PROBABILITIES OF MISCLASSIFICATION LINKED WITH INVERSE HYPERBOLIC SINE NORMAL (IHSN) DISTRIBUTION

Awogbemi Clement Adeyeye[1] and Olowu Abiodun Rafiu[2]

[1]Statistics Programme, National Mathematical Centre, Abuja, Nigeria

[2]Mathematics Programme, National Mathematical Centre, Abuja, Nigeria

E-mail of corresponding author: awogbemiadeyeye@yahoo.com

Cite this article:

Awogbemi C.A., Olowu A.R. (2021), Probabilities of Misclassification Linked with Inverse Hyperbolic Sine Normal (IHSN) Distribution. Advanced Journal of Science, Technology and Engineering 1(1), 42-51. DOI: 10.52589/AJSTE-BL0DW0R7.

Manuscript History

Received: 2 Oct 2021

Accepted: 23 Oct 2021

Published: 30 Dec 2021

ABSTRACT: *Probability of misclassification occurs when there is a choice of criteria that is not favourable for classification. The probabilities of misclassification associated with a family of Johnson's system, the Inverse Hyperbolic Sine Normal distribution, was developed in this study. The distribution theory and rules, along with the formulation of the system, were generated. It was asserted that the estimation of the parameters of the system could be demystified if one or more variables under consideration are distributed normally.*

KEYWORDS: Probability of Misclassification, Inverse Hyperbolic Sine Normal, Johnson's System, Transformation, Normality.

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

INTRODUCTION

Contributions on probabilities of misclassification associated with different distributions abound in the literature (Amoh & Kocherlakota, 1986; Mahmoud & Moustafa, 1995; Awogbemi & Onyeagu, 2019). Notable among these are the associations between probabilities of misclassification and Johnson's system of distribution, with the exclusion of inverse hyperbolic sine normal distribution. This exclusion has since created a gap that has not been resolved (Awogbemi et al., 2017; Awogbemi & Urama, 2020).

Inverse hyperbolic sine transformation was first suggested by Johnson (1949a), originally proposed by Johnson (1949b) and applied by Burbidge et al. (1988). The distribution of inverse hyperbolic sine models non-normality of marginal distributions. The interest is to transform a random variable in a way that allows for non-normality, but with explicit joint distribution function. The transformation maps the non-normal variables into a joint normal distribution that allows for contemporaneous interdependence (Octavio et al., 2011).

The purpose of this study is to generate the distribution theory and rules related to inverse hyperbolic sine normal, formulate the probabilities of misclassifications and probability density function associated with the inverse hyperbolic sine normal distribution, and also estimate hyperbolic sine normal.

Preliminaries

Let the hyperbolic sine be defined by $(x) = \frac{e^{x}-e^{-x}}{2}$ with an inverse of

$sinh^{-1}(y) = ln\,[y+(1+y^2)^{\frac{1}{2}}]$ *and* modification $f(x) = \frac{expexp\,(\vartheta x)\,-expexp\,(-\theta x)}{2\theta}$,

$where\ \ f^{-1}(y) = \frac{ln\,[\theta y+(\vartheta^2+1)^{1/2}]}{\theta}$ (Burbidge, 1988).

The modified hyperbolic sine ensures that the normal distribution becomes a special case as θ approaches zero. Using L' Hospital's rule, a linear function is generated as

$$\frac{expexp\,(\theta x)\,-exp\,(-\theta x)}{\theta} = 0\,\frac{[expexp\,(\theta x)\,-exp\,(-\theta x)]}{\frac{\partial}{\partial\theta}(\partial\theta)}$$

$$0\,\frac{[expexp\,(\theta x)\,-exp\,(-\theta x)]}{2}$$

$$= x$$

Thus, a random variable X is an inverse hyperbolic sine normal if $Y = sinh^{-1}(X)$ is normal, $-\infty < X < \infty$. The probability density function of Y is expressed as

$$f(y) = \frac{1}{\sigma_{y\sqrt{2\pi}}}\,exp\,(-\frac{1}{2\sigma_y^2}[y-u_{y]}]^2), -\infty < y < \infty \qquad (1)$$

where u_y, σ_y satisfy the conditions that $\sigma_y > 0$.

