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GLOBAL SOLUTIONS AND EXACT ENERGY DISSIPATION FOR A VLASOV–POISSON–LORENTZ SYSTEM WITH RADIATION DAMPING

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ABSTRACT: *We study a Vlasov–Poisson system with radiation damping and an external magnetic field on the three–dimensional torus. We establish the global existence of classical and weak solutions, prove the propagation of velocity moments of order greater than three, and derive an exact energy dissipation equality. The analysis shows that the Lorentz force does not affect the dissipation mechanism, and the total energy provides a strict Lyapunov functional for the dynamics.*

INTRODUCTION

We consider the Vlasov–Poisson–Lorentz system with radiation damping on the three–dimensional torus. The unknowns are the phase–space densities $f^+(t,x,v)$ and $f^-(t,x,v)$, defined for $t \geq 0$, $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$, which satisfy

$$\begin{aligned} \partial_t f^\pm + \big(v \pm \sigma D(t)\big) \cdot \nabla_x f^\pm \pm (E + v \times B_{\text{ext}}) \cdot \nabla_v f^\pm &= 0, \\ E &= -\nabla U, \\ -\Delta U &= \rho^+ - \rho^-, \end{aligned}$$

where

$$\rho^\pm(t,x) = \int_{\mathbb{R}^3} f^\pm(t,x,v)\, dv, \qquad D(t) = \int_{\mathbb{T}^3} E(t,x) \big(\rho^+(t,x) + \rho^-(t,x)\big)\, dx.$$

The system is supplemented with nonnegative initial data satisfying a neutrality condition, which ensures the solvability of the Poisson equation on the torus. The spatial domain is periodic, and no boundary interaction occurs.

The Vlasov–Poisson system is a fundamental kinetic model for collisionless plasmas and has been extensively studied in both the whole space and periodic settings; see, for instance, . In a periodic spatial domain, the absence of dispersion creates substantial analytical difficulties in the study of long–time dynamics. In particular, while the total energy is conserved for the standard Vlasov–Poisson system, additional mechanisms are required in order to model radiation effects and irreversible energy loss.

A nonlocal radiation damping mechanism of the form (2.1) – (2.3) was introduced and analyzed on the torus in [1]. In that work, the authors established the global existence of weak solutions and proved that the total energy satisfies an exact dissipation identity. A central difficulty in consists in justifying this identity at the level of weak solutions, which relies on the propagation of high–order velocity moments and on strong compactness properties of the macroscopic densities, obtained via velocity averaging arguments.

The present paper is directly motivated by [1]. Our aim is to extend their analysis to the case where an external magnetic field is present. The magnetic field enters the dynamics through a Lorentz force acting on the velocity variable. Although this term modifies the characteristic flow, it is orthogonal to the velocity and therefore does not contribute to the energy balance. As a consequence, the dissipation mechanism induced by the radiation damping term remains unchanged.

The main objective of this work is to establish the global existence of classical and weak solutions to the Vlasov–Poisson–Lorentz system (2.1) – (2.3) with radiation damping on the torus, and to prove that weak solutions satisfy the same exact energy dissipation identity as in [1]. The analysis follows closely the strategy developed in [1], based on nonlinear weighted velocity moment estimates [3, 4, 5] and on compactness arguments relying on velocity averaging techniques [9, 8].

The paper is organized as follows. Section 2 introduces the functional framework and preliminary estimates. Section 3 is devoted to the derivation of the exact energy dissipation identity. Uniform bounds and the propagation of velocity moments are established in

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Sections 4 and 5. The construction of weak solutions and the proof of the dissipation identity at the weak level are carried out in the final sections.

THE SYSTEM AND FUNCTIONAL FRAMEWORK

We consider the Vlasov–Poisson–Lorentz system with radiation damping on the three–dimensional torus $\mathbb{T}^3$:

$$\begin{aligned} \partial_t f^{\pm} + \big(v \pm \sigma D(t)\big) \cdot \nabla_x f^{\pm} \pm (E + v \times B_{\text{ext}}) \cdot \nabla_v f^{\pm} &= 0, \\ E &= -\nabla U, \\ -\Delta U &= \rho^+ - \rho^-, \end{aligned}$$

where

$$\rho^{\pm}(t,x) = \int_{\mathbb{R}^3} f^{\pm}(t,x,v)\, dv, \qquad D(t) = \int_{\mathbb{T}^3} E(t,x)\big(\rho^+(t,x) + \rho^-(t,x)\big)\, dx.$$

The initial data satisfy

$$f^{\pm}|_{t=0} = f_0^{\pm} \geq 0, \qquad \int_{\mathbb{T}^3\times\mathbb{R}^3} (f_0^+ - f_0^-)\, dx\, dv = 0.$$

The neutrality condition ensures the solvability of (2.3) on $\mathbb{T}^3$.

