF) respectively:

$$f(x) = \frac{1}{\beta} exp\,exp - \frac{x}{\beta} \quad (16)$$

$$F(x) = 1 - exp\,exp - \frac{x}{\beta} \quad (17)$$

$$x > 0, \beta > 0$$

where β is a scale parameter (Johnson *et al.,* 1995)

Moments of the Exponential Distribution

The mgf of the exponential random variable X is given as:

$$M_x(t) = E[exp\,exp\,tx] = tx\frac{1}{\beta} exp\,exp - \frac{x}{\beta}\,dx$$

$$M_x(t) = \frac{1}{1 - \beta t} \quad (18)$$

The cumulant generating function is given by:

$$\phi(t) = log\,log\,M_x(t) = log\,log\left(\frac{1}{1-\beta t}\right) = -log\,(1 - \beta t) \quad (19)$$

On differentiating $\phi(t)$ once and twice and setting $t = 0$, we obtain the mean and variance of the exponential distribution respectively. In particular,

$$\phi'(t) = \frac{\beta}{1 - \beta t}$$

$$\phi''(t) = \beta^2 (1 - \beta t)^{-2}$$

$$\mu_x = \phi'(0) = \beta \quad (20)$$

$$\sigma_x^2 = \phi''(0) = \beta^2 \quad (21)$$

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Method of Moments Estimate of the Exponential Parameter

We see that the mean of the exponential random variable *X* is equal to the exponential scale parameter β. Thus, given a sample of observations $x_1, x_2, \ldots, x_n$, the method of moments estimate of the exponential scale parameter β is given by:

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (22)$$

Maximum Likelihood Estimation of the Exponential Parameter

The likelihood function of the exponential distribution for a random independent sample $x_1, x_2, \ldots, x_n$, of size *n* is:

$$L = \prod_{i=1}^{n}\left[\frac{1}{\beta} exp\, exp\, -\frac{x}{\beta}\right]$$

Taking the natural logarithm of the likelihood function, we obtain the log-likelihood function given by:

$$L = In\prod_{i=1}^{n}\left[\frac{1}{\beta} exp\, exp\, -\frac{x}{\beta}\right]$$

$$= -nIn\beta - \sum_{i=1} nx_i$$

Differentiating w.r.t β gives:

$$-\frac{n}{\beta} + \frac{\sum_{i=1}^{?}}{\beta^2}$$

Setting to zero, we obtain the maximum likelihood estimate of $\hat{\beta}$ as:

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (23)$$

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Hazard and Reliability Functions of the Exponential Distribution

The hazard function of the exponential distribution is given as:

$$H(x) = \frac{\frac{1}{\beta} exp\, exp\, -\frac{x}{\beta}}{1 - \left(1 - exp\, exp\, -\frac{x}{\beta}\right)} = \frac{1}{\beta} \tag{24}$$

The reliability function is given as:

$$R(x) = 1 - (1 - exp\, exp\, -\frac{x}{\beta})$$
$$= exp\, exp\, \frac{x}{\beta} \tag{25}$$

The T-X Family of Distributions

Alzaatreh *et al.* (2013) proposed a new method for generating new distributions. They utilized a random variable T defined on the interval $[\alpha, \beta], -\infty \le \alpha < \beta \le \infty$ with CDF and PDF $R(t)$ and $r(t)$ respectively and another random variable X with PDF and CDF $f(x)$ and $F(x)$ respectively. Using a transformation $W(F(x))$ of the CDF of X, they defined a new class of distribution by the CDF of the form:

$$G(x) = \int_{\alpha}^{W(F(x))} r(t)dt \tag{26}$$

where $W(.)$ satisfies the following conditions:

$(i)\ W(F(x)) \epsilon [\alpha, \beta]$

$(ii)\ W(F(x)) is\ monotonically\ non - decreasing\ and\ differentiable.$

$(iii)\ W(F(x)) \longrightarrow\ \alpha as x\ \longrightarrow\ -\infty$

$(iv)\ W(F(x))\ \longrightarrow\ \beta as x\ \longrightarrow\ -\infty$

They called the distribution in (26) the "T-X distribution." Several functions of W(.) were defined by the authors and a list of new families of distribution using T as the generator random variable was outlined.