Transforming $Y = sinh^{-1}(X)$, the density function of X translates to

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

$$f(x) = \frac{1}{(x^2+1)^{1/2}\sigma_y\sqrt{2\pi}} \exp[-\frac{1}{2\sigma_y^2}(sinh^{-1}(x) - u_y)^2], -\infty < y < \infty \quad (2)$$

It is assumed that σ_y^2 *emanates from populations* π_1 and π_2 with X∈ $\pi_1, \mu_y = \mu_{1y}$ *and* X∈ π_2, $\mu_y = \mu_{2y}$.

METHODS

Conditional Probabilities of Misclassification for Inverse Hyperbolic Sine Normal

When an observation from π_1 is misclassified, the conditional probability of misclassification of IHSN is expressed as

$$e_{12}(\underline{x}_1, \underline{x}_2) = \int_{\frac{(\underline{x}_1,\underline{x}_2)}{2}}^{\infty} \frac{1}{(x^2+1)^{\frac{1}{2}}\sigma_{y\sqrt{2\pi}}} exp\,[-\frac{1}{2\sigma_y^2}(sinh^{-1}(x) - \mu_{1y})^2]dx,\ when\ \underline{x}_1 < \underline{x}_2$$

$$= \int_{-\infty}^{\frac{(\underline{x}_1,\underline{x}_2)}{2}} \frac{1}{(x^2+1)^{\frac{1}{2}}\sigma_{y\sqrt{2\pi}}} exp\,[-\frac{1}{2\sigma_y^2}(sinh^{-1}(x) - \mu_{1y})^2]dx,\ when\ \underline{x}_1 \geq \underline{x}_2 \quad (3)$$

By transforming the variable of integration, $k = \frac{sinh^{-1} - \mu_{1y}}{\sigma_y}$ and setting $\alpha = \frac{(\underline{x}_1,\underline{x}_2)}{2}$,

$$e_{12}(\underline{x}_1, \underline{x}_2) = \int_{[sinh^{-1}(\alpha) - \mu_{1y}]/\sigma_y}^{\infty} \frac{1}{\sqrt{2\pi}} exp\ exp\left[-\frac{t^2}{2}\right] dt, when\ \underline{x}_1 < \underline{x}_2$$

$$= \int_{-\infty}^{[sinh^{-1}(\alpha) - \mu_{1y}]/\sigma_y} \frac{1}{\sqrt{2\pi}} exp\ exp\left[-\frac{t^2}{2}\right] dt, when\ \underline{x}_1 \geq \underline{x}_2 \quad (4)$$

Equation (4) can be expressed as

$$e_{12}(\underline{x}_1, \underline{x}_2) = 1 - \Phi[\frac{sinh^{-1}(\alpha) - \mu_{1y}}{\sigma_y}]\ when\ \underline{x}_1 < \underline{x}_2$$

$$= \Phi[\frac{sinh^{-1}(\alpha) - \mu_{1y}}{\sigma_y}]\ when\ \underline{x}_1 \geq \underline{x}_2 \quad (5)$$

Substituting for α in equation (5), we have the conditional probability of misclassification when an observation from population π_1 is misclassified.

$$e_{12}(\underline{x}_1, \underline{x}_2) = 1 - \Phi[\frac{sinh^{-1}\left(\frac{(\underline{x}_1,\underline{x}_2)}{2}\right) - \mu_{1y}}{\sigma_y}], when\ \underline{x}_1 < \underline{x}_2$$

$$= \Phi[\frac{sinh^{-1}\left(\frac{(\underline{x}_1,\underline{x}_2)}{2}\right) - \mu_{1y}}{\sigma_y}],\ when\ \underline{x}_1 \geq \underline{x}_2 \quad (6)$$

