



GRAPH TOPOLOGICAL INDICES AND ENERGIES OF A MIXED HOURGLASS GRAPH

Kekere Syaka Yusuf, Babarinsa Olayiwola*, and Johnson Fatokun.

Department of Mathematical Sciences,
Federal University Lokoja, P.M.B 1154, Kogi, Nigeria.

*Corresponding Author's Email: olayiwola.babarinsa@fulokoja.edu.ng;
Tel.: +2348060032554

Cite this article:

K. S., Yusuf, B., Olayiwola, J., Fatokun (2026), Graph Topological Indices and Energies of a Mixed Hourglass Graph. African Journal of Mathematics and Statistics Studies 9(1), 101-107. DOI: 10.52589/AJMSS-JVC7MCCV

Manuscript History

Received: 26 Dec 2025

Accepted: 28 Jan 2026

Published: 19 Mar 2026

Copyright © 2026 The Author(s).

This is an Open Access article distributed under the terms of Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0), which permits anyone to share, use, reproduce and redistribute in any medium, provided the original author and source are credited.

ABSTRACT: *Spectral graph theory has witnessed the development of topological indices based on degrees as effective methods of graph structure analysis, particularly in terms of graph energies. The current work explores degree-weighted spectral invariants (Nirmala and Sombor energies) on the mixed hourglass graph, which is topologically synthesized as a structure arising from the mixed adjacency representation of the WH factorization. The mixed hourglass graph is introduced as the graph containing both undirected edges and arcs, of which the vertex degrees serve as the central concept in terms of constructing the Nirmala and Sombor matrices. The relationship between the Nirmala and Sombor energies is the similarity between their graph spectra, and that the energies are even.*

MSC 2020 subject classification: 05C50, 05C90, 05C09, 15A23

KEYWORDS: WH factorization; Mixed hourglass graph; Topological index; Graph energy.



INTRODUCTION

A mixed adjacency matrix $M(G) = [m_{ij}]$ of a mixed graph G is an $n \times n$ matrix indexed by the vertices (v_1, \dots, v_n) , where $m_{ij} = 1$ if $v_i v_j \in E(G)$, $m_{ij} = -1$ if $v_i v_j \in A(G)$, and $m_{ij} = 0$ Otherwise, see [1, 2]. The mixed adjacency matrix formed the mixed graph. A mixed graph is the orientation of a subset of the undirected graph, with its edge set being the combination of the arcs and undirected edges (e_1, \dots, e_n) [3]. If a finite graph G possesses an even number of vertices and odd degree (number of edges) then the degree of the vertices is twice the number of edges. The degrees of vertices form several vertex-degree-based graph invariants called topological indices.

An index graph is a connectivity index with a mathematical formula that can be applied to any graph that models some molecular structure [4]. Nowadays, we have 200 topological indices of mathematics used in chemistry and other fields. These indices can form adjacency matrices with unique eigenvalues and the energy [5]. For instance, the Sombor index, $S^*(G)$, in Equation (1) is constructed by using Euclidean metrics and the Nirmala index, $N^*(G)$, (see Equation 2)

is a somewhat simplified variant of the Sombor index (that replaces $\sqrt{d(v_i)^2 + d(v_j)^2}$ with $\sqrt{d(v_i) + d(v_j)}$).

$$S^*(G) = \sum_{v_i v_j \in E(G)} \sqrt{d(v_i)^2 + d(v_j)^2} \quad (1)$$

where d_{v_i}, d_{v_j} are degrees of adjacent vertices v . Sombor matrix is a degree-weighted $n \times n$ matrix such that

$$S(G)_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$N^*(G) = \sum_{v_i v_j \in E(G)} \sqrt{d(v_i) + d(v_j)} \quad (2)$$

where d_{v_i}, d_{v_j} are degrees of adjacent vertices v . Nirmala matrix is a $n \times n$ matrix such that

$$\begin{cases} \sqrt{d_i + d_j} & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

The spectrum of a graph is the set of the eigenvalues of its adjacency matrix. The spectrum and energy of Sombor and Nirmala can be found in [5-7]. Ivan Gutman introduced, although originated from theoretical chemistry, the graph energy from an adjacency matrix [8, 9]. Therefore, mixed energy $E_M(G)$ of a mixed graph G is the sum of the absolute values of eigenvalues, $\lambda_i(G)$, of the adjacency matrix of the graph with order n [10].