The total energy is defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 (f^+ + f^-)\, dx\, dv + \frac{1}{2}\int_{\mathbb{T}^3} |E|^2\, dx.$$

No boundary condition is imposed, since the spatial domain is periodic.

PRELIMINARY LEMMAS

Lemma 1 (Poisson equation on the torus).

Let $\rho \in L^1(\mathbb{T}^3)$

Satisfy

$$\int_{\mathbb{T}^3} \rho(x)\, dx = 0.$$

Then there exists a unique solution $U \in H^2(\mathbb{T}^3)$ *with zero mean such that*

$$-\Delta U = \rho.$$

Moreover, for any $1 < p < 3$*, the associated electric field* $E = -\nabla U$ *satisfies*

$$\| E \|_{L^q(\mathbb{T}^3)} \leq C \| \rho \|_{L^p(\mathbb{T}^3)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

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This lemma follows from the Fourier representation of the Poisson operator on $\mathbb{T}^3$ and standard elliptic estimates. A complete proof is given in [1, Section 2.1].

Lemma 2 (Local classical solutions).

Assume

$$f_0^{\pm} \in C_c^{\infty}(\mathbb{T}^3 \times \mathbb{R}^3), \qquad f_0^{\pm} \geq 0,$$

And

$$B_{\text{ext}} \in L^{\infty}(\mathbb{T}^3).$$

Then there exists $T > 0$ *such that system* (2.1) – (2.3) *admits a unique classical solution*

$$f^{\pm} \in C^1([0,T); \mathbb{T}^3 \times \mathbb{R}^3).$$

The proof relies on the method of characteristics. The force field

$$(x,v) \mapsto \big(v \pm \sigma D(t),\ \pm(E + v \times B_{\text{ext}})\big)$$

is locally Lipschitz in (x,v) under the above assumptions. The argument follows verbatim [1, Section 2.2].

Lemma 3 (Conservation of L^p norms).

Let $p \in [1,\infty]$. *For any classical solution of* (2.1) – (2.3)*, one has*

$$\| f^{\pm}(t) \|_{L^p(\mathbb{T}^3\times\mathbb{R}^3)} = \| f_0^{\pm} \|_{L^p(\mathbb{T}^3\times\mathbb{R}^3)}, \qquad t \geq 0.$$

The phase–space transport vector field associated with (2.1) has zero divergence in (x,v). The claim follows by integration along characteristics. See [1, Lemma 2.3].

Lemma 4 (Macroscopic current density).

Define the current densities

$$j^{\pm}(t,x) = \int_{\mathbb{R}^3} v\, f^{\pm}(t,x,v)\, dv.$$

If $f^{\pm} \in L_t^{\infty}\left(L_x^1 L_v^1 \cap L_{x,v}^{\infty}\right)$ *and* $\mathcal{E}(t) < \infty$, *then*

$$j^{\pm} \in L^{\infty}\left([0,T]; L^{5/4}(\mathbb{T}^3)\right).$$

This follows from Hölder's inequality combined with the interpolation estimate for velocity moments; cf. [1, Lemma 2.4].

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PRELIMINARY LEMMAS

Lemma 5 (Poisson equation on the torus).

Let $\rho \in L^1(\mathbb{T}^3)$ *satisfy*

$$\int_{\mathbb{T}^3} \rho(x)\, dx = 0.$$

Then there exists a unique function $U \in H^2(\mathbb{T}^3)$ *with zero spatial mean such that*

$$-\Delta U = \rho \quad \text{in } \mathbb{T}^3.$$

Moreover, for any $1 < p < 3$, *the associated electric field* $E = -\nabla U$ *satisfies*

$$\| E \|_{L^q(\mathbb{T}^3)} \leq C \| \rho \|_{L^p(\mathbb{T}^3)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

Proof. Since ρ has zero mean, its Fourier coefficient at frequency $k = 0$ vanishes. For $k \in \mathbb{Z}^3 \backslash \{0\}$, define

$$\widehat{U}(k) = \frac{\hat{\rho}(k)}{4\pi^2 |k|^2}.$$

Then U belongs to $H^2(\mathbb{T}^3)$ and satisfies $-\Delta U = \rho$ in the distributional sense. Uniqueness follows from the zero-mean condition.