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

Thus, the conditional probability of misclassification for IHSN when an observation from population π_2 is misclassified is given by

$$e_{21}(\underline{x}_1,\underline{x}_2) = \Phi\left[\frac{sinh^{-1}\left(\frac{(\underline{x}_1,\underline{x}_2)}{2}\right)-\mu_{2y}}{\sigma_y}\right], when\ \underline{x}_1 < \underline{x}_2$$

$$= 1-\Phi\left[\frac{sinh^{-1}\left(\frac{(\underline{x}_1,\underline{x}_2)}{2}\right)-\mu_{2y}}{\sigma_y}\right],\ when\ \underline{x}_1 \geq \underline{x}_2 \qquad (7)$$

When $X \in \pi_i$, $E(X) = -d\ sinh\ sinh\ (\mu_{iy})$, $Var(X) = \frac{1}{2}[d^2-1]\{-d^2$ $cosh\ cosh\ (2\mu_{iy})\ +1\}$, $i = 1, 2;$
where $d = exp\ exp\ \left[\frac{\sigma_y^2}{2}\right]$, (see Johnson, 1949).

Distribution of Conditional Probability of Misclassification

From equation (5), $e_{21}(\underline{x}_1,\underline{x}_2) \leq z\ \ when$

$$\underline{x}_1 < \underline{x}_2\ and\ \ \underline{x}_1+\underline{x}_2\ \geq 2\ sinh\ sinh\ [\mu_{1y}-\sigma_y\Phi^{-1}_{(z)}]\ \ or$$

$$\underline{x}_1 \geq \underline{x}_2\ and\ \ \underline{x}_1+\underline{x}_2\ \leq 2\ sinh\ sinh\ [\mu_{1y}-\sigma_y\Phi^{-1}_{(z)}] \qquad (8)$$

$$Pr\ \{e_{21}(\underline{x}_1,\underline{x}_2) \leq z\} = Pr\ \{(\underline{x}_2-\underline{x}_1) > 0, (\underline{x}_1+\underline{x}_2) \geq 2\ sinh\ sinh\ [\mu_{1y}-\sigma_y\ \Phi^{-1}(z)]\}$$
$$+ Pr\ \{(\underline{x}_2-\underline{x}_1)\ \leq 0, (\underline{x}_1+\underline{x}_2) <\ 2\ sinh\ sinh\ [\mu_{1y}-\sigma_y\ \Phi^{-1}(z)]\} \qquad (9)$$

$Let\ M = \underline{x}_2 - \underline{x}_1\ and\ N = \underline{x}_1 + \underline{x}_2$ so that

$$Pr\ \{e_{21}(\underline{x}_1,\underline{x}_2) \leq z\} = Pr\ Pr\ \{M > 0, N > b_1\}\ + Pr\ Pr\ \{M \leq 0, N < b_2\}\ , \qquad (10)$$

$where\ b_1\ = 2sinh\ [\mu_{1y}-\sigma_y\Phi^{-1}(z)]$ and $b_2\ = 2sinh\ [\mu_{1y}+\sigma_y\Phi^{-1}(z)]$.

By also standardizing M and N, we have,

$$Pr\ \{e_{21}(\underline{x}_1,\underline{x}_2) \leq z\}$$
$$= Pr\ Pr\ \left\{\frac{M-\mu_m}{\sigma_m} > -\frac{\mu_m}{\sigma_m}, \frac{N-\mu_n}{\sigma_n} > \frac{b_1-\mu_n}{\sigma_n}\right\} +$$
$$Pr\ Pr\ \left\{\frac{M-\mu_m}{\sigma_m} < -\frac{\mu_m}{\sigma_m}, \frac{N-\mu_n}{\sigma_n} < \frac{b_2-\mu_n}{\sigma_n}\right\}$$

$$= \ Pr\ Pr\ \left\{Z_1 > -\frac{\mu_m}{\sigma_m}, Z_1 > \frac{b_1-\mu_n}{\sigma_n}\right\}\ + Pr\ Pr\ \left\{Z_2 < -\frac{\mu_m}{\sigma_m}, Z_2 < \frac{b_2-\mu_n}{\sigma_n}\right\} \qquad (11)$$

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

Approximation of the Distribution of Sample Means using Normal Distribution

The distribution of $\underline{x}_1, \underline{x}_2$ for large samples n_1, n_2 is approximated using normal distribution to have $(Z_1\ Z_2\ ) = SBVN[(0\ 0\ ), (1\ \rho^*\ \rho^*\ 1\ )]$.