$$E_M(G) = \sum_{i=1}^n |\lambda_i(G)|$$

PRELIMINARIES ON MIXED HOURGLASS GRAPH

The concept of mixed hourglass-adjacency matrix was coined from the hourglass matrix, which is obtained from quadratic interlocking factorization through modified Cramer's rule, see [11, 12]. An hourglass matrix is a nonsingular matrix of order $n(n \geq 3)$ with nonzero entries from the t th to the $(n - i + 1)$ th element of the i th and $(n - i + 1)$ th row of the matrix and 0's otherwise, for $i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$ [13]. A mixed hourglass-adjacency matrix $M(G)$ of a mixed hourglass graph G is a matrix $M(G)_{n \times n} = (h_{i,j})_{n \times n}$ defined as

$$M(G) = \{1 \text{ if } v_i, v_j \text{ is an edge} \quad -1 \text{ if } v_i, v_j \text{ is an arc} \quad 0 \text{ otherwise}$$

For $i = 1, 2, \dots, n$, $j = n - 1 + i$, and $n \geq 3$.

Thus, a mixed hourglass graph $G = (V, E, A)$ is an ordered triple consisting of a set of vertices $V(G) = \{v_1, v_2, \dots, v_n\}$, a set of undirected edges $E(G) = \{e_1, e_2, \dots, e_n\}$, and a set of directed arcs $A(G)$ [14].

Lemma 2.1. [14] The total number of undirected edges, $e(G)$, in a mixed hourglass graph is

$$e(G) = \frac{n - \beta_n^*}{2}$$

where

$$\beta_n^* = |(n + 1) \bmod 2 - 1|.$$

Corollary 2.2. [14] Let $A(G)$ be the number of arcs in a mixed hourglass graph G . Then

$$A(G) = \frac{n^2 - 2n + \beta_n^*}{2}$$

Theorem 2.3. [12] Let $E_M(G)$ be the mixed energy of a mixed hourglass graph G of order n . Then

$$E_M(G) = n - \beta_n^*$$

where

$$n - \beta_n^* = \{n \text{ if } n \text{ is even} \quad n - 1 \text{ if } n \text{ is odd}$$

Several results have been established on the mixed hourglass graph. Though the energy of a mixed hourglass graph was obtained; the Nirmala and Sombor energies of the mixed hourglass graph have not been considered.



NIRMALA AND SOMBOR ENERGY

To obtain the Nirmala and Sombor energy from the graph indices, we compute the degrees of the vertices, D_G , of a mixed hourglass graph as follows.

order	Vertices								
3							1	2	1
4					1		3	3	1
5				1	3		4	3	1
6			1	3	5		5	3	1
7			1	3	5	6	5	3	1
8		1	3	5	7	7	5	3	1
9	1	3	5	7	8	7	5	3	1

Theorem 2.4. The sum of the degrees of the vertices in a mixed hourglass graph $\sum_{i=1}^n D_G$ is

$$\sum_{i=1}^n D_G = \frac{n^2 - \beta_n^*}{2} \tag{2}$$

Proof. A mixed hourglass graph has a sink node located at the $v_{\frac{n+1}{2}}$ with $(n - 1)$ degree vertex. However, two vertices ($v_{\frac{n}{2}}$ and $v_{\frac{n+2}{2}}$) with $(n - 2)$ in-degree is even with each degree of the vertex having $(n - 1)$. That is,

$$\{v_{\frac{n}{2}} + v_{\frac{n+2}{2}} = 2(n - 1) \text{ if } n \text{ is even } v_{\frac{n+1}{2}} = n - 1 \text{ if } n \text{ is odd}$$

Based on the structure of the mixed hourglass graph, there are two-degree vertices (v_1 and v_n) with value 1, two-degree vertices (v_2 and v_{n-1}) with value 3, two degree vertices (v_3 and v_{n-2}) with value 5, two degree vertices (v_4 and v_{n-3}) with value 7 till two degree vertices (v_i and v_{n-i+1}) with value $[2(i - 1) + 1]$, for $i = 1, 2, \dots, n$. Therefore, the sum of the degrees of the vertices is

$$\sum_{i=1}^n D_G = \begin{cases} \sum_{i=1}^n [2(i - 1) + 1] + 2(n - 1) & \text{if } n \text{ is even} \\ \sum_{i=1}^n [2(i - 1) + 1] + n - 1 & \text{if } n \text{ is odd} \end{cases}$$

Evaluating $\sum_{i=1}^n D_G$ by the sum of an arithmetic progression, it is obvious that $\sum_{i=1}^n D_G \geq \frac{n^2}{2}$. Thus, $\sum_{i=1}^n D_G = \frac{n^2}{2}$ if n is even and $\sum_{i=1}^n D_G > \frac{n^2}{2} = \frac{n^2-1}{2}$. To harmonize the numerator, we introduce $\beta_n^* = |(n + 1) \bmod 2 - 1|$ from Lemma 2.1. This implies that $\beta_n^* = 1$ if n is odd and $\beta_n^* = 0$ when n is even. In all, we have

$$\sum_{i=1}^n D_G = \frac{n^2 - \beta_n^*}{2}$$

■



Corollary 2.5. The sum of the degrees of the vertices of a mixed hourglass graph is even.