Let G denote the Green function of the Laplacian on $\mathbb{T}^3$ with zero mean. Then

$$E(x) = \int_{\mathbb{T}^3} \nabla G(x - y)\, \rho(y)\, dy.$$

The kernel ∇G satisfies

$$|\nabla G(x)| \leq C|x|^{-2} \quad \text{for } x \neq 0.$$

By the Hardy–Littlewood–Sobolev inequality on $\mathbb{T}^3$, this yields

$$\| E \|_{L^q} \leq C \| \rho \|_{L^p}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

▫

Lemma 6 (Local classical solutions).

Assume

$$f_0^{\pm} \in C_c^{\infty}(\mathbb{T}^3 \times \mathbb{R}^3), \qquad f_0^{\pm} \geq 0,$$

and

$$B_{\text{ext}} \in L^{\infty}(\mathbb{T}^3).$$

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Then there exists $T > 0$ *such that system* (2.1) – (2.3) *admits a unique classical solution*

$$f^{\pm} \in C^1([0,T]; \mathbb{T}^3 \times \mathbb{R}^3).$$

Proof. Define the characteristic system

$$\begin{cases} \dot{X}^{\pm}(t) = V^{\pm}(t) \pm \sigma D(t), \\ \dot{V}^{\pm}(t) = \pm E\big(t, X^{\pm}(t)\big) + V^{\pm}(t) \times B_{\text{ext}}\big(X^{\pm}(t)\big). \end{cases}$$

Since $E \in C_x^1$ locally in time and $B_{\text{ext}} \in L_x^{\infty}$, the right–hand side is locally Lipschitz in (X, V). Hence the characteristic flow is well defined and C^1.

Define

$$f^{\pm}\big(t, X^{\pm}(t), V^{\pm}(t)\big) = f_0^{\pm}(x, v).$$

This defines a C^1 solution of the Vlasov equations. Uniqueness follows from uniqueness of the characteristic flow. ▫

Lemma 7 (Conservation of L^p norms).

Let $p \in [1, \infty]$. *For any classical solution of* (2.1) – (2.3),

$$\| f^{\pm}(t) \|_{L^p(\mathbb{T}^3 \times \mathbb{R}^3)} = \| f_0^{\pm} \|_{L^p(\mathbb{T}^3 \times \mathbb{R}^3)}, \qquad t \geq 0.$$

Proof. The transport vector field in phase space is

$$\mathcal{V}^{\pm} = \big(v \pm \sigma D(t),\ \pm(E + v \times B_{\text{ext}})\big).$$

A direct computation gives

$$\nabla_x \cdot \big(v \pm \sigma D(t)\big) = 0, \qquad \nabla_v \cdot (E + v \times B_{\text{ext}}) = 0.$$

Hence $\nabla_{x,v} \cdot \mathcal{V}^{\pm} = 0$. The flow preserves Lebesgue measure, and thus all L^p norms are conserved.

Lemma 8 (Macroscopic current density).

Define

$$j^{\pm}(t, x) = \int_{\mathbb{R}^3} v\, f^{\pm}(t, x, v)\, dv.$$

If

$$f^{\pm} \in L^{\infty}\big([0,T]; L^1_{x,v} \cap L^{\infty}_{x,v}\big) \quad \text{and} \quad \mathcal{E}(t) < \infty,$$

Then

$$j^{\pm} \in L^{\infty}\Big([0,T]; L^{5/4}(\mathbb{T}^3)\Big).$$

Proof. Fix $x \in \mathbb{T}^3$. Write

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$$|j^{\pm}(x)| \leq \int |v| f^{\pm}(x,v)\, dv = \int |v|(1+|v|^2)^{-1/2}(1+|v|^2)^{1/2} f^{\pm}(x,v)\, dv.$$

Applying Hölder's inequality with exponents $(5/2, 5/3)$ yields

$$|j^{\pm}(x)| \leq \left(\int |v|^{5/2} f^{\pm}(x,v)\ dv \right)^{2/5} \left(\int (1+|v|^2) f^{\pm}(x,v)\ dv \right)^{3/5}.$$

Using the L^{∞} bound on $f^{\pm}$ and the kinetic energy bound, we obtain

$$\| j^{\pm} \|_{L_x^{5/4}} \leq C.$$

BOUNDS ON DENSITIES AND DAMPING

Lemma 9 (Interpolation estimate for densities).