Equation (11) is thus expressed as

$$Pr\ \{e_{21}\left(\underline{x}_1, \underline{x}_2\right) \leq z \approx H_{\rho^*}\left[\frac{\mu_m}{\mu_m}, \frac{\mu_n - b_1}{\sigma_v}\right] + H_{\rho^*}\left[-\frac{\mu_m}{\mu_m}, \frac{b_2 - \mu_n}{\sigma_n}\right], \tag{12}$$

where $\mu_m = -q\left[sinh\ sinh\ \left(\mu_{2y}\right)\ - sinh\ sinh\ \left(\mu_{1y}\right)\right]$,

$$\mu_n = -q\left[sinh\ sinh\ \left(\mu_{1y}\right)\ - sinh\ sinh\ \left(\mu_{2y}\right)\right],$$

$$\sigma_m^2 = \sigma_n^2 = \frac{1}{2}(q^2 - 1)\left[-\frac{q^2 cosh\ (2\mu_{1y})}{n_1} - \frac{q^2 cosh\ (2\mu_{2y})}{n_2}\right] + \frac{1}{n_1} + \frac{1}{n_2} \text{ and}$$

$$\sigma_{mn} = \frac{1}{2}(q^2 - 1)\left[\frac{q^2 cosh\ (2\mu_{1y})}{n_1} - \frac{q^2 cosh\ (2\mu_{2y})}{n_2}\right] - \frac{1}{n_1} + \frac{1}{n_2}$$

ALITER:

In the alternative, $Pr\ \{e_{21}\left(\underline{x}_1, \underline{x}_2\right) \leq z\} \approx H[h_{11}, h_{21}; \rho^*] + B[h_{12}, h_{22}; \rho^*]$, (13)

where $h_{11} = \frac{\{-q\left[sinhsinh\ \left(\mu_{2y}\right) - sinhsinh\ \left(\mu_{1y}\right)\right]\}}{\sigma_m} = -h_{12}$,

$$h_{21} = \frac{\left[-q\left[sinhsinh\ \left(\mu_{2y}\right) + sinhsinh\ \left(\mu_{1y}\right)\right] - 2sinh\ \{\mu_{1y} - \sigma_y \Phi^{-1}(z)\}\right]}{\sigma_m}$$

$$h_{22} = \frac{\left[2sinh\ \{\sigma_y \Phi^{-1}(z) + \mu_{1y}\} + q\left(\mu_{2y}\right) + sinhsinh\ \left(\mu_{1y}\right)\}\right]}{\sigma_m}$$

$$\rho^* = \frac{\left[\frac{\{-q^2 coshcosh\ (2\mu_{2y}) + 1\}}{n_2} - \frac{\{q^2 coshcosh\ (2\mu_{1y}) + 1\}}{n_1}\right]}{\left[\frac{\{-q^2 coshcosh\ (2\mu_{1y}) + 1\}}{n_1} + \frac{\{-q^2 coshcosh\ (2\mu_{2y}) + 1\}}{n_2}\right]}$$

From the distribution function in equation (13),

$$P(z) = \int_{-\infty}^{h_{21}(z)} \int_{-\infty}^{h_{11}} f(z_1, z_2) dz_1 dz_2 + \int_{-\infty}^{h_{22}(z)} \int_{-\infty}^{-h_{11}} f(z_1, z_2) dz_1 dz_2 \tag{14}$$