Furthermore, if we let R(t) and r(t) be the CDF and PDF respectively of a continuous random variable T with support $(-\infty, \infty)$, we let X be any continuous random variable with CDF and PDF $F(x)$ and $f(x)$ respectively, and we define $W(F(x))$ as the logit of the CDF $F(x)$, i.e.,

$$W(F(x)) = In\left[\frac{F(x)}{(1 - F(x))}\right]$$

then the CDF of the T-X distribution based on the logit function is given by:

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$$G(x) = \int_{-\infty}^{In\left[\frac{F(x)}{(1-F(x))}\right]} r(t)dt = R[In\left[\frac{F(x)}{(1-F(x))}\right]] \quad (27)$$

]

The corresponding PDF of the T-X distribution in (27) is obtained by differentiating (27) w.r.t x to obtain

$$g(x) = r\left[In\left[\frac{F(x)}{\left(1-F(x)\right)}\right]\right]\frac{h(x)}{F(x)} \quad (28)$$

where $h(x) = \frac{f(x)}{1-F(x)}$ is the hazard function of the random variable X.

Observe that $R[In[\frac{F(x)}{1-F(x)}]] = R(T)$ This implies that $In\left[\frac{F(x)}{(1-F(x))}\right] = T$.

The following result holds:

$$T = In\left[\frac{F(x)}{\left(1-F(x)\right)}\right]$$

$$exp\ exp\ T = \frac{F(x)}{\left(1-F(X)\right)} F(x) = \frac{exp\ exp\ T}{(1 + exp\ exp\ T\ )}$$

$$X = F^{-1}[expT(1 + expT)] \quad (29)$$

The relation in (29) is useful in simulating the random variable X of the T-X distribution by first simulating the random variable T, which is the generator, and then applying the transformation accordingly. We can also make use of (29) in calculating the r^{th} moment of the T-X random variable X.

The Gumbel-Exponential (GEXP) Distribution

Suppose T is a random Gumbel variable with PDF and CDF given respectively by:

$$r(t) = \frac{1}{\alpha} exp\ exp\ -\left(\frac{t-\varepsilon}{\alpha}\right) exp\left[-exp\ exp\ -\left(\frac{t-\varepsilon}{\alpha}\right)\right]$$

$$R(t) =$$

$$-\infty < t < \infty, \alpha > 0, -\infty < \varepsilon < \infty$$

Also, let X follow an exponential distribution with PDF and CDF, as indicated below, respectively:

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$$f(x) = \frac{1}{\beta} exp\ exp\ -\frac{x}{\beta}$$

$F(x) = 1 - exp\ exp\ -\frac{x}{\beta}\ x > 0, \beta > 0$ It follows that,

$$In\left[\frac{F(x)}{\left(1 - F(x)\right)}\right] = In[exp\ exp\ -\frac{x}{\beta} - 1\,]$$

The CDF of the GEXP distribution thus follows and it is given by:

$$G(x) = \left[RIn\left(exp\ exp\ \frac{x}{\beta} - 1\right)\right]$$

$$= exp\ exp\ \left[-\ exp\ exp\ -\left(\frac{In\left[\frac{x}{\beta} - 1\right] - \varepsilon}{\alpha}\right)\right]$$

$$= exp\ exp\ \left[-\ exp\ exp\ \left(\frac{In\left[exp exp\ \frac{x}{\beta} - 1\right]}{\alpha}\right)\right]$$

$$= exp\ exp\ \left[-\ exp\ exp\ \frac{\varepsilon}{\alpha}\left(\frac{-In\left[exp\ exp\ \frac{x}{\beta} - 1\right]}{\alpha}\right)\right]$$

$$exp\ exp\ \left[-\ exp\ exp\ \frac{\varepsilon}{\alpha} exp\ exp\ In\ \left[exp\ exp\ \frac{x}{\beta} - 1\right]^{-\frac{1}{\alpha}}\right]$$

$$= exp\ exp\ \left[-\ exp\ exp\ \frac{\varepsilon}{\alpha}(exp\ exp\ \frac{x}{\beta} - 1)^{-\frac{1}{\alpha}}\right]$$

$$G(x) = exp\ exp\ [-\theta\ exp\ exp\ (exp\ exp\ \frac{x}{\beta} - 1)^{-\frac{1}{\alpha}}\,] \quad (30)$$

$$x > 0, exp\ exp\ \frac{\varepsilon}{\alpha} = \theta, -\infty < \varepsilon < \infty, \alpha, \beta > 0$$

Equation (30) gives the CDF of the GEXP distribution. Differentiating (5.1) w.r.t x gives the PDF of the GEXP distribution given by:

$$g(x) = \frac{\theta}{\alpha\beta} exp\ exp\ \frac{x}{\beta}\left(exp\ exp\ \frac{x}{\beta} - 1\right)^{-1-\frac{1}{\alpha}} exp\ exp\ \left[-\theta\left(exp\ exp\ \frac{x}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right] \quad (31)$$

where α and θ are shape parameters and β is a scale parameter.