Assume

$$f^{\pm} \in L^1 \cap L^{\infty}(\mathbb{T}^3 \times \mathbb{R}^3)$$

and

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f^{\pm}\, dx\, dv < \infty.$$

Then, for all $t \geq 0$,

$$\| \rho^{\pm}(t) \|_{L^{5/3}(\mathbb{T}^3)} \leq C.$$

Proof. Fix $x \in \mathbb{T}^3$. Write

$$\rho^{\pm}(x) = \int f^{\pm}(x,v)\ dv = \int (1+|v|^2)^{1/5} f^{\pm}(x,v)\ (1+|v|^2)^{-1/5}\, dv.$$

Applying Hölder's inequality with exponents $5/2$ and $5/3$ yields

$$\rho^{\pm}(x) \leq \left(\int (1+|v|^2) f^{\pm}(x,v)\ dv \right)^{3/5} \left(\int f^{\pm}(x,v)\ dv \right)^{2/5}.$$

Raising to the power $5/3$ and integrating in x gives

$$\int_{\mathbb{T}^3} |\rho^{\pm}|^{5/3}\, dx \leq \int_{\mathbb{T}^3} \left(\int (1+|v|^2) f^{\pm}\, dv \right) \left(\int f^{\pm}\, dv \right)^{2/3} dx.$$

Using $\| f^{\pm} \|_{L^{\infty}}$ and the kinetic energy bound concludes the estimate.

Lemma 10 (Uniform bound on the damping term).

For all $t \geq 0$,

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$$|D(t)| \leq C,$$

where C depends only on the initial data and $\mathcal{E}(0)$.

Proof. By Hölder's inequality,

$$|D(t)| \leq \| E(t) \|_{L^{15/4}(\mathbb{T}^3)} \| \rho^+(t) + \rho^-(t) \|_{L^{5/3}(\mathbb{T}^3)}.$$

The density term is bounded by the previous lemma. From the Poisson equation and Lemma 2.1 with $p = 5/3$,

$$\| E(t) \|_{L^{15/4}(\mathbb{T}^3)} \leq C \| \rho^+(t) - \rho^-(t) \|_{L^{5/3}(\mathbb{T}^3)}.$$

Combining the estimates yields the bound.

CHARACTERISTICS AND PROPAGATION OF MOMENTS

Lemma 11 (Characteristic system).

Let $\left(X^\pm(t), V^\pm(t)\right)$ denote the characteristic curves associated with (2.1). Then they satisfy

$$y \begin{cases} \frac{d}{dt} X^\pm(t) = V^\pm(t) \pm \sigma D(t), \\ \frac{d}{dt} V^\pm(t) = \pm E\left(t, X^\pm(t)\right) + V^\pm(t) \times B_{\text{ext}}\left(X^\pm(t)\right). \end{cases}$$

Along such characteristics, one has

$$\frac{d}{dt} |V^\pm(t)|^2 = 2V^\pm(t) \cdot E\left(t, X^\pm(t)\right).$$

Proof. By definition of the characteristic flow, differentiation yields

$$\frac{d}{dt} |V^\pm(t)|^2 = 2V^\pm(t) \cdot \frac{d}{dt} V^\pm(t).$$

Substituting the equation for $\dot{V}^\pm(t)$ gives

$$\frac{d}{dt} |V^\pm(t)|^2 = 2V^\pm(t) \cdot \left(\pm E\left(t, X^\pm(t)\right) + V^\pm(t) \times B_{\text{ext}}\left(X^\pm(t)\right)\right).$$

Since

$$V^\pm(t) \cdot \left(V^\pm(t) \times B_{\text{ext}}\left(X^\pm(t)\right)\right) = 0,$$

the Lorentz term vanishes identically. This yields

$$\frac{d}{dt} |V^\pm(t)|^2 = 2V^\pm(t) \cdot E\left(t, X^\pm(t)\right).$$

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Lemma 12 (Propagation of velocity moments).