Proof. This is the consequence of Theorem 2.4, since $\sum_{i=1}^n D_G$ is a divisor of 2. That

$$2 \sum_{i=1}^n D_G = n^2 - \beta_n^*$$

■

Corollary 2.6. Let $E_M(G)$ and $\sum_{i=1}^n D_G$ be the mixed energy, and the sum of the degrees of the vertices of a mixed hourglass graph G . Then

$$E_M(G) - \sum_{i=1}^n D_G < 0 \#$$

Proof. Recall, from Theorem 2.3, that

$$E_M(G) = n - \beta_n^*$$

and

$$\sum_{i=1}^n D_G = \frac{n^2 - \beta_n^*}{2}$$

Thus,

$$E_M(G) - \sum_{i=1}^n D_G = n - \beta_n^* - \left(\frac{n^2 - \beta_n^*}{2} \right) = \frac{2n - n^2 - \beta_n^*}{2} = \frac{-n^2 + 2n}{2} - \left(\frac{\beta_n^*}{2} \right)$$

Since

$$\beta_n^* = |(n+1) \bmod 2 - 1| = \{0 \text{ if } n \text{ is even } 1 \text{ if } n \text{ is odd}\}$$

then the value of $\left(\frac{\beta_n^*}{2} \right)$ is insignificant. That is

$$\frac{\beta_n^*}{2} = \{0 \text{ if } n \text{ is even } \frac{1}{2} \text{ if } n \text{ is odd}\}$$

But

$$-n^2 + 2n < 0,$$

provided that $n \neq 1, 2$. This confirms that $n \geq 3$ exist for every mixed hourglass graph. Thus

$$E_M(G) - \sum_{i=1}^n D_G < 0$$

■



Since the indices are established based on the degree of the vertices, thus, the sum of the degrees of the vertices is crucial in establishing the Nirmala and Sombor energies through the eigenvalues obtained from the adjacency matrices of their indices, as shown in Table 1 and Figure 1.

Table 1: The Values of Mixed, Nirmala, and Sombor Energy and their Graph Order

Order	Mixed energy	Nirmala energy	Sombor energy
3	2	2.82	2.82
4	4	7.72	11.30
5	4	7.72	11.30
6	6	14.04	25.44
7	6	14.04	25.44
8	8	21.52	45.24
9	8	21.52	45.24
10	10	30	66.86