Therefore, the probability density function of the probability of misclassification denoted by

$$p(z) = \int_{-\infty}^{h_{11}} f[z_1, h_{21}(z)] h'_{21}(z) dz_1 + \int_{-\infty}^{-h_{11}} f[z_1, h_{22}(z)] h'_{22}(z) dz_1 \tag{15}$$

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

Expected Probability of Misclassification of IHSN

Let the expected (unconditional) probability of misclassification of $e_{21}\left(\underline{x}_1, \underline{x}_2\right)$ be defined by

$$E[Pr\ \{e_{21}\left(\underline{x}_1, \underline{x}_2\right)\} = Pr\ \{\left(\underline{x}_2 - \underline{x}_1\right) > 0, X - \frac{1}{2}(\underline{x}_1 + \underline{x}_2) > 0/X \in \pi_1$$

$$+ Pr\ \{\left(\underline{x}_2 - \underline{x}_1\right) \leq 0, X - \frac{1}{2}(\underline{x}_1 + \underline{x}_2) \leq 0/X \in \pi_1 \qquad (16)$$

(see Awogbemi, 2018).

Also, let the distributions of $\underline{x}_1, \underline{x}_2$ *and* X be independent with

$$g(x) = \frac{1}{(x^2+1)^{\frac{1}{2}}} exp\ [-\frac{1}{2\sigma_y^2}(sinh^{-1}(x) - \mu_{1y})^2], -\infty < x < \infty \qquad (17)$$

The parameters μ_{1y}, σ_y satisfy the conditions, $-\infty < \mu_{1y} < \infty$, σ_y.

By Central Limit Theorem (CLT),

$\underline{x}_1 \approx N(\mu_1, \frac{\sigma_1^2}{n_1})$, $\mu_1 = E\left(\underline{x}_1\right) = -qsinh\left(\mu_{1y}\right)$ and

$$\sigma_1^2 = V\left(X \middle| X\epsilon\pi_1\right) = \frac{1}{2}(q^2 - 1)[-q^2\ cosh\ cosh\ \left(2\mu_{1y}\right) + 1]$$

$\underline{x}_2 \approx N(\mu_2, \frac{\sigma_2^2}{n_2})$, $\mu_2 = E\left(\underline{x}_2\right) = -qsinh\left(\mu_{2y}\right)$ and

$$\sigma_2^2 = V\left(X \middle| X\epsilon\pi_2\right) = \frac{1}{2}(q^2 - 1)[-q^2\ cosh\ cosh\ \left(2\mu_{2y}\right) + 1],$$

where $q = exp\ (\frac{\sigma_y^2}{2})$.

The RHS of equation (16) is determined conditionally and the conditionality is also removed so that the second term of equation (16) is now expressed as

$$Pr\ Pr\ \left\{\left(\underline{x}_2 - \underline{x}_1\right) > 0, X - \frac{1}{2}\left(\underline{x}_1 + \underline{x}_2\right) \geq 0\right\}$$

$$= \int_{-\infty}^{\infty} \{\left(\underline{x}_2 - \underline{x}_1\right) > 0, \underline{x}_1 + \underline{x}_2 \leq 2x \middle| x\} g(x) dx \qquad (18)$$

Let the unconditional probability corresponding to the probability in the integrand in a form involving the bi-variate normal distribution function, be written as $p(x)$ so that

$$p(x) = Pr\ Pr\ \{M > 0, n < 2x\}, \qquad (19)$$

where $M = \underline{x}_2 - \underline{x}_1, N = \underline{x}_2 + \underline{x}_1$, and $(M\ N) \approx [(\mu_m\ \mu_n), (\sigma_m^2\ \sigma_{mn}\ \sigma_{mn}\ \sigma_n^2)]$.