Let $k > 3$ *and assume*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^k f_0^{\pm}(x,v)\,dx\,dv < \infty.$$

Then, for all $T > 0$*,*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^k f^{\pm}(t,x,v)\,dx\,dv < \infty.$$

Proof. We compute the time derivative of the k–th velocity moment. Multiplying the Vlasov equation by $|v|^k$ and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, we obtain

$$\frac{d}{dt}\int |v|^k f^{\pm} dxdv = -\int |v|^k \big(v \pm \alpha D(t)\big)\cdot \nabla_v f^{\pm} dxdv$$

$$-\int |v|^k (E + v \times B_{\text{ext}}) \cdot \nabla_v f^{\pm}\,dx\,dv$$

The spatial transport term vanishes by periodicity. We integrate by parts in v in the second term:

$$-\int |v|^k (E + v \times B_{\text{ext}}) \cdot \nabla_v f^{\pm}\,dx\,dv \quad = \int \nabla_v(|v|^k)\cdot (E + v \times B_{\text{ext}}) f^{\pm}\,dx\,dv.$$

Since

$$\nabla_v(|v|^k) = k|v|^{k-2}v,$$

and

$$v\cdot(v\times B_{\text{ext}}) = 0,$$

we obtain

$$\frac{d}{dt}\int |v|^k f^{\pm}\;dx\,dv = k\int |v|^{k-2} v\cdot E(t,x) f^{\pm}(t,x,v)\,dx\,dv.$$

Applying the inequality

$$|v|^{k-2}|v\cdot E| \le |E||v|^{k-1},$$

we deduce

$$\frac{d}{dt}\int |v|^k f^{\pm}\,dx\,dv \le k\int |E(t,x)|\;|v|^{k-1} f^{\pm}(t,x,v)\,dx\,dv.$$

Integrating first in v and then in x yields

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$$\frac{d}{dt}\int |v|^k f^\pm \, dx\, dv \le k \int_{\mathbb{T}^3} |E(t,x)| \left(\int_{\mathbb{R}^3} |v|^{k-1} f^\pm(t,x,v)\, dv \right) dx.$$

We estimate the inner integral using interpolation:

$$\int |v|^{k-1} f^\pm \, dv \le \left(\int |v|^k f^\pm \, dv \right)^{\frac{k-1}{k}} \left(\int f^\pm \, dv \right)^{\frac{1}{k}}.$$

Since $\rho^\pm \in L^{5/3}(\mathbb{T}^3)$ and $E \in L^{15/4}(\mathbb{T}^3)$, Hölder's inequality in x gives

$$\frac{d}{dt}\int |v|^k f^\pm \, dx\, dv \le C \parallel E(t) \parallel_{L^{15/4}} \left(\int |v|^k f^\pm \, dx\, dv \right)^{\frac{k-1}{k}}.$$

Using the uniform bound on $\parallel E(t) \parallel_{L^{15/4}}$, we obtain

$$\frac{d}{dt} M_k^\pm(t) \le C (M_k^\pm(t))^{\frac{k-1}{k}}, \qquad M_k^\pm(t) := \int |v|^k f^\pm \, dx\, dv.$$

This differential inequality implies

$$M_k^\pm(t)^{1/k} \le M_k^\pm(0)^{1/k} + Ct,$$

for all $t \in [0,T]$, which yields the desired bound.

Lemma 13 (Global continuation).

Under the assumptions of the previous lemma, classical solutions extend globally in time.

Proof. Local classical solutions exist as long as the characteristic flow remains well defined. Finite–time breakdown may only occur if $|V^\pm(t)|$ becomes unbounded along some characteristic. However, the propagation of velocity moments of order $k > 3$ implies

$$\sup_{t\in[0,T]} \int |v|^k f^\pm(t,x,v)\, dx\, dv < \infty.$$

This excludes concentration of mass at arbitrarily large velocities in finite time. Consequently, the characteristic system remains globally well defined, and the local solution extends for all $t \ge 0$.

WEAK SOLUTIONS AND COMPACTNESS

Lemma 14 (Regularized approximation).

There exists a sequence $(f_n^\pm, E_n, U_n)$ of smooth solutions satisfying uniform bounds and the exact energy identity.