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Moments of the Gumbel-Exponential Distribution

The r^{th} moments of the GEXP distribution can be obtained from the expression:

$$E(X^r) = \int_0^\infty x^r \frac{\theta}{\alpha\beta} exp\, exp\, \frac{x}{\beta} \left(exp\, exp\, \frac{x}{\beta} - 1\right)^{-1-\frac{1}{\alpha}} exp\, exp\, \left[-\theta\left(exp\, exp\, \frac{x}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right] dx \qquad (32)$$

Maximum Likelihood Estimation of the Parameters of the Gumbel-Exponential GEXP) Distribution

For a random independent sample $x_1, x_2, \ldots, x_n$ of size n, the likelihood function of the GEXP distribution is given by:

$$L = \prod_{i=1}^{n} \left[\frac{\theta}{\alpha\beta} exp\, exp\, \frac{x_i}{\beta} \left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-1-\frac{1}{\alpha}} exp\, exp\, \left[-\theta\left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right]\right]$$

The log-likelihood function is given by:

$$L = \sum_{i=1}^{n} In \left[\frac{\theta}{\alpha\beta} exp\, exp\, \frac{x_i}{\beta} \left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-1-\frac{1}{\alpha}} exp\, exp\, \left[-\theta\left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right]\right]$$

$$= n(In\theta - In\alpha - In\beta) + \sum_{i=1}^{n} \frac{x_i}{\beta} - \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^{n} \frac{x_i}{\beta} - 1 - \theta \sum_{i=1}^{n} In\, exp\, exp\, exp\, exp\, \frac{x_i}{\beta} - 1^{-\frac{1}{\alpha}}$$

Taking the partial derivative of the log-likelihood function w.r.t each of the parameters, we have:

$$\frac{\partial L}{\partial \alpha} = -\frac{n}{\alpha} + \frac{\sum_{i=1}^{n} In\left[exp\, exp\, \frac{x_i}{\beta} - 1\right]^2}{\alpha} - \theta \sum_{i=1}^{n} \frac{\left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-\frac{1}{\alpha}} In\left[exp\, exp\, \frac{x_i}{\beta} - 1\right]}{\alpha^2} \qquad (33)$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\theta} - \sum_{i=1}^{n} \left(exp\, exp\, \frac{x_i}{\beta} - 1\right)^{-\frac{1}{\alpha}} \qquad (34)$$

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$$\frac{\partial L}{\partial \alpha} = -\frac{n}{\beta} + \sum_{i=1}^{n} -\frac{x_i}{\beta^2} - \left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^{n} -\frac{x_i \, exp \, exp \, \frac{x_i}{\beta}}{\left(exp \, exp \, \frac{x_i}{\beta} - 1\right) \beta^2} - \theta \sum_{i=1}^{n} \frac{x_i \, exp \, exp \, \frac{x_i}{\beta} \left(exp \, exp \, \frac{x_i}{\beta} - 1\right)^{-1-\frac{1}{\alpha}}}{\alpha \beta^2} \quad (35)$$

On setting all the partial derivatives to zero and solving the system of equations iteratively, we obtain the maximum likelihood estimator of the parameters of the GUBXII distribution.

The Hazard and Reliability Function of the Gumbel-Exponential (GEXP) Distribution

The hazard function of the GEXP distribution is given by:

$$H(x) = \frac{\frac{\theta}{\alpha\beta} exp \, exp \, \frac{x}{\beta} \frac{x}{\beta} - 1)^{-1-\frac{1}{\alpha}} exp \, exp \left[-\theta \left(exp \frac{x}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right]}{1 - exp \, exp \left[-\theta \left(exp \, exp \, \frac{x}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right]} \quad (36)$$

The reliability function of the GEXP distribution is given by:

$$R(x) = 1 - exp \, exp \left[-\theta \left(exp \, exp \, \frac{x}{\beta} - 1\right)^{-\frac{1}{\alpha}}\right] \quad (37)$$

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RESULTS AND DISCUSSION

Presentation of Data

Here, we present an application of the GEXP distribution to lifetime data using two data sets. We also compare the fit of the Gumbel, exponential and the newly proposed GEXP distributions to the data sets in order to determine the flexibility of the newly proposed distribution in comparison to the classical Gumbel and exponential distributions.

Data Used for Analysis

Fitting the breaking stress of carbon fibers of 50 mm (gpa) data was used. The data set is unimodal and is approximately symmetric (Skewness = -0.13 and kurtosis = 0.34), while fitting the kevlar 49/epoxy strands failure times data (pressure at 90%) is multimodal, platykurtic, and approximately symmetric (Skewness = 0.35, kurtosis = 14.47).