Figure 1: Mixed, nirmala, and sombor energy of the mixed hourglass graph

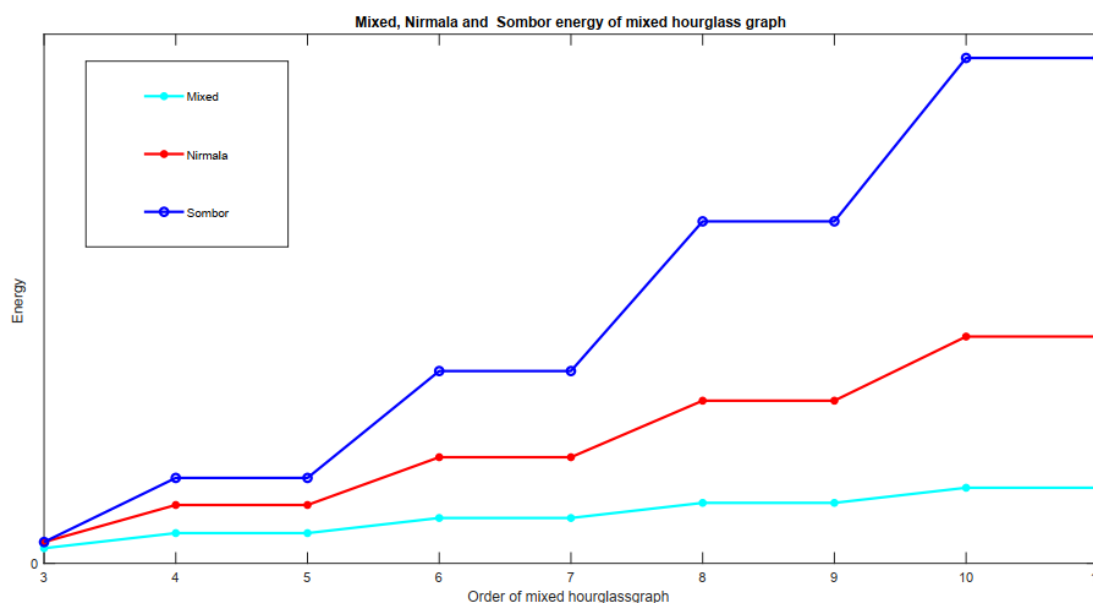


Figure 1 illustrates the Nirmala energy and Sombor energy as stairs or zigzags due to the similarity in values between the odd and even orders of the graph. However, the mixed energy from the graph is steep. The higher the order of the graph, the wider the differences between the values of the Nirmala energy and the Sombor energy. In all, it can be deduced that all values of graph energies are even.



Corollary 2.7. Let $E_M(G)$, $E_N(G)$ and $E_S(G)$ be the mixed energy, Nirmala and Sombor energy of a mixed hourglass graph G . Then

$$E_M(G), E_N(G), \text{ and } E_S(G) \text{ are even.}$$

Proof. This is a direct consequence of Theorem 2.3 (for $E_M(G)$) and Theorem 4.2.1 (for $E_N(G)$, and $E_S(G)$ by considering Corollary 2.5), which is well represented in Table 1.

CONCLUSION

This work extends the spectral analysis of mixed hourglass graphs by calculating and characterizing their Nirmala and Sombor energies, two new degree-based generalizations of the classical graph energy. Explicit derivations show that the sum of vertex degrees is even, and hence, this yields even values of Nirmala and Sombor energies for several graph orders. The staircase patterns occurring in energy values over successive orders mirror the fact that Nirmala and Sombor matrices possess common spectral properties. Comparisons with mixed energy provide inequalities and dependencies on structure repeatedly. These results not only reinforce the theoretical structure for degree-weighted topological indices but also speak to their potential when modeling hybrid networks, such as those in chemistry and communication systems.

REFERENCES

1. Beck, M., et al., *On weak chromatic polynomials of mixed graphs*. Graphs and Combinatorics, 2015. 31: p. 91-98.
2. Babarinsa, O., et al., *On Eigenvalues of Hermitian-Adjacency Matrix*. Communications in Mathematics and Applications, 2020. 11(2): p. 215.
3. Zhang, J. and H. Kan, *On the minimal energy of graphs*. Linear Algebra and its Applications, 2014. 453: p. 141-153.
4. Randić, M., *On the history of the Randić index and emerging hostility toward chemical graph theory*. MATCH Commun. Math. Comput. Chem, 2008. 59(1): p. 5-124.
5. Gutman, I., *Geometric approach to degree-based topological indices: Sombor indices*. MATCH Commun. Math. Comput. Chem, 2021. 86(1): p. 11-16.
6. Gutman, I., *Some basic properties of Sombor indices*. Open Journal of Discrete Applied Mathematics, 2021. 4(1): p. 1-3.
7. Kulli, V., *Nirmala index*. International Journal of Mathematics Trends and Technology-IJMTT, 2021. 67.
8. Li, X., Y. Shi, and I. Gutman, *Graph energy*. 2012: Springer Science & Business Media.
9. Gutman, I., *Hyperenergetic and hypoenergetic graphs*. Selected Topics on Applications of Graph Spectra, Math. Inst., Belgrade, 2011: p. 113-135.
10. Alawiah, N., et al., *New upper bounds on the energy of a graph*. MATCH Commun. Math. Comput. Chem, 2018. 79(2): p. 287-301.
11. Babarinsa, O. and H. Kamarulhaili, *Modification of Cramer's rule*. Journal of Fundamental and Applied sciences, 2017. 9(5): p. 556-567.
12. Babarinsa, O. and H. Kamarulhaili, *Mixed Energy of a Mixed Hourglass Graph*. Communications in Mathematics and Applications, 2019. 10(1): p. 45-53.
13. Babarinsa, O. and H. Kamarulhaili. *Quadrant interlocking factorization of hourglass matrix*. in *AIP Conference Proceedings*. 2018. Kuantan: AIP Publishing.
14. Babarinsa, O. and H. Kamarulhaili. *Mixed hourglass graph*. in *AIP Conference Proceedings*. 2019. Penang, Malaysia: AIP Publishing LLC.