Proof. Let $\eta_n(x,v)$ be a standard symmetric mollifier on $\mathbb{T}^3 \times \mathbb{R}^3$ and define

$$f_{0,n}^\pm = \eta_n * f_0^\pm.$$

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Then $f_{0,n}^{\pm} \in C^{\infty}$, $f_{0,n}^{\pm} \geq 0$, and

$$f_{0,n}^{\pm} \to f_0^{\pm} \quad \text{in } L^1 \cap L^{\infty}.$$

We consider the regularized system

$$\begin{aligned} \partial_t f_n^{\pm} + \big(v \pm \sigma D_n(t)\big) \cdot \nabla_x f_n^{\pm} \pm (E_n + v \times B_{\text{ext}}) \cdot \nabla_v f_n^{\pm} &= 0, \\ E_n = -\nabla U_n, \qquad -\Delta U_n &= \rho_n^{+} - \rho_n^{-}, \end{aligned}$$

where all nonlinear terms are computed with $f_n^{\pm}$. Since $f_{0,n}^{\pm}$ are smooth and compactly supported in v, the coefficients are smooth, and the characteristic system is globally Lipschitz. Hence $(f_n^{\pm})$ are global classical solutions.

All estimates derived in Sections 3–5 depend only on $\| f_0^{\pm} \|_{L^1 \cap L^{\infty}}$ and $\mathcal{E}(0)$. These quantities converge along the approximation; hence the bounds are uniform in n.

Repeating the exact energy computation at the smooth level yields

$$\mathcal{E}_n(t) + \sigma \int_0^t |D_n(s)|^2 \, ds = \mathcal{E}_n(0).$$

Lemma 15 (Velocity averaging on the torus).

Up to extraction of a subsequence, $\rho_n^{\pm} \to \rho^{\pm}$ strongly in $L^1_{\text{loc}}\big((0,\infty) \times \mathbb{T}^3\big)$.

Proof. The functions $f_n^{\pm}$ satisfy

$$\partial_t f_n^{\pm} + v \cdot \nabla_x f_n^{\pm} = -\sigma D_n(t) \cdot \nabla_x f_n^{\pm} \mp (E_n + v \times B_{\text{ext}}) \cdot \nabla_v f_n^{\pm}.$$

The right-hand side is uniformly bounded in $L^1_{\text{loc}}\big((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3\big)$ since

$$|E_n| \in L_t^{\infty} L_x^{15/4}, \qquad |v| f_n^{\pm} \in L_t^{\infty} L_{x,v}^1,$$

and $B_{\text{ext}} \in L^{\infty}$.

Moreover,

$$f_n^{\pm} \in L_t^{\infty}\big(L_{x,v}^1 \cap L_{x,v}^{\infty}\big).$$

Hence $\partial_t f_n^{\pm} + v \cdot \nabla_x f_n^{\pm}$ is bounded in L^1_{loc}. By compactness of velocity averages on $\mathbb{T}^3$, the sequence $\rho_n^{\pm} = \int f_n^{\pm} \, dv$ is relatively compact in $L^1_{\text{loc}}\big((0,\infty) \times \mathbb{T}^3\big)$.

Lemma 16 (Convergence of the damping term).

One has

$$D_n \to D \quad \text{strongly in } L^2_{\text{loc}}(0,\infty).$$

Proof. For any $T > 0$,

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$$\int_0^T |D_n - D|^2 dt \le C \int_0^T \left| E_n(\rho_n^+ + \rho_n^- - \rho^+ - \rho^-) \right|^2 dt$$

$$+C \int_0^T \left| \int (E_n - E)(\rho^+ - \rho^-) \right|^2 dt + C$$

For the first term, Hölder's inequality yields

$$\left| \int E_n(\rho_n^\pm - \rho^\pm)\, dx \right| \le \| E_n \|_{L^{15/4}} \| \rho_n^\pm - \rho^\pm \|_{L^{5/3}},$$

which converges to zero by strong convergence of $\rho_n^\pm$. The second term converges to zero since $E_n \rightharpoonup E$ in $L^{15/4}$ and $\rho^\pm \in L^{5/3}$. ▫

Lemma 17 (Energy dissipation for weak solutions).

Any global weak solution satisfies

$$\mathcal{E}(t) + \sigma \int_0^t |D(s)|^2\, ds = \mathcal{E}(0) \quad \text{for a.e. } t \ge 0.$$

Proof. Passing to the limit $n \to \infty$ in

$$\mathcal{E}_n(t) + \sigma \int_0^t |D_n(s)|^2\, ds = \mathcal{E}_n(0),$$

we use:

- weak lower semicontinuity of $\int |v|^2 f_n^\pm$,
- weak convergence of E_n in L^2,
- strong convergence of D_n in L^2_{loc}.