Thus, $p(x) = Pr\ \{Z_1 > -\frac{\mu_m}{\sigma_m}, Z_2 \leq \frac{2x - \mu_n}{\sigma_n}\}$

$$= \int_{-\frac{\mu_m}{\sigma_m}}^{\infty} \int_{-\infty}^{\frac{2x-\mu_n}{\sigma_n}} h_\rho(z_1, z_2) dz_1 dz_2$$

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

$$= \int_{-\infty}^{\frac{\mu_m}{\sigma_m}} \int_{-\infty}^{\frac{2x-\mu_n}{\sigma_n}} h_{-\rho}(z_1, z_2) dz_1 dz_2$$

$$= H_{-\rho}\left[\frac{\mu_m}{\sigma_m}, \frac{2x-\mu_n}{\sigma_n}\right] \quad (20)$$

Considering the equivalence of the third term of equation (18) so that

$$Pr\, Pr\, \{(\underline{x}_2 - \underline{x}_1) \leq 0, (\underline{x}_1 + \underline{x}_2) \geq 2x\} = \int_{-\infty}^{\infty} p'(x) g(x) dx, \quad (21)$$

where $p'(x) = Pr\, Pr\, (U \leq 0, V \leq 2x)$

$$= Pr\, \{Z_1 \leq -\frac{\mu_m}{\sigma_m}, z_2 \geq \frac{2x-\mu_n}{\sigma_n}\}$$

$$= \int_{-\infty}^{\frac{\mu_m}{\sigma_m}} \int_{\frac{2x-\mu_n}{\sigma_n}}^{\infty} h_{\rho}(z_1, z_2) dz_1 dz_2$$

$$= \int_{-\infty}^{\frac{\mu_m}{\sigma_m}} \int_{\infty}^{\frac{2x-\mu_n}{\sigma_n}} h_{\rho}(z_1, z_2) dz_1 dz_2$$

$$= H_{-\rho}\left[-\frac{\mu_m}{\sigma_m}, \frac{\mu_n - 2x}{\sigma_n}\right] \quad (22)$$

Therefore, the expected probability of misclassification IHSN is expressed as

$$E(\{e_{21}(\underline{x}_1, \underline{x}_2))\} = \int_{-\infty}^{\infty} g(x)\left[H_{-\rho}\left(\frac{\mu_m}{\sigma_m}, \frac{2x-\mu_n}{\sigma_n}\right) + H_{-\rho}\left(-\frac{\mu_m}{\sigma_m}, \frac{\mu_n-2x}{\sigma_n}\right)\right], \quad (23)$$

where $g(x) = \frac{1}{(x^2+1)^{\frac{1}{2}}\sigma_y\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma_y^2}(sinh^{-1}(x) - \mu_{1y})^2\right], \infty < x < -\infty$

Transformation of Random Variables Using Inverse Hyperbolic Sine

The basic fundamental involves estimating the parameters that transform a random variable into another distribution with desirable characteristics. In this instance, transformation of a random variable allows for normality.

Let $y = b(x)$ be a monotonic transformation that can be used to define a distribution for y $[f_y(y)]$ based on the distribution of x $[f_x(x)]$. Then the distribution of y is expressed as

$$f_y(y) = \left|\frac{db^{-1}(y)}{dy}\right| f_x[b^{-}(y)], \quad (24)$$

where $f_y(y)$ is the marginal distribution of y, b^{-1} is the inverse mapping based on $y = b(x)$, and $f_x(x)$ is the distribution of x.

Given an IHS function, a univariate inverse hyperbolic sine distribution is expressed as

$$b(y_k, \varphi) = \frac{ln\,[\varphi y_k + (y_k^2+1)^{\frac{1}{2}}]}{\varphi}, \quad (25)$$

where y_k is the observed variable and φ is a parameter that allows the distribution to be leptokurtic. The form of transformation used is

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

$$U_k = y_k - \hat{y}_k \tag{26}$$

$$V_k = \frac{ln\,[\varphi U_k + (U_k^2\varphi^2+1)^{\frac{1}{2}}]}{\varphi} \tag{27}$$

$$e_k' = V_k - \mu, \tag{28}$$

where y_k is the deviation from the deterministic model, $\hat{y}_k$ is the prediction, V_k is the transformed deviation and μ is the centrality constant which allows for skewness along with φ.