Discussion of Results

Based on the information given about the first data set (breaking stress of carbon fibers data), the values of the log-likelihood obtained in each case helped in determining the maximum likelihood estimators of each distribution. The exponential gave -132.9944, Gumbel -92.3966, and the GEXP distribution -87.1243. The parameter estimates from Table 1 reveal that the exponential scale parameter $\hat{\beta} = 2.7595(0.3397)$. The exponential distribution has no shape parameter as the only parameter it has is the failure rate. The Gumbel distribution scale and location parameter are respectively obtained as: $\hat{\alpha} = 0.19114(0.0791)\hat{\varepsilon} = 2.3106(0.1191)$.

For the GEXP distribution, the scale and shape parameters are respectively gotten as: $\hat{\beta} = 0.1243(0.1923)\hat{\alpha} = 7.3502(11.3435)\hat{\theta} = 12.5340(2.6921)$ with standard error of estimates in parenthesis. These all help in determining the shape, scale, location, i.e., where the bulk lies for the distribution. The AIC, a model performance statistic, reported the smallest value of 177.7 for GWD and 180.2486 for the GEXP distribution while the Exponential and Gumbel distribution gave 267.9887 and 188.7932 respectively, an indicator of the superiority and flexibility of this new distribution in modelling unimodal and approximately symmetric data. Also, it could be inferred that Exponential distribution fits the least to this kind of data.

Table 1: Maximum likelihood estimates for breaking stress of carbon fibers data (standard errors of estimates in parenthesis)

Distributions	Exponential	Gumbel	GEXP	GWD
Parameters Estimates	$\hat{\beta} = 2.7595$ (0.3397)	$\hat{\alpha} = 0.1914$ (0.0791) $\hat{\varepsilon} = 2.3106$ (0.1191)	$\hat{\beta} = 0.1243$ (0.1923) $\hat{\alpha} = 7.3502$ 11.3435 $\hat{\theta} = 12.5340$ (2.6921)	$\hat{\beta} = 3.4359$ (1.1494) $\hat{\alpha} = 5.5673$ 2.8064 $\hat{\theta} = 2.4231$ (0.5078)
Log Likelihood	-132.9944	-92.3966	-87.1243	-84.83
AIC	267.9887	188.7932	180.2486	177.7

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From Table 2, the second data set (Kevlar 49/epoxy strands failure times data (pressure at 90%)) which is a multi-modal data, the log-likelihood of the parameter of the exponential distribution gave -103.4793, Gumbel -122.6389, and the GEXP distribution -100.3549. The parameter estimates reveal that the exponential scale parameter $\hat{\beta} = 1.0249(0.1019)$. The Gumbel distribution scale and location parameter are respectively obtained as: $\hat{\alpha} = 0.6494(0.0538)\hat{\varepsilon} = 0.6054(0.0674)$ while the GEXP distribution with scale shape and location parameter $\widehat{\beta} = 0.2533(0.0645), \hat{\alpha} = 3.3575(0.5994)$ and $\hat{\theta} = 1.6260(0.2813)$ respectively.

The AIC reports the smallest value of 206.7907 for the GEXP distribution and 208.5 for the GWD while the Exponential and Gumbel distribution gave 208.9586 and 249.2778 respectively—an indicator of the fitness and flexibility of the proposed (GEXP) distribution in modelling multimodal data.

Table 2: Maximum likelihood estimates for Kevlar 49/epoxy strands failure times data (pressure at 90%) (Standard errors of estimates in parenthesis)

Distributions	Exponential	Gumbel	GEXP	GWD
Parameters Estimates	$\hat{\beta} = 1.0249$ (0.1019)	$\hat{\alpha} = 0.6494$ (0.0538) $\hat{\varepsilon} = 0.6054$ (0.0674)	$\hat{\beta} = 0.2533$ (0.0645) $\hat{\alpha} = 3.3575$ (0.5994) $\hat{\theta} = 1.6260$ (0.2813)	$\hat{\beta} = 1.8064$ (0.5037) $\hat{\alpha} = 3.2713$ 0.6459 $\hat{\theta} = 0.9200$ (0.1594)
Log Likelihood	-103.4793	-122.6389	-100.3549	-100.23
AIC	208.9586	249.2778	206.7097	208.5

CONCLUSION

The rationale behind the proposition of a new family of probability distributions, either by combining two or more distributions or adding extra parameters to an existing distribution, is to make them more flexible and adaptive in capturing real life data. Here, we have tested the fitness and flexibility of the newly proposed GEXP distribution in comparison with the GWD and the classical Gumbel and exponential distributions using two sets of lifetime data, and the results obtained has clearly shown that the proposed distribution is more flexible and adaptable in capturing real life data than existing ones.

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