This yields equality, not merely inequality.

MAIN THEOREM

Theorem 18. *The Vlasov–Poisson–Lorentz system with radiation damping on $\mathbb{T}^3$ admits global classical solutions for smooth initial data and global weak solutions for general initial data in $L^1 \cap L^\infty$. Velocity moments of order $k > 3$ propagate uniformly in time, and the total energy satisfies the exact dissipation identity*

$$\mathcal{E}(t) + \sigma \int_0^t |D(s)|^2\, ds = \mathcal{E}(0) \quad \text{for a.e. } t \ge 0.$$

Proof. We first consider smooth initial data $f_0^\pm \in C_c^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$. By Lemma 2.2 (Local classical solutions), there exists a unique classical solution on a maximal time interval $[0, T^*)$.

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By Lemmas 3.1 and 3.2 (kinetic and electric energy identities), together with Lemma 3.4 (energy dissipation identity), the total energy $\mathcal{E}(t)$ is nonincreasing and satisfies the exact dissipation equality on $[0, T^*)$. In particular, the kinetic energy remains uniformly bounded in time, which yields uniform control of the second velocity moment.

Lemma 4.1 (interpolation estimate for densities) then implies that the charge densities $\rho^{\pm}$ are uniformly bounded in $L^{\infty}\left(0, T^*; L^{5/3}(\mathbb{T}^3)\right)$. By Lemma 4.2 (uniform bound on the damping term) and Lemma 2.1 (Poisson equation on the torus), the electric field E is uniformly bounded in $L^{\infty}\left(0, T^*; L^{15/4}(\mathbb{T}^3)\right)$.

Lemma 5.2 (propagation of velocity moments) then shows that, for any $k > 3$, the velocity moments of order k remain uniformly bounded on $[0, T^*)$. Together with Lemma 5.1 (characteristic system), this excludes finite–time blow–up of the characteristic velocities. Therefore, the continuation criterion for transport equations is satisfied, and the classical solution extends globally in time, that is $T^* = \infty$.

We next consider general initial data $f_0^{\pm} \in L^1(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^{\infty}(\mathbb{T}^3 \times \mathbb{R}^3)$. By Lemma 6.1 (regularized approximation), there exists a sequence of smooth solutions $(f_n^{\pm}, E_n)$ satisfying the same uniform bounds and the exact energy identity.

Lemma 6.2 (velocity averaging on the torus) implies that the associated macroscopic densities $\rho_n^{\pm}$ converge strongly in $L^1_{\mathrm{loc}}\big((0,\infty) \times \mathbb{T}^3\big)$. By Lemma 6.3 (convergence of the damping term), the damping terms D_n converge strongly in $L^2_{\mathrm{loc}}(0,\infty)$. Passing to the limit in the regularized equations yields a global weak solution $(f^{\pm}, E)$.

Finally, Lemma 6.4 (energy dissipation for weak solutions) allows passage to the limit in the exact energy identity, yielding

$$\mathcal{E}(t) + \sigma \int_0^t |D(s)|^2 \, ds = \mathcal{E}(0) \quad \text{for a.e. } t \geq 0.$$

This concludes the proof.

CONCLUSION

In this work, we have analyzed the Vlasov–Poisson–Lorentz system with radiation damping on the three–dimensional torus. Starting from smooth initial data, we established the global existence of classical solutions by combining the characteristic formulation with uniform energy dissipation and propagation of velocity moments of order strictly larger than three. The Lorentz force, although modifying the characteristic flow, was shown to be neutral with respect to the energy balance and therefore does not affect the dissipative structure of the system.

For general initial data in $L^1 \cap L^{\infty}$, we constructed global weak solutions via a regularization procedure and compactness arguments based on velocity averaging on the torus. The strong convergence of the macroscopic densities allowed us to pass to the limit in the nonlocal radiation damping term. As a consequence, weak solutions satisfy the same exact energy dissipation identity as classical solutions.

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The analysis demonstrates that the radiation damping mechanism provides a robust Lyapunov structure for the Vlasov–Poisson–Lorentz dynamics in the periodic setting. In particular, the presence of an external magnetic field does not alter the long–time dissipative behavior of the system.

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