The univariate transformation is extended to multivariate space as

$$f_y(y) = f_x[b^{-}(y)] \begin{vmatrix} \frac{\partial b_1^{-1}(y)}{\partial y_1} & \cdots & \frac{\partial b_1^{-1}(y)}{\partial y_n} \\ \frac{\partial b_2^{-1}(y)}{\partial y_1} & \cdots & \frac{\partial b_2^{-1}(y)}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial b_m^{-1}(y)}{\partial y_1} & \cdots & \frac{\partial b_m^{-1}(y)}{\partial y_n} \end{vmatrix}, \tag{29}$$

where $b^{-1}(y)$ is a vector inverse mapping function and $b_i^{-1}(y)$ is the inverse mapping function function for the i^{th} element from the mapping function. The Jacobian matrix of the transformation is

$$\frac{\partial b_i^{-1}(y)}{\partial y_j} = [1 + (\varphi_i y_i)^2]^{-\frac{1}{2}}, \;\; i = j$$

$$\frac{\partial b_i^{-1}(y)}{\partial y_j} = 0, \;\; i \neq j \tag{30}$$

With the determinant of the Jacobian as the product of the diagonal elements, the multivariate density function is generated as

$$f_{u_k} = \frac{1}{(2\pi)^{\frac{1}{2}}}|\Omega|^{-\frac{1}{2}} exp\left[-\frac{1}{2}(V_k-\mu)\Omega^{-1}(V_k-\mu)'\right]\prod_{i=1}^{m} [1 + (U_{ik}\varphi_i)^2]^{-\frac{1}{2}}$$

$$V_k = \left(\frac{ln\{U_{1k}\varphi_1 + [(U_{1k}\varphi_1)^2+1]\}^{\frac{1}{2}}}{\varphi_1}\; \frac{ln\{U_{2k}\varphi_2 + [(U_{2k}\varphi_2)^2+1]\}^{\frac{1}{2}}}{\varphi_2} \cdots \frac{ln\{U_{2k}\varphi_2 + [(U_{mk}\varphi_m)^2+1]\}^{\frac{1}{2}}}{\varphi_m}\right), \tag{31}$$

where Ω is the variance matrix of the transformed residuals, V_k is the vector of transformed residuals, μ is the vector of non-centrality parameters and φ is the vector of peakedness parameters.

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

Estimation of Inverse Hyperbolic Sine System

A maximum likelihood estimation method is being used in this study to estimate the parameters of the probability density function in equation (31). This procedure jointly estimates the parameters of the linear trend and the non-normal transformation. The individual series is first tested using the parametric procedure proposed by Bera and Jarque (1982). In order to estimate the non-normality parameters, the ordinary least squares estimate of the trend and standard errors are used as initial estimates.

The ordinary least squares generates

$$\hat{m} = (X'X)^{-1}(X'y)$$

$$\hat{e} = y - X\hat{b} \qquad (32)$$

where $\hat{m}$ denotes the estimated trend parameters, X is a matrix consisting of a column of 1s and a linear trend column, y is the observed value, and $\hat{e}$ is a vector of estimated residual.

The maximum likelihood method is being used to fit φ and μ based on the residuals as follows:

(a)$L_a = -\frac{k}{2} ln(\sigma^2) - \frac{1}{2}\sum_{k=1}^{k} \quad \frac{(V_k-\mu)^2}{\sigma^2} - \frac{1}{2}\ ln\ (1 + \hat{e}_k^2\varphi^2)$

$$V_k = \frac{ln[\hat{e}_k^2\varphi + (\hat{e}_k^2\varphi^2+1)^2]}{\varphi} \qquad (33)$$

The estimated deviations in (33) are used based on estimated trend parameters, which by construction are efficient. Thus, the results of equation (33) are used as starting values in the likelihood function.

(b) $L_b = -\frac{k}{2} ln(\sigma^2) - \frac{1}{2}\sum_{k=1}^{k} \quad \frac{(V_k-\mu)^2}{\sigma^2} - \frac{1}{2}\ ln\ (1 + \hat{e}_k^2\varphi^2)$

$$U_k = y_k - \alpha_o + \alpha_1 \text{k}$$

$$V_k = \frac{ln[V_k\varphi + (U_k^2\varphi^2+1)^{\frac{1}{2}}]}{\varphi} \qquad (34)$$

In particular, $2(L_a - L_b) \sim \chi^2_{(2)}$.

(c) The individual maximum likelihood results are used as starting points to estimate the system. Using the log of equation (31) as likelihood function,

$$L_c = -\frac{k}{2} ln|\Omega| - \frac{1}{2}\sum_{k=1}^{k} \quad (V_k - \mu)\Omega^{-1}(V_k - \mu)' - \frac{1}{2}\sum_{k=1}^{k} \quad \sum_{i=1}^{m} \quad (1 + U_{i,k}^2\varphi_i^2)$$

$$U_{ik} = y_{ik} - \alpha_{io} - \alpha_{il}k, i = 1, \dots m$$

$$z_{ik} = ln\ \frac{[U_{ik}\varphi_i] + (V_{ik}^2\varphi^2+1)^{\frac{1}{2}}]}{\varphi_i} \qquad (35)$$

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7

Equation (35) may be complex to maximize, but the estimation can be demystified if one or more of the variables are normally distributed. For instance, if the first variable is normal, then $\varphi \rightarrow 0$, equation (35) becomes linear and the Jacobian transformation also turns to 1.

CONCLUSION

The rules and theory related to the distribution have been presented with an alternative approach for the approximation of the sample mean using normal distribution. Normality is a special case of transformation, and it is feasible to test directly for using estimated parameters. It is established that the estimation of the parameters of the inverse hyperbolic sine system can be demystified if one or more of the variables are distributed normally. The transformation of the system to normality consists of an explicit interaction term.

REFERENCES

Amoh, R.K. & Kocherlakota, K. (1986). Errors of Misclassification Associated with the Inverse Gaussian Distribution, *Journal of Communication in Statistics,* 15(2), pp. 589-612.

Awogbemi, C.A. & Onyeagu, S.I. (2019). Errors of Misclassification Associated with Edgeworth Series Distribution. *American Journal of Theoretical and Applied Statistics,* 8(6), pp 203- 213.

Awogbemi, C.A. & Urama, K.U. (2020). On Probabilities of Misclassification Related to Logit- normal Distribution. *African Journal of Mathematics and Statistics Studies,* 3(2), pp 22-77.

Awogbemi, C.A. (2019). Errors of Misclassification Associated with Edgeworth Series Distribution, Unpublished Ph.D Dissertation, Department of Statistics, Faculty of Physical Sciences, Nnamdi Azikiwe University, Awka, Nigeria.

Awogbemi, C.A., Urama, K.U. & Alagbe, S.A. (2017). Probabilities of Misclassification Associated with Log-normal Distribution, *ABACUS (Mathematics Science Series*), 44(2), pp 88-97.

Burbidge, J.B. , Magee, L. & Robb, A.L. (1988). Alternative Transformation to Handle Extreme Values of the Dependent Variable. *Journal of the American Statistical Association*, 83, pp 123-127.

Johnson, N.L. (1949a). Systems of Frequency Curves Generated by Methods of Translation. *Biometrika*, 36, pp 149-177.

Johnson, N.L. (1949b). Bivariate Distributions based on Simple Translation System, *Biometrika*, 36, pp 297-304.

Mahmoud, M.A. & Moustafa, H.M. (1995). Errors of Misclassification Associated with Gamma Distribution, *Journal of Mathematical Computing* Modeling, 22(3), pp 105-119.

Octavio, A.R., Charles , B.M, & William, G.B. (2011). Estimation and Use of the Inverse Hyperbolic Sine Transformation to Model Non-normal Correlated Random Variables, *Journal of Applied Sciences*, 21(4), pp 289-304.

Article DOI: 10.52589/AJSTE-BL0DW0R7
DOI URL: https://doi.org/10.52589/AJSTE-BL0DW